A PRODUCT IN HOMOTOPY THEORY

HARUO SUZUKI

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1. Introduction. H. Samelson conjectured, in his paper [1] that the Whitehead product in homotopy groups satisfies an analogous relation to the Jacobi identity in Lie algebras. This is stated also by A. L. Blakers and W. S. Massey [6]. We refer to the relation as the Jacobi identity in Whitehead products.

The present paper proves the identity for elements of dimension >1. For this purpose we introduce a new product in homotopy groups of an H-space (See section 3 below and J.-P. Serre [2]) by means of the product operation of the space. We call the product an H-product. It is connected to the Pontrjagin product of homology groups (cf. L. Pontrjagin [4], H. Hopf [5]) and is interesting itself (see section 3, Proposition 2 below).

This product is bilinear for elements of dimension $\geqq 2$ and is not associative but under some additional conditions[1] satisfies a modified form of the Jacobi identity. In the lacet spaces [2] the relation holds and is translated to the Jacobi identity in Whitehead products of the original space, using certain isomorphisms. These isomorphisms are Eilenberg's suspension for homotopy groups (see section 2 below) in a fiber space of paths starting from a fixed point.

2. Preliminaries. Let X be an arcwise connected topological space and x_0 be a fixed point in it. We consider a space whose elements are paths beginning at x_0 with compact-open topology and denote it by E. A mapping which associates each element of E with its terminal point, is continuous and denoted by P. Moreover it it well known that E is a fiber space with a base space X, projection P and a fiber, the lacet space Ω_X relative to x_0 (see J.-P. Serre [2]).

Let p and n be integers such that $1<p\leqq n, f$ be a mapping from an n-dimensional cube I^n (an n-fold product space of $I=[0,1]$) into X such that $f(\dot{I}^n)=x_0$ where $\dot{I}^n$ is the boundary of I^n. Under these notations we define a mapping T_pf of I^{n-1} into Ω_X by the formula

(1) $$T_pf(x_1, \dots x_{n-1})(t) = f(x_1, \dots x_{p-1}, t, x_p, \dots, x_{n-1}),$$

(this definition has its sense if only the faces $x_p=0$ and $x_p=1$ of I^n go into x_0). T_p is one-to-one and induces a homomorphism of $\pi_n(X, x_0)$ into $\pi_{n-1}(\Omega_X, x_0)$ for $n>1$, where x_0 is also a constant path $I \rightarrow x_0 \in X$. We also denote this homomorphism by T_p. Let Σ_p be the inverse of T_p;

(2) $$\Sigma_pf'(x_1, \dots, x_{p-1}, t, x_p, \dots x_{n-1}) = f'(x_1, \dots, x_{n-1})(t),$$

where f' is a mapping of I^{n-1} into Ω_X, then we have

1) §5, Theorem 3 in this paper.

(3) $$\Sigma_p T_p f = f.$$

A homomorphism of homotopy groups induced by Σ_p is denoted by Σ_p.

PROPOSITION 1. *T_p is an isomorphism of $\pi_n(X, x_0)$ onto $\pi_{n-1}(\Omega_X, x_0)$*[2)] *and Σ_p in its inverse.*

The proof is trivial. Moreover we have the relations $T_p = (-1)^{p+q}T_q$ $(1 < p, q \leqq n)$, $T_n = \partial P_*^{-1}$ which were shown by H. Samelson [1], where ∂ is the boundary homomorphism of the homotopy group $\pi_n(E, \Omega_X, x_0)$ to $\pi_{n-1}(\Omega_X, x_0)$ (this is an isomorphism onto, P_* is an isomorphism of $\pi_n(E, \Omega_X, x_0)$ onto $\pi_n(X, x_0)$ induced by the projection P. Hence a relation $T_p = (-1)^{n+p}\partial P_*^{-1}$ holds.

T_n is the transgression and Σ_n the Eilenberg's suspension for n and $(n-1)$ dimensional homotopy groups (cf. J.-P. Serre [2, pp. 453]). For the sake of convenience we write T, Σ instead of T_n, Σ_n respectively.

REMARK. The isomorphism T_n was given by W. Hurewicz [9] for the first time.

COROLLARY 1. *If A is a subset of X containing x_0, then for $n > 2$ we have*

$$\pi_n(X, A, x_0) \approx \pi_{n-1}(\Omega_X, \Omega_A, x_0).$$

The isomorphism is induced by T_p $(p < n)$.

PROOF. Consider the exact homotopy sequences of pairs (X, A, x_0) and $(\Omega_X, \Omega_A, x_0)$. T_p induces a homomorphism of the first sequence to the second. In fact, in the diagram $(n > 2)$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_n(X, x_0) & \xrightarrow{j_*} & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \xrightarrow{i_*} & \pi_{n-1}(X, x_0) & \longrightarrow \cdots \\
 & \downarrow T_p & & \downarrow T_p & & \downarrow T_p & & \downarrow T_p & \\
\cdots \longrightarrow & \pi_{n-1}(\Omega_X, x_0) & \xrightarrow{j_*} & \pi_{n-1}(\Omega_X, \Omega_X, x_0) & \xrightarrow{\partial} & \pi_{n-2}(\Omega_A, x_0) & \xrightarrow{i_*} & \pi_{n-2}(\Omega_X, x_0) & \longrightarrow \cdots
\end{array}$$

homomorphisms of each square are commutative. Making use of Proposition 1 above and the five lemma (Eilenberg-Steenrod [7]), our result is obtained immediately.

COROLLARY 2. *For a triad $(X; A, B, x_0)$, where $x_0 \in A \cap B$, and for $n > 3$, we obtain*

$$\pi_n(X; A, B, x_0) \approx \pi_{n-1}(\Omega_X; \Omega_A, \Omega_B, x_0)$$

The isomorphism is induced by $T_p (2 < p < n)$.

The proof is analogous to that of the Corollary 1 above.

3. A new product in homotopy groups of the *H*-space.

DEFINITION 1. We call a space X with a product operation $\vee$, satisfying following conditions, an H-space and denote it by $(X, \vee)$:

2) If Ω_X is arcwise connected i.e. X is a simply connected space, we can take x_0 as the base point ot homotopy groups of Ω_X without any loss of generality. Even if X has not this property, as for isomorphism T_p, it is enough to consider the arcwise connected component containing x_0, therefore the condition is not so restrictive.

(H. 1). The mapping $(x, y) \to x \vee y$ is a continuous mapping of the space $X \times X$ into X.

(H. 2). There exists a *fixed* point $x_0 \in X$ such that $x_0 \vee x_0 = x_0$ and the continuous mappings of X into itself: $x \to x \vee x_0$, $x \to x_0 \vee x$ are homotopic to the identical mapping of X by two *fixed* homotopies $H_l(x, t)$, $H_r(x, t)$ which leave the point x_0 invariant (cf. J. P. Serre [2, PR 474]).

REMARK. This definition is somewhat different from that of J.-P. Serre. The latter treats the homology theory, therefore it does not need to fix the point x_0 and the homotopies of (H. 2).

For example, Topological groups and lacet spaces become H-spaces. In topological groups the operation of multiplication is regarded as $\vee$, the unit element as x_0 and the two homotopies of (H. 2) are trivial. In lacet spaces an ordinary product of paths [10, VIII, § 46, pp. 217-8] is considered as $\vee$, a fixed constant path as x_0 and the two homotopies of (H. 2) are these induced by a homotopic transformation of parameters, which remove the constant path at one end point [10, VIII, § 46, pp. 217-8]. These homotopies in lacet spaces play a fundamental role to prove the modified form of the Jacobi identity for the H-product (see Theorems 1, 2).

Let X be an arcwise connected space and f_n, g_n be mappings from the n-dimensional cube I^n into the space X such that the restrictions of these mappings on $\dot{I}^n$ agree, i. e. $f_n | \dot{I}^n = g_n | \dot{I}^n$. Similarly to the theory of S. Eilenberg [8, § 1], we define a mapping $d(f_n, g_n)$ of an n-dimensional sphere S^n to X as follows: $d(f_n, g_n) | I^n_+$ is induced by f_n, $d(f_n, g_n) | I^n_-$ is induced by g_n, where I^n_+, I^n_- are two copies of I^n identified on the boundaries and represent upper and lower hemispheres of S^n respectively. Hence we have $I^n_+ \cup I^n_- = S^n$ and $I^n_+ \cap I^n_- = S^{n-1}$, the latter is an $(n-1)$ dimensional equatorial sphere of S^n. We take $(0, \ldots, 0) \in S^{n-1}$ as a pole of S^n and describe an element of $\pi_n(X, x_0)$ determined by $d(f_n, g_n)$ as $\mathrm{d}(f_n, g_n)$.

We define here that the two singular n-cubes (i. e. continuous mappings of Euclidean n-cubes) f_n, f'_n are the same if there exists a homeomorphism λ of the Euclidean n-cubes preserving its orientation such that $f_n = f'_n \lambda$. For any singular n-cubes f_n, g_n and a homeomorphism λ of the n-cubes such that $f_n | \dot{I}^n = g_n \lambda | \dot{I}^n$, we can define a mapping $d(f_n, g_n \lambda)$ and an element $\mathrm{d}(f_n, g_n \lambda)$ of $\pi_n(X, x_0)$ determined by it.

Now let f be a mapping from I^p into X such that $f(\dot{I}^p) = x_0$ and g be that from I^q into X such that $g(\dot{I}^q) = x_0$. Let α be an element of $\pi_p(X, x_0)$ determined by f and β be that of $\pi_q(X, x_0)$ determined by g. We define a mapping $f \vee g$ of $I^p \times I^q$ into X by a formula

$$f \vee g(x, y) = f(x) \vee g(y)$$

for $x \in I^p, y \in I^q$. We deform a partial mapping $f \vee g | (I^p \times I^q)^{\bullet}$ to a mapping which coincides with $f(x)$ on $I^p \times \dot{I}^q$ and with $g(y)$ on $\dot{I}^p \times I^q$. This is established as follows. The mapping $f \vee g$ on $\dot{I}^p \times \dot{I}^q$ is always a constant x_0, therefore we apply the homotopies (relative to x_0) of condition (H. 2) to both of $I^p \times \dot{I}^q$ and

$\dot{I}^p \times I^q$ independently and obtain the desired homotopy. Thus we have extended the mapping $f \vee g$ of $I^p \times I^q \times 0$ identified with $I^p \times I^q$ to that of $I^p \times I^q \times 0 \cup (I^p \times I^q)^{\cdot} \times I = \overline{I^{p+q}}$. We denote it by $f \nabla g$. $\overline{I^{p+q}}$ is homeomorphic to a $(p+q)$-dimensional Euclidean cube[3], hence $f \nabla g$ determines a singular cube.

Let $\lambda_{p,q}$ be a homeomorphism of $I^p \times I^q \times I$ onto $I^q \times I^p \times I$ defined by $\lambda_{p,q}(x, y, t) = (y, x, t)$ for all $x \in I^p, y \in I^q$ and $t \in I$. We consider $d(f\nabla g, (g\nabla f)\lambda_{p,q}) \in \pi_{p+q}(X, x_0)$ i.e. a homotopy class of $d(f\nabla g, (g\nabla f)\lambda_{p,q})$ by homotopies which map the point $(0, \ldots, 0) \times (0, \ldots, 0) \times 1 \in \dot{\overline{I^{p+q}}} = \overline{I^{p+q}_+} \cap \overline{I^{p+q}_-}$ always to x_0. It is shown that the element is uniquely determined by α, β and this operation is linear for elements of dimension > 1.

Let f' be another mapping of α and g' be that of β. Let $F(x, t)$ and $G(y, t)$ give these homotopies $f \simeq f', g \simeq g'$ relative to x_0 ($x \in I^p, y \in I^q$ and $t \in I$). We define a mapping of $I^p \times I^q \times 0 \times I \cup (I^p \times I^q)^{\cdot} \times I \times I$ into X by the formulas

$$
\begin{array}{ll}
F(x,t) \vee G(y,t), & \text{if } x \times y \times 0 \times t \in I^p \times I^q \times 0 \times I, \\
H_l(F(x,t), s), & \text{if } x \times y \times s \times t \in I^p \times \dot{I}^q \times I \times I, \\
H_r(G(y,t), s), & \text{if } x \times y \times s \times t \in \dot{I}^p \times I^q \times I \times I.
\end{array}
$$

This gives a homotopy $f\nabla g \simeq f'\nabla g'$ which maps $(0, \ldots, 0) \times (0, \ldots, 0) \times 1$ always to x_0. The homotopies of the mappings $f\nabla g, (g\nabla f)\lambda_{p,q}$ defined above agree on the boundary $\dot{\overline{I^{p+q}}}$. Hence we obtain the homotopy

$$d(f\nabla g, (g\nabla f)\lambda_{p,q}) \simeq d(f'\nabla g', (g'\nabla f')\lambda_{p,q})$$

relative to x_0. This proves that $d(f\nabla g, (g\nabla f)\lambda_{p,q})$ is determined by α and β.

Let α_1, α_2 be elements of $\pi_p(X, x_0)$ such that $\alpha = \alpha_1 + \alpha_2$ and f_1, f_2 be mappings of I^p into X such that $f_1(\dot{I}^p) = f_2(\dot{I}^q) = x_0$. We define a mapping $f_{1,2}$ by

$$
\begin{aligned}
f_{1,2}(x_1, \ldots, x_p) &= f_1(2x_1, \ldots, x_p) && \text{if } 0 \leqq x_1 \leqq 1/2, \\
&= f_2(2x_1 - 1, \ldots, x_p) && \text{if } 1/2 \leqq x_1 \leqq 1.
\end{aligned}
$$

This belongs to α. Let S^{p+q} be a $(p+q)$-dimensional sphere. We shrink its equatorial sphere to a point and identify the two spheres thus obtained with two copies $[\overline{I^{p+q}_+} \cup \overline{I^{p+q}_-}]_1, [\overline{I^{p+q}_+} \cup \overline{I^{p+q}_-}]_2$ of $\overline{I^{p+q}_+} \cup \overline{I^{p+q}_-}$, where the points $[(1, 0, \ldots, 0) \times (0, \ldots, 0) \times 1]_1$, $[(0, \ldots, 0) \times (0, \ldots, 0) \times 1]_2$ coincide with the point shrunk. We describe the shrinking followed by $d(f_1\nabla g, (g\nabla f_1)\lambda_{p,q})$ and $d(f_2\nabla g, (g\nabla f_2)\lambda_{p,q})$ on the two spheres respectively, as $F_{1,2}$.

Next we identify the part $1 \times I^{p+q-1} \times I$ of $[\overline{I^{p+q}_+}]_1$ with $0 \times I^{p+q-1} \times I$ of $[\overline{I^{p+q}_+}]_2$ and retract it to $1 \times (I^{p+q-1} \times 0 \times \dot{I}^{p+q-1} \times 1)$. This is a deformation retract. Similarly we consider this operation for $[\overline{I^{p+q}_-}]_1$, $[\overline{I^{p+q}_-}]_2$. A space thus obtained is clearly homeomorphic to $\overline{I^{p+q}_+} \cup \overline{I^{p+q}_-}$.

Let θ be a composite mapping of identifications and homeomorphisms

3) A homeomorphism is given as follows: we project the set $I^p \times I^q \times 0 \cup (I^p \times I^q) \times I$ to a hyperplane $x_{p+q+1} = 1$ from a point $(\frac{1}{2}, \ldots, \frac{1}{2}, 2)$.

stated above, from S^{p+q} onto $\overline{I_+^{p+q}} \cup \overline{I_-^{p+q}}$. We have easily

$$F_{1,2} \simeq d(f_{1,2}\nabla g, (g\nabla f_{1,2})\lambda_{p,q})\theta,$$

where this homotopy maps the point $(0, \ldots, 0) \times (0, \ldots, 0) \times 1$ always to x_0. Since the degree of θ is $+1$, $F_{1,2}$ and $d(f_{1,2}\nabla g, (g\nabla f_{1,2})\lambda_{p,q})$ represent the same element of $\pi_{p+q}(X, x_0)$. If $\omega(\alpha_1)$ is an automorphism of $\pi_{p+q}(X, x_0)$ induced by a closed path $F_{1,2}|[I \times (0, \ldots, 0) \times (0, \ldots, 0) \times 1]_1 = f_1 | I \times (0, \ldots, 0)$, $F_{1,2}$ determines

$$\mathrm{d}(f_1\nabla g, (g\nabla f_1)\lambda_{p,q}) + \omega(\alpha_1)\mathrm{d}(f_2\nabla g, (g\nabla f_2)\lambda_{p,q}).$$

For $p > 0$ ω is trivial.

Similarly this holds for β. Thus the linearity is proved.

DEFINITION 2. To any elements $\alpha \in \pi_p(X, x_0)$, $\beta \in \pi_q(X, x_0)$ we associate an element $(-1)^p d(f\nabla g, (g\nabla f)\lambda_{p,q})$ of $\pi_{p+q}(X, x_0)$ and call it an H-product of α and β and denote it by $< \alpha, \beta >$.

We show some properties of this product in the following Propositions.

PROPOSITION 2. *Let h be the Hurewicz natural homomorphism of $\pi_n(X, x_0)$ into $H_n(X)$ and $*$ be the Pontrjagin product. We have the relation*

$$h < \alpha, \beta > = (-1)^p\{h\alpha * h\beta - (-1)^{pq} h\beta * h\alpha\}. \tag{4}$$

PROOF. If we regard the mappings $f\nabla g$, $(g\nabla f)\lambda_{p,q}$, $d(f\nabla g, (g\nabla f)\lambda_{p,q})$ as cubic singular cycles we have

$$d(f\nabla g, (g\nabla f)\lambda_{p,q}) = f\nabla g - (-1)^{pq} g\nabla f.$$

By means of a natural deformation retract we obtain the relation $f\nabla g \sim f \vee g$ (homologous) and this determines $h\alpha * h\beta$. Thus the result is proved.

PROPOSITION 3. *If a topological group G is abelian, then the H-product in homotopy groups of G is trivial.*

This is a direct consequence of the Definition 2.

4. A relation between the *H*-product and the Whitehead product. In this section, we consider how to derive the Whitehead product from our H-product. Let X be an arcwise connected space and f be a mapping of I^{p+1} into X such that $f(\dot{I}^{p+1}) = x_0$ and g be that of I^{q+1} into X such that $g(\dot{I}^{q+1}) = x_0$ where x_0 is a fixed point of X. Let α and β be elements of homotopy groups determined by the mappings f and g respectively i. e. $\alpha \in \pi_{p+1}(X, x_0)$, $\beta \in \pi_{q+1}(X, x_0)$.

The Whitehead product $[\alpha, \beta]$ of α and β (see [3]) is defined as an element of $\pi_{p+q+1}(X, x_0)$ determined by a mapping h of $(I^{p+1} \times I^{q+1})^{\cdot}$ into X such that

$$\begin{aligned} h(x, y) &= f(x) \qquad \text{if } x \in I^{p+1}, y \in \dot{I}^{q+1}, \\ &= g(y) \qquad \text{if } x \in \dot{I}^{p+1}, y \in I^{q+1}. \end{aligned}$$

For the sake of convenience, we describe the mapping h as $[f, g]$.

PROPOSITION 4. *In every H-space the Whitehead product is null.*

PROOF The mapping $f\nabla g$ gives a null homotopy of $h=[f,g]$.

Let f' be a transgression of the mapping f, namely a mapping of I^p into a lacet space Ω_X of X based on x_0 such that $f'(\dot{I}^p)=x_0$[4], and similarly g' be a transgression of g i. e. $Tf=f'$, $Tg=g'$. The elements of homotopy groups determined by f' and g' are $T\alpha\in\pi_p(\Omega_X,x_0)$ and $T\beta\in\pi_q(\Omega_X,x_0)$. We denote them by α' and β' respectively.

THEOREM 1. *Let α, β be as above. Then we have the formula*

$$T[\alpha,\beta]=<T\alpha,T\beta>. \tag{5}$$

PROOF. We deform the mapping $\Sigma d(f'\nabla g',(g'\nabla f')\lambda_{p,q})$ and will show that the element of $\pi_{p+q+1}(X,x_0)$ determined by it, coincides with $(-1)^p[\alpha,\beta]$.

Now, we construct a mapping $[\varphi_s]_+$ of $E=\overline{I^{p+q}}\times I(0\leqq s\leqq 1)$ onto itself. (It is not always necessary that the mapping is continuous). On $\overline{I^{p+q}}\times t$, for any $x\times y\in(I^p\times I^q)^{\cdot}$ we map the line segment with end points $(1/2,\dots,1/2)\times(1/2,\dots,1/2)\times 0\times t, x\times y\times 0\times t$ onto a broken line segment (tree) with vertices $(1/2,\dots,1/2)\times(1/2,\dots,1/2)\times 0\times t, x\times y\times 0\times t, x\times y\times s(1-2t)\times t$ for $0\leqq t\leqq 1/2$ and onto that with vertices $(1/2,\dots,1/2)\times(1/2,\dots 1/2)\times 0\times t, x\times y\times 0\times t, x\times y\times s(2t-1)\times t$, for $1/2\leqq t\leqq 1$, linearly about length. On $\dot{I}^{p+q}\times I\times I$, for any $x\in I^p-\dot{I}^p$, $y\in\dot{I}^q$, $1/2\leqq t\leqq 1$, in $x\times y\times I\times I$ we map the interval $x\times y\times[0,2t-1]\times t$ onto the interval $x\times y\times[s(2t-1), 2t-1]\times t$ linearly, and the interval $x\times y\times r\times[0,1/2(1+r)]$ onto the line segment with end points $x\times y\times(r+s(1-r))\times 0$, $x\times y\times r\times 1/2(1+r)$ $(0\leqq r\leqq 1)$, linearly about length. For $x\in\dot{I}^p$, $y\in I^q-\dot{I}^q$ the mapping is defined similarly by inverting the value t. For any $x\in\dot{I}^p$, $y\in\dot{I}^q$ we map the interval $x\times y\times[0,1-2t]\times t$ onto $x\times z\times[s(1-2t), 1-2t]\times t$ for $0\leqq t\leqq 1/2$ and $x\times y\times[0,2t-1]\times t$ onto $x\times y\times[s(2t-1),2t-1]\times t$ for $1/2\leqq t\leqq 1$ linearly. We define a mapping $[\varphi_s]_-$ by $[\varphi_s]_-(x,y,s,t)=[\varphi_s]_+(x,y,s,1-t)$.

Let S^{p+q+1} be a $(p+q+1)$-dimensional sphere represented by two copies E_+,E_- of the cube E by identifying their boundaries and φ_s be a mapping of S^{p+q+1} onto itself. It is one-to-one for s-values $0\leqq s<1$, but is not continuous on $\dot{I}^p\times\dot{I}^q\times I\times I$. However $[\Sigma d(f'\nabla g', (g'\nabla f')\lambda_{p,q}]\varphi_s^{-1}$ is defined for $0\leqq s\leqq 1$ and continuous and gives a homotopy of the mapping $\Sigma d(f'\nabla g',(g'\nabla f')\lambda_{p,q})$.

There exists a homeomorphism of $\varphi_1(E_+)$ onto $(I^p\times I^q)\times(I\times 0\cup 1\times I)\cup(I^p\times I^q)^{\cdot}\times$ (the triangle with vertices $(0,0),(1,0),(1,1)$ in $I\times I$) as follows: for any $x\in I^p$, $y\in I^q$ line segments $\varphi_1(x\times y\times 0\times[0,1/2])$ and $\varphi_1(x\times y\times 0\times[1/2,1])$ go onto $x\times y\times I\times 0$ and $x\times y\times 1\times I$ obviously piecwise linearly. For $(x,y)\in(I^p\times I^q)^{\cdot}$, $\varphi_1(x\times y\times I\times I)$ which is $x\times y\times$ (the triangle with vertices$(1,0),(0,1/2),(1,1)$ in $I\times I$) goes onto $x\times y\times$ (the triangle with vertices $(0,0),(1,0),(1,1)$) by an affine transformation which maps the vertices $(1,0),(0,1/2)$,

4) x_0 means also the constant path $I\to x_0\in X$.

$(1,1)$ to $(0,0),(1,0),(1,1)$ in $I\times I$ respectively. Under this homeomorphism line segments $\varphi_1\left(x\times y\times s\times\left[0,\frac{1+s}{2}\right]\right)$, $\varphi_1\left(x\times y\times s\times\left[\frac{1+s}{2},1\right]\right)$ for any $x\in I^p-\dot I^p, y\in\dot I^q$ go onto the broken line segment with vertices $x\times y\times 0\times 0$, $x\times y\times 1\times s$, $x\times y\times 1\times 1$ and $\varphi_1\left(x\times y\times s\times\left[0,\frac{1-s}{2}\right]\right)$, $\varphi_1\left(x\times y\times s\times\left[\frac{1-s}{2},1\right]\right)$ for any $x\in\dot I^p, y\in I^q-\dot I^q$ go onto that with vertices $x\times y\times 0\times 0$, $x\times y\times(1-s)\times 0$, $x\times y\times 1\times 1$ and $\varphi_1(x\times y\times s\times\left(\left[0,\frac{1-s}{2}\right]\cup\left[\frac{1-s}{2},\frac{1+s}{2}\right]\cup\left[\frac{1+s}{2},1\right]\right))$ for any $x\in\dot I^p$, $y\in\dot I^q$ go onto that with vertices $x\times y\times 0\times 0$, $x\times y\times(1-s)\times 0, x\times y\times 1\times s, x\times y\times 1\times 1$, piecewise linearly. Similarly $\varphi_1(E_-)$ is homeomorphic to $I^p\times I^q\times(0\times I\cup I\times 1)\cup(I^p\times I^q)^{\boldsymbol{\cdot}}\times$ (the triangle with vertices $(0,0),(0,1),(1,1)$ in $I\times I$). These induce a homeomorphism ϕ' of $\varphi_1(S^{p+q+1})$ onto $(I^p\times I^q\times I\times I)^{\boldsymbol{\cdot}}$. Let ϕ be a mapping ϕ' followed by the transformation η: $(I^p\times I^q\times I\times I)\to(I^p\times I\times I^q\times I)$ defined by $\eta(x,y,s,t)=(x,s,y,t)$. This is a homeomorphism with the degree $(-1)^p$. From the construction the relation

$$[\Sigma d(f'\nabla g',(g'\nabla f')\lambda_{p,q})]\,\varphi_1^{-1}=[f,g]\phi$$

is obtained. Therefore for any $\alpha'\in\pi_p(\Omega_X,x_0)$, $\beta'\in\pi_q(\Omega_X,x_0)$ we have

$$\Sigma<\alpha',\beta'>=[\Sigma\alpha',\Sigma\beta'],$$

and this means that for any $\alpha\in\pi_{p+1}(X,x_0)$, $\beta\in\pi_{q+1}(X,x_0)$

$$T[\alpha,\beta]=<T\alpha,T\beta>.$$

5. **The Jacobi identities in homotopy groups.** Let X be an arcwise connected H-space and x_0, H_l, H_r be those of Definition 1. We suppose that mappings $f\colon I^p\to X$, $g\colon I^q\to Y$, $h\colon I^r\to X$, $f(\dot I^p)=g(\dot I)=h(\dot I^r)=x_0$ represent $\alpha\in\pi_p(X,x_0)$, $\beta\in\pi_q(X,x_0)$, $\gamma\in\pi_r(X,x_0)$ respectively. Let $\overline{I^p}$ be $I^p\times 0\cup I^p\times I_p$ where I_p is $[0,1]$ with the index p. Briefly we set $I^p\times 0=I^p, I^p\times I_p=O^p$, hence $\overline{I^p}=I^p\cup O^p$. First we construct two mappings $F_{f,(g,h)}$, $F_{(f,g),h}$ of a $(p+q+r)$-dimensional cube $E_{p,q,r}=\overline{I^p}\times\overline{I^q}\times\overline{I^r}$ into X as follows. Let $\bar x$ be an arbitrary element of $\overline{I^p}$. We have $\bar x=x$ for $\bar x\in I^p\times 0$ and $x=\bar x\times t_p(x\in\dot I^p, t_p\in I_p)$ for $\bar x\in O^p$ and similarly for $\bar y\in\overline{I^q}$, $\bar z\in\overline{I^r}$. We define

$$(6)\qquad F_{f,(g,h)}(\bar x,\bar y,\bar z)\qquad\qquad F_{(f,g),h}(\bar x,\bar y,\bar z)$$

$$=\begin{cases} f(x)\vee(g(y)\vee h(z)) \\ f(x)\vee H_r(h(z),t_q) \\ f(x)\vee H_l(g(y),t_r) \\ H_r(g(y)\vee h(z),t_p) \\ H_r(H_r(h(z),t_q),t_p) \\ H_r(H_l(g(y),t_r),t_p) \\ \\ x_0 \end{cases} =\begin{cases} (f(x)\vee g(y)\vee h(z) & \text{on } I^p\times I^q\times I^r, \\ H_l(f(x),t_q)\vee h(z) & \text{on } I^p\times O^q\times I^r, \\ H_l(f(x)\vee g(y),t_r) & \text{on } I^p\times I^q\times O^r, \\ H_r(g(y),t_p)\vee h(z) & \text{on } O^p\times I^q\times I^r, \\ & \text{on } O^p\times O^q\times I^r \\ H_l(H_r(g(y),t_p),t_r) & \text{on } O^p\times I^q\times O^r, \\ H_l(H_l(f(x),t_q),t_r) & \text{on } I^p\times O^q\times O^r, \\ x_0 & \text{on } O^p\times O^q\times O^r, \end{cases}$$

on $I^p \times O^q \times O^r$, $F_{f,(g,h)}$ maps all points of the triangle with vertices $x \times y \times 0 \times z \times 0$, $x \times y \times 1 \times z \times 0$, $x \times y \times 0 \times z \times 1$ $(x \in I^p, y \in \dot{I}^q, z \in \dot{I}^r)$ to $f(x) \vee x_0$ and all points of a line segment connecting $x \times y \times 1 \times z \times t$, $x \times y \times t \times z \times 1$ to $H_t(f(x), t)$. On $O^p \times O^q \times I^r$ we define $F_{(f,g),h}$ by a method analogous as above. These two mappings agree on the boundary $\dot{E}_{p,q,r}$ of $E_{p,q,r}$. Moreover we define such a pair of mappings for every order of suffixes f, g, h.

LEMMA 1. *If X is a lacet space, x_0 is a constant path and H_l, H_r are homotopies induced by a homotopic transformation of parameters which remove the constant path x_0 at one end* (*see section* 3), *then $F_{f(g,h)}, F_{(f,g),h}$ are homotopic leaving the mappings on the boundary $\dot{E}_{p,q,r}$ fixed. This holds good for any order of f, g, h.*

PROOF. For any points of $E_{p,q,r}$ paths of its images by the two mappings change each other by means of a homotopic transformation of parameters and we can define this transformation continuously on the whole $E_{q,p,r}$.

Let C_D^{q+r} be a $(q+r)$-dimensional cube and ρ_D be a mapping of it onto $\overline{I_+^{q+r}} \cup \overline{I_-^{q+r}}$, which maps $\dot{C}_D^{q+r}$ to $(0, \ldots, 0) \times (0, \ldots, 0) \times 1$ and is a homeomorphism on $C_D^{q+r} - \dot{C}_D^{q+r}$. We set

$$D_{\langle f, \langle g,h\rangle\rangle} = d[f\nabla(d(g\nabla h, (h\nabla g)\lambda_{q,r})\rho_0, \{d(g\nabla h, (h\nabla g)\lambda_{q,r})\rho_0)\nabla f\}\lambda_{p,q+r}]$$

and denote its inverse image sphere by S^{p+q+r}. Similarly we can define $D_{\langle g, \langle h,f\rangle\rangle}$, $D_{\langle h, \langle f,g\rangle\rangle}$.

We construct a mapping $i_{q,r}$ of $[\overline{I^q} \times \overline{I^r}]_+ \cup [\overline{I^q} \times \overline{I^r}]_-$ onto $\overline{I_+^{q+r}} \cup \overline{I_-^{q+r}} \cup (0, \ldots, 0) \times (0, \ldots, 0) \times 1 \times I$ int he following way. First we define a mapping $[i_{q,r}]_+$from $[\overline{I^q} \times \overline{I^r}]$ onto $\overline{I_+^{q+r}} \cup (0, \ldots, 0) \times (0, \ldots, 0) \times 1 \times I$ by

$$[i_{q,r}]_+(x \times y) = \begin{cases} y \times z & \text{if } \bar{y} = y \in I^q,\ \bar{z} = z \in I^r. \\ y \times z \times t_q & \text{if } \bar{y} = y \times t_q \in O^q,\ \bar{z} = z \in I^r, \\ y \times z \times t^r & \text{if } \bar{y} = y \times I^q,\ \bar{z} = z \times t_r \in O^r, \end{cases}$$

In $O^q \times O^r$, for any $y \in \dot{I}^q, z \in \dot{I}^r$ and $0 \leqq t \leqq 2$ we identify the line segment $\{y \times t_q \times z \times t_r \mid t_q + t_r = t\}$ to a point represented by $y \times t \times z \times 0$ for $0 \leqq t \leqq 1$ and by $y \times 1 \times z \times (t-1)$ for $1 \leqq t \leqq 2$. Let U be a neighborhood of $(0, \ldots, 0) \times (0, \ldots. 0)$ on $\dot{I}^q \times \dot{I}^r$, consisting of all points whose distances from $(0, \ldots, 0) \times (0, \ldots, 0)$ are less than $1/2$. For any $(y, z) \notin U$ we identify $y \times 1 \times z \times I$ to $y \times 1 \times z \times 0$. In U for any $(y, z) \in \dot{U}$ let $l_{y,z}$ be a line segment with end points $y \times 1 \times z \times 1$, $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times 1$ and $l'_{y,z}$ be that connecting $y \times 1 \times z \times 0$, $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times 0$. We consider a homeomorphism of $l_{y,z}$ onto $l'_{y,z} \cup (0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times I$ such that the part of $l_{y,z}$ from $(0, \ldots, 0,) \times 1 \times (0, \ldots 0) \times 1$ to its center goes onto $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times I$ and the other part onto $l'_{y,z}$ linearly. We identify every line segment connecting the two points corresponding under the above homeomorphism to its end point belonging to $l'_{y,z} \cup (0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times I$. We map $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times I$ onto $(0, \ldots, 0) \times (0, \ldots, 0) \times 1 \times I$ in the obvious way. Thus $[i_{q,r}]_+$ is defined on the whole $[\overline{I^q} \times \overline{I^r}]_+$.

Similarly $[i_{q,r}]_-$ is defined.

Let C_F^{q+r} be a $(q+r)$-dimensional cube and ρ_F be a homeomorphism of it onto a $(q+r)$-dimensional cell $i_{q,r}^{-1}[I_+^{q+r} \cup I_-^{q+r}]$. Let $i_{q,r}^C$ be a mapping of C_F^{q+r} onto C_D^{q+r} such that $i_{q,r}\,\rho_F = \rho_D i_{q,r}^C$. This is uniquely determined in $C_F^{q+r} - \dot{C}_F^{q+r}$ and is extended naturally to a mapping of C_F^{q+r}. We construct a mapping $\overline{\rho_F}$ of $\overline{C_F^{q+r}}$ onto $[I^q \times I^r]_+ \cup [\overline{I^q \times I^r}]_-$ by $\overline{\rho_F}(u \times 0) = \rho_F(u)$ on $C_F^{q+r} \times 0$ $(u \in C_F^{q+r})$ and defining $\overline{\rho_F}(u \times t)$ on $\dot{C}_F^{q+r} \times I$ as a point dividing the line segment with end points, $\rho_F(u)(u \in \dot{C}_F^{q+r})$ and $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times 1$ in the ratio $t : 1-t$. A mapping $\overline{\rho_D}$ of $\overline{C_D^{q+r}}$ onto $[I_+^{q+r} \cup I_-^{q+r}] \cup (0, \ldots, 0) \times (0, \ldots, 0) \times 1 \times I$ is defined by

$$\overline{\rho_D}(v \times 0) = \rho_D(v) \text{ on } C_D^{q+r} \times 0 (v \in C_D^{q+r}),$$
$$\overline{\rho_D}(v \times t) = \rho_D(v) \times t \text{ on } \dot{C}_D^{q+r} \times I (v \in \dot{C}_D^{q+r}).$$

Let $\overline{i_{q,r}^C}$ be a mapping of $\overline{C_F^{q+r}}$ onto $\overline{C_D^{q+r}}$ defined by

$$\overline{i_{q,r}^C}(u \times 0) = i_{q,r}^C(u) \qquad \text{on } C_F^{q+r} \times 0,$$
$$\overline{i_{q,r}^C}(u \times t) = i_{q,r}^C(u) \times t \quad \text{on } \dot{C}_F^{q+r} \times I.$$

Easily we have

$$i_{q,r}\,\overline{\rho_F} = \overline{\rho_D}\,\overline{i_{q,r}^C}.$$

There exist two mappings F_1, F_2 of $\overline{I^q \times C_F^{q+r}}$ induced by $F_{f,(g,h)}$, $F_{f,(h,g)}$ and $F_{(g,h),f}, F_{(h,g),h}$ respectively. We set $F_{<f,<g,h>>} = d(F_1, F_2)$ and describe its inverse image sphere by S_F^{p+q+r}. Similarly $F_{<g,<h,f>>}$ and $F_{<h,<f,g>>}$ can be defined.

We construct a mapping i of S_F^{p+q+r} onto S_D^{p+q+r} as follows:

$$\begin{aligned} i(\bar{x} \times \bar{u}) &= x \times \overline{i_{q,r}^C}(u) \qquad \text{if } \bar{x} = x \in I^p,\ u \in \overline{C_F^{q+r}}, \\ &= x \times \overline{i_{q,r}^C}(\bar{u}) \times t \text{ if } \bar{x} = x \times t \in O^p,\ \bar{u} \in C_F^{q+r} \times 0, \\ &= x \times i_{q,r}^C(u) \times t \text{ if } \bar{x} = x \times t_p \in O^p, \bar{u} = u \times t_c \in \dot{C}_F^{q+r} \times I \\ &\quad \text{such that } t_p = t \text{ and } t_c \in [0, t] \text{ or } t_p \in [0, t], t_c = t. \end{aligned}$$

i induces mappings of $[\overline{I^p \times C_F^{q+r}}]_+$ onto $[\overline{I^p \times C_D^{q+r}}]_+$ and of $[\overline{I^p \times C_F^{q+r}}]_-$ onto $[\overline{I^p \times C_D^{q+r}}]_-$ i.e. mapping of S_F^{p+q+r} onto S_D^{p+q+r}.

LEMMA 2. *We have the following relation.*

(7) $$D_{<f,<g,h>>}\, i \simeq F_{<f,<g,h>>},$$

where this homotopy maps the point $(0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times 1 \times (0, \ldots, 0) \times 1 \in S_F^{p+q+r}$ *always to* x_0.

PROOF. The mapping i is an identification mapping i_1 of the four parts of S_F^{p+q+r} induced by that of $E_{p,q,r}$ above, followed by an orientation preserving homeomorphism of $i_1(S_F^{p+q+r})$ onto S_D^{p+q+r}. Changing the values on each line segment of S_F^{p+q+r} which is identified to a point by i_1, for the value of its end point continuously in regard to a parameter τ $(0 \leqq \tau \leqq 1)$, $D_{<f,<g,h>>}\, i$, $F_{<f,<g,h>>}$ are homotopic relative to the point x_0, where the base point of S_F^{p+q+r} is $(0,\ldots,0) \times 1 \times (0,\ldots,0) \times 1 \times (0,\ldots,0) \times 1$.

REMARK. In lacet spaces, this lemma is proved directly, using the homotopic transformations of parameters of paths (see the proof of Lemma 1).

Since i is a mapping between the $(p+q+r)$-dimensional spheres with the degree 1, $D_{<f,<g,h>>}$, $F_{<f,<g,h>>}$ represent the same element of $\pi_{p+q+r}(X, x_0)$.

For the mappings of F's the following result is obtained.

LEMMA 3. *Let* dF *be an element of a homotopy group represented by a mapping* F *of a sphere. We have the relation*

(8) $$\mathrm{d}\,F_{<f,<g,h>>} + (-1)^{p(q+r)}\mathrm{d}\,F_{<g,<h,f>>} + (-1)^{r(p+q)}\mathrm{d}\,F_{<h,<f,g>>} = 0.$$

PROOF. Let $\lambda^{p,q,r}_{p',q',r'}$ be a homeomorphism of $E_{p,q,r}$ onto $E_{p',q',r'}(\{p,q,r\} = \{p',q',r'\})$ defined by the permutation (p',q',r') of (p,q,r) and $\overline{\lambda}^{p,q,r}_{p',q',r'}$ be a homeomorphism of S_F^{p+q+r} induced by $\lambda^{p,q,r}_{p',q',r'}$. Let Γ_1 be a space consisting of three $(p+q+r)$-dimensional spheres which are copies of S_F^{p+q+r} and have a base point $(0,\ldots,0) \times 1 \times (0,\ldots,0) \times 1 \times (0,\ldots,0) \times 1$ in common. Let G be a proper identification of a $(p+q+r)$-dimensional sphere S^{p+q+r} to Γ_1 followed by $F_{<f<g,h>>}$, $F_{<g,<h,f>>}\,\overline{\lambda}^{p,q,r}_{q,r,p}$ and $F_{<h,<f,g>>}\,\overline{\lambda}^{p,q,r}_{r,p,q}$ on each S_F^{p+q+r} respectively. S_F^{p+q+r} consists of four inverse images of $E_{p,q,r}$ under $1 \times \overline{\rho}_F$ (1 = identity mapping of $\overline{I^p}$). We identify each inverse images in Γ_1 to copies of $E_{p,q,r}$. Let Γ_2 be a space constructed by this operation from Γ_1. Then $F_{<f,<g,h>>}$ is this identification followed by the mappings $F_{f,(g,h)}$, $F_{f,(h,g)}\lambda^{p,q,r}_{p,r,q}$, $F_{h,(f,g)}\lambda^{p,q,r}_{r,p,q}$ and $F_{h,(g,f)}\lambda^{p,q,r}_{r,p,q}$ from four copies of $E_{p,q,r}$ respectively. Similarly such decompositions hold for $F_{<g,<h,f>>}\overline{\lambda}^{p,q,r}_{q,r,p}$ and $F_{<h,<f,g>>}\overline{\lambda}^{p,q,r}_{r,p,q}$. Moreover, six pairs of mappings from copies of $E_{p,q,r}$ on Γ_2 i. e. $F_{f,(g,h)}$ and $F_{(f,g)h}$, $F_{g,(h,f)}\lambda^{p,q,r}_{q,r,p}$ and $F_{(g,h),f}\lambda^{p,q,r}_{q,r,p}$, etc. agree on the boundary $\dot{E}_{p,q,r}$ respectively. Let Γ_3 be a space obtained by identifying the boundaries of each two copies of $E_{p,q,r}$ on Γ_2 paired as above. The space consists of six spheres identified properly on their equatorial spheres. Hence G is a composition of the identification S^{p+q+r} onto Γ_3 and the six mappings $d(F_{f,(g,h)}, F_{(f,g),h})$, $d(F_{g,(h,f)}\lambda^{p,q,r}_{q,r,p}, F_{(g,h)f}\lambda^{p,q,r}_{q,r,p})$, etc. of six spheres on Γ_3 respectively. The latter is homotopic to the constant mapping x_0 by Lemma 1 and the homotopy extension property of a finite polyhedron [11, pp. 501].

THEOREM 2. *Let* X *be an arcwise connected space and* Ω_X *be its lacet space based on a fixed point* $x_0 \in X$. *For any three elements* $\alpha \in \pi_p(\Omega_X, x_0)$, $\beta \in \pi_q(\Omega_X, x_0)$, $\gamma \in \pi_r(\Omega_X, x_0)$ *we have the Jacobi identity in H-products*:

(9) $$(-1)^{(p+1)r} < \alpha, <\beta,\gamma>> + (-1)^{(q+1)p} < \beta, <\gamma,\alpha>> + (-1)^{(r+1)q} < \gamma, <\alpha,\beta>> = 0$$

PROOF. From (7), (8) and Definition 2 the relation follows easily, using the bilinearity of H-product for elements of dimension $\geqq 2$.

COROLLARY 2.1. *The Jacobi identity in Whitehead produdts for elements of dimension* >1 *holds* i. e. *for any set of elements* $\alpha' \in \pi_{p+1}(X, x_0)$, $\beta' \in \pi_{q+1}(X, x_0)$ *and* $\gamma' \in \pi_{r+1}(X, x_0)$, *where* $p, q, r > 0$, *we have the relation*

$$(-1)^{(p+1)r}[\alpha', [\beta', \gamma']] + (-1)^{(q+1)p}[\beta', [\gamma', \alpha']] + (-1)^{(r+1)q}[\gamma', [\alpha', \beta']] = 0. \tag{10}$$

This is the Samelson's conjecture.

PROOF. From Theorem 2, this is immediately shown using Theorem 1.

When we take a topological space as an H-space (See section 3, example), the result of Theorem 2 is also obtained. The procedure of the proof of this fact is analogous to that of Theorem 2 and more easy.

The proof of theorem 2 can be applied for the H-space in which the result of Lemma 2 is satisfied. Therefore the theorem is stated in the following general form.

THEOREM 3. *Let* X *be an arcwise connected H-space and* x_0, H_l, H_r *be those of Definition* 1. *If for any mappings* $f: I^p \to X, g: I^q \to X, h: I^r \to X$ *such that* $f(\dot{I}^p) = g(\dot{I}) = h(\dot{I}^r) = x_0$ *we have the homotopy* $F_{f,(g,h)} \simeq F_{(f,g),h}$ *leaving the mappings on* $\dot{E}_{p,q,r}$ *fixed and similar relations for all permutations of suffixes* f, g, h, *then the Jacobi identity in H-products* (9) *holds.*

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.