SOME TRIGONOMETRICAL SERIES, X

SHIN-ICHI IZUMI

(Received April 20, 1954)

1. Let $\varphi(t)$ be an even integrable function with period 2π. Lebesgue's convergence test of the Fourier series of $\varphi(t)$ at $t = 0$ reads as follows [1]:

THEOREM. *If*

$$\int_0^t |\varphi(u)| du = o(t) \ as \ t \to 0 \tag{1}$$

and

$$\lim_{h\to 0} \int_h^\pi \frac{|\varphi(t+h) - \varphi(t)|}{t} dt = 0, \tag{2}$$

then the Fourier series of $\varphi(t)$ converges at $t = 0$.

The condition (1) was generalized by S. Pollard [2], J. J. Gergen [3], G. Sunouchi [4] (*cf.* [5]) and many other writers. On the other hand the condition (2) was generalized by S. Pollard in the form

$$\lim_{k\to\infty} \lim_{u\to 0} \int_{ku}^\pi \frac{|\varphi(t+u) - \varphi(t)|}{t} dt = 0. \tag{3}$$

The object of this part is to show that the absolute value sign may be omitted with some modification[1].

2. THEOREM 1. *Let $\varphi(t)$ be an integrable function. If* (1) *holds and*

$$n\int_0^{\pi/n} dt \left| \sum_{k=1}^{(n-1)/2} \int_{t+2k\pi/n}^{t+(2k+1)\pi/n} \frac{\varphi(v) - \varphi(v - \pi/n)}{v} dv \right| = o(1) \tag{4}$$

as $n \to \infty$, then the Fourier series of $\varphi(t)$ converges at $t = 0$.

PROOF. We may suppose that $\varphi(0) = 0$ and $\varphi_1(\pi) = 0$, where

$$\varphi_1(t) = \int_0^t \varphi(u)\, du.$$

Then[2]

1) When this paper is written up, Prof. G. Sunouchi let the author know a paper of H. E. Bray, Rice Inst. Pamphlet, 1953, which is in the same direction as this paper but the result does not overlap. He shows that sin $1/t$ does not satisfy the condition (2), but its Fourier series converges to zero at $t = 0$. One can show that sin $1/t$ satisfies the condition (4).

2) $S_n(x)$ is the nth partial sum of Fourier series of $\varphi(t)$ at $t = x$.

$$S_n(o) = \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{\sin nt}{t}dt + o(1)$$

$$= -\frac{2}{\pi}\int_0^\pi \varphi_1(t)\left[\frac{n\cos nt}{t} - \frac{\sin nt}{t^2}\right]dt + o(1)$$

$$= -\frac{2}{\pi}\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) + o(1) = -\frac{2}{\pi}(I_1 + I_2) + o(1),$$

say. By (1), $I_1 = o(1)$. After Salem [6]

$$I_2 = \int_0^{\pi/n}\left[n\cos nt\sum_{k=1}^{n-1}(-1)^k\frac{\varphi_1(t+k\pi/n)}{t+k\pi/n}\right.$$

$$\left.-\sin nt\sum_{k=1}^{n-1}(-1)^k\frac{\varphi_1(t+k\pi/n)}{(t+k\pi/n)^2}\right]dt = I_{21} - I_{22},$$

say. We may suppose n odd, and then

$$I_{21} = n^2\int_0^{\pi/n}\cos nt\left[\frac{2\pi}{n}\sum_{k=1}^{(n-1)/2}\frac{\varphi_1(t+2k\pi/n)}{t+2k\pi/n}\right.$$

$$\left.-\frac{2\pi}{n}\sum_{k=1}^{(n-1)/2}\frac{\varphi_1(t+(2k-1)\pi/n)}{t+(2k-1)\pi/n}\right]dt = n^2\int_0^{\pi/n}\cos nt.\, J_{21}\,dt,$$

say. Then

$$J_{21} = \int_t^{t+\pi}\frac{\varphi_1(u)}{u}du - \sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/n}\left[\frac{\varphi_1(u)}{u} - \frac{\varphi_1(t+2k\pi/n)}{t+2k\pi/n}\right]du$$

$$-\int_{t+\pi/n}^{t+\pi+\pi/n}\frac{\varphi_1(u)}{u}du + \sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/n}\left[\frac{\varphi_1(u-\pi/n)}{u-\pi/n}\right.$$

$$\left.-\frac{\varphi_1(t+(2k-1)\pi/n)}{t+(2k-1)\pi/n}\right]du$$

$$= \left(\int_t^{t+\pi/n} - \int_{t+\pi}^{t+\pi-\pi/n}\right)\frac{\varphi_1(u)}{u}du - \sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/2}$$

$$\left[\frac{\varphi_1(u)}{u} - \frac{\varphi_1(u-\pi/n)}{u-\pi/n} - \frac{\varphi_1(t+2k\pi/n)}{t+2k\pi/n} + \frac{\varphi(t+2k-2)\pi}{t+(2k-1)\pi/n}\right]du$$

$= J_{211} - J_{212}$.

Now

$$n^2\int_0^{\pi/n}\cos nt\, J_{211}\,dt = n^2\int_0^{\pi/n}\cos nt\int_t^{t+\pi/n}\frac{\varphi_1(u)}{u}du$$

$$-n^2\int_0^{\pi/n}\cos nt\,dt-\int_{t+\pi}^{t+\pi+\pi/n}\frac{\varphi_1(u)}{u}du=o(1)$$

by (1). The integrand of J_{212} is

$$\frac{\varphi_1(u)}{u}-\frac{\varphi_1(u-\pi/n)}{u-\pi/n}-\frac{\varphi_1(t+2k\pi/n)}{t+2k\pi/n}+\frac{\varphi_1(t+(2k-1)\pi/n)}{t+(2k-1)\pi/n}$$

$$=-\frac{\pi}{n}\varphi_1(u)\frac{(u-(t+2k\pi/n))\,(t+(2k-1)\pi/n+u)}{u(u-\pi/n)(t+2k\pi/n)(t+(2k-1)\pi/n)}$$

$$+\Big(\frac{\varphi_1(u)-\varphi_1(u-\pi/n)}{u-\pi/n}+\frac{\varphi_1(u)-\varphi_1(t+2k\pi/n)}{t+2k\pi/n}$$

$$+\frac{\varphi_1(t+(2k-1)\pi/n)-\varphi_1(u)}{t+(2k-1)\pi/n}\Big)=K_{k1}+K_{k2}.$$

We get

$$\left|n^2\int_0^{\pi/n}\cos nt\,dt\sum_{k=1}^{(n-1)2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/n}K_{k1}\,du\right|$$

$$\leqq An^2\int_0^{\pi/n}dt\cdot\frac{1}{n^2}\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}\frac{|\varphi_1(u)|}{u^3}du=o\Big(\int_0^{\pi/n}dt\sum_{k=1}^{(n-1)/2}\frac{n^2}{k^2}\cdot\frac{1}{n}\Big)=o(1)$$

by (1). Further

$$n^2\int_0^{\pi/n}\cos nt\,dt\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/n}K_{k2}\,du$$

$$=n^2\int_0^{\pi/n}\cos nt\,dt\sum_{k=1}^{(n-1)/2}\frac{1}{t+2k\pi/n}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}[\varphi_1(u)-\varphi_1(u-\pi/n)$$

$$-\varphi_1\,(t+2k\pi/n)+\varphi_1(t+(2k-1)\pi/n)]du+o(1),$$

which is less than, in absolute value,

$$n\int_0^{\pi/n}dt\left|\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}\frac{\varphi(v)-\varphi(v-\pi/n)}{v}dv\right|+o(1),$$

thus we have proved $I_{21}=o(1)$.

Concerning I_{22}, estimation may be similarly carried out and we get $I_{22}=o(1)$, and then

$$I_2=I_{21}+I_{22}=o(1).$$

Hence the theorem is completely proved.

3. From the proof of theorem 1, we get the following theorem:

THEOREM 2. *If* (1) *holds, then a necessary and sufficient condition that the Fourier series of* $\varphi(t)$ *converges at* $t=0$, *is*

$$\lim_{n\to\infty}n\int_0^{\pi/n}\cos nt\,dt\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+2(k+1)\pi/n}\frac{\varphi(v)-\varphi(v-\pi/n)}{v}dv=0. \qquad (5)$$

(2) implies (4) and (4) implies (5). Hence the necessary and sufficient condition (5) does not seem to be non-interesting. But it is of course disirable that the condition does not contain the terms cosnx and sinnx.

REFERENCES

[1] A. ZYGMUND, Trigonometrical series, 1936.
[2] S. POLLARD, Journ. London Math. Soc., 2(1926)
[3] J. J. GERGEN, Quarterly Journ. (2), 1(1930)
[4] G. SUNOUCHI, Tôhoku Math. Journ. (2)-1-3(1950-2)
[5] S. IZUMI, Tôhoku Math. Journ., in the press.
[6] R. SALEM, Comptes Rendus, 207(1938).

MATHEMATICAL INSITITUTE, TOKYO TORITSU UNIVERSITY.