SOME TRIGONOMETRICAL SERIES XI

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In this paper we prove two kinds of integrability theorems of trigonometrical series.

1. We shall prove the following theorem:

THEOREM 1. *If*

$$\sum_{k=n}^{2n} |\Delta a_k| = O(1) \tag{1}$$

and the sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{2}$$

converges boundedly in the interval (δ, π) *for any* $\delta > 0$, *then a necessary and sufficient condition for the convergence of the Cauchy integral*

$$\int_{\to 0}^{\pi} g(x)dx, \tag{3}$$

is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \tag{4}$$

Necessity of the condition was proved in the paper [1][1, 2], so that it is sufficient to prove the sufficiency of the condition[1]. By the hypothesis

$$\int_{x}^{\pi} g(t)dt = \sum_{n=1}^{\infty} \frac{a_n}{n} (\cos n\pi - \cos nx) \qquad (0 < x < \pi).$$

Hence the existence of (3) is equivalent to that of

$$\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos nx \tag{5}$$

If (4) converges, then, for any $\delta > 0$, there is an N such that

$$\left| \sum_{n=p}^{q} \frac{a_n}{n} \right| < \delta \quad (q > p > N).$$

Let $N < [1/x] = \lambda$, and write

1) Cf. R. P. Boas [2].

2) In [1], Theorem 2 is trivial.

$$\sum_{n=1}^{\infty}\frac{a_n}{n}\cos nx=\sum_{n=1}^{N-1}+\sum_{n=N}^{\lambda-1}+\sum_{n=\lambda}^{\infty}=S_1+S_2+S_3.$$

Since $1-\cos nx\uparrow$ $(N\leqq n<\lambda)$, we have

$$\left|S_2-\sum_{n=N}^{\lambda-1}\frac{a_n}{n}\right|=\left|\sum_{n=N}^{\lambda-1}\frac{a_n}{n}(1-\cos nx)\right|$$

$$\leqq(1-\cos 1)\max_{\lambda>p\geqq N}\left|\sum_{n=p}^{N}\frac{a_n}{n}\right|<\delta,$$

and

$$\left|S_1-\sum_{n=1}^{N-1}\frac{a_n}{n}\right|=\left|\sum_{n=1}^{N-1}\frac{a_n}{n}(1-\cos nx)\right|\to 0$$

as $x\to 0$. For sufficiently large r, we put

$$S_3=\sum_{n=\lambda}^{\infty}=\sum_{n=\lambda}^{\lambda-1}+\sum_{n=r\lambda}^{\infty}=S_{3,1}+S_{3,2}.$$

We get

$$S_{3,1}=\sum_{n=\lambda}^{r\lambda-1}\Delta\frac{a_n}{n}\cdot\frac{\sin(n+1/2)x}{2\sin x/2}$$

$$-\frac{a_\lambda}{\lambda}\frac{\sin(\lambda+1/2)x}{2\sin x/2}+\frac{a_{r\lambda-1}}{r\lambda-1}\frac{\sin(r\lambda-1/2)x}{2\sin x/2}$$

$$=\sum_{n=\lambda}^{r\lambda-1}\frac{\Delta a_n}{n}\frac{\sin(n+1/2)x}{2\sin x/2}+o(1)$$

$$=S'_{3,1}+o(1).$$

Let $\lambda_1,\lambda_2,\ldots,\lambda_\mu$ be the extremum points of $\sin(n+1/2)x/n$ for $\lambda\leqq n\leqq r\lambda-1$, then

$$\lambda_k=(k+1/2)\pi/x+1/2+o(1),$$
$$\mu=O(r),$$

and hence

$$S'_{3,1}=\sum_{n=\lambda}^{\lambda_1}+\sum_{k=1}^{\mu-1}\sum_{n=\lambda_k}^{\lambda_{k+1}-1}+\sum_{n=\lambda_\mu}^{r\lambda-1}$$

$$=o\left(\frac{1}{x}\sum_{k=1}^{\mu}\frac{1}{\lambda_k}\right)=o(\log r).$$

Further

$$S_{3,2}=\sum_{n=r\lambda}^{\infty}\Delta\frac{a_n}{n}\cdot\frac{\sin(n+1/2)x}{2\sin x/2}+o(1)$$

$$= O\Big(\frac{1}{x}\sum_{k=0}^{\infty}\sum_{n=2^k r\lambda}^{2^{k+1}r\lambda}\frac{|\Delta a_n|}{n}\Big) = O(1/r).$$

Hence

$$\lim_{x\to 0} S_3 = \lim_{x\to 0} S_{3,1} + \lim_{x\to\infty}\lim_{x\to 0} S_{3,2} = 0.$$

Thus we have proved (5).

We can prove similarly the following theorem[1]:

THEOREM 2. *If the series*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx \tag{6}$$

converges boundedly in the interval (δ, π) *for any* $\delta > 0$ *or* (6) *is the Fourier series of* $f(x)$, *and further if*

$$(\mathrm{A})\sum_{n=0}^{\infty} a_n = 0,$$

$$\sum_{\nu=n}^{2n} |a_\nu| = O(1),$$

then the existence of the Cauchy integral

$$\int_{\to 0}^{\pi} \frac{f(t)}{t}\, dt$$

is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \frac{s_n}{n}.$$

THEOREM 3. *Let* $0 < \alpha < 1$. *If*

$$n^{1-\alpha} a_n \to 0 \ (n \to \infty),$$

$$\sum_{\nu=n}^{2n} |\Delta a_\nu| = O(1/n^{1-\alpha}),$$

then the existence of the limit

$$\lim_{x\to 0} f_\alpha(x) = \lim_{x\to 0} \frac{1}{\Gamma(\alpha)} \int_x^{\pi} (x-t)^{\alpha-1} f(t)dt,$$

where $f(x)$ *is defined by* (6), *is equivalent to the convergence of the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$$

2. THEOREM 4. *Let* $0 < \alpha \leqq 1$ *and* $0 < \beta \leqq 1$. *If*

$$\sum_{\nu=1}^{n} \nu a_\nu = O(n^\alpha), \tag{1}$$

1) Cf [1].

(2) $$\sum_{\nu=n}^{\infty} |\Delta a_\nu| = O(1/n^\beta),$$

then the integrals

$$\int_0^\pi \frac{f(x)}{x^\gamma}\,dx \quad and \quad \int_0^\pi \frac{g(x)}{x^\gamma}\,dx$$

exist, where $\gamma < 2\alpha/(\alpha+\beta)$ *and*

(3) $$f(x) = \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad g(x) = \sum_{\nu=1}^{\infty} a_\nu \sin \nu x.$$

This is an improvement of [3].

For the proof, we write

$$g(x) = \sum_{\nu=1}^{\infty} a_\nu \sin \nu x = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty} = S_1 + S_2.$$

Since

$$\Delta \frac{\sin nx}{n} = O(x/n),$$

we get

$$S_1 = \sum_{\nu=1}^{n} a_\nu \sin \nu x = \sum_{\nu=1}^{n} \nu a_\nu \frac{\sin \nu x}{\nu}$$

$$= \sum_{\nu=1}^{n-1} t_\nu \Delta \frac{\sin \nu x}{\nu} + t_n \frac{\sin nx}{n},$$

where $t_\nu = \sum_{\mu=1}^{\nu} \mu a_\mu$. Hence, by (1),

$$S_1 = O(xn^\alpha) + o(1).$$

By (2)

$$S_2 = \sum_{\nu=n+1}^{\infty} a_\nu \sin \nu x = O\left(\frac{1}{x}\sum_{\nu=n}^{\infty} |\Delta a_\nu|\right) = O(1/xn^\beta).$$

Thus, if

(4) $$\pi/(n+1)^{(\alpha+\beta)/2} \leqq x \leqq \pi/n^{(\alpha+\beta)/2},$$

then

(5) $$g(x) = S_1 + S_2 = O(n^{(\alpha-\beta)/2}),$$

$$\int_0^\pi \frac{g(x)}{x^\gamma}\,dx = \sum_{n=1}^{\infty} \int_{\pi/(n+1)^{(\alpha+\beta)/2}}^{\pi/n^{(\alpha+\beta)/2}}$$

$$= O\left(\sum_{n=1}^{\infty} n^{(\alpha-\beta)/2} n^{(\alpha+\beta)\gamma/2} n^{-1-(\alpha+\beta)/2}\right) = O(1).$$

Concerning cosine series, proof runs quite similarly as M. Sato [4] has proved.

THEOREM 5. *Let* $0 < \beta < \alpha < 1$. *If* (1) *and* (2) *hold, then* (3) *belongs to the class* L^γ *where* $\gamma < (\alpha + \beta)/(\alpha - \beta)$.

Proof runs similarly as Theorem 5. By (4) and (5),

$$\int_0^\pi |g(x)|^\gamma dx = \sum_{k=1}^\infty \int_{\pi/(n+1)^{(\alpha+\beta)/2}}^{\pi/n^{(\alpha+\beta)/2}} |g(x)|^\gamma dx$$

$$= O\left(\sum_{k=1}^\infty n^{\gamma(\alpha-\beta)/2} n^{-1-(\alpha+\beta)/2}\right) = O(1).$$

Proof is also similar for cosine series.

REFERENCES

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