# SOME TRIGONOMETRICAL SERIES, IX

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This paper consists of two independent parts. The first part concerns uniform convergence of Fourier series and the second gives an approximation formula.

## PART I.

1. R. Salem [1] has given a very general test for uniform convergence of Fourier series, which includes known criteria. By his idea, T. Kawata and the author [2] have given a test for uniform Cesàro summability of Fourier series. The last test, as is shown in [2], contains theorems due to Zygmund, Wiener-Marcinkiewicz and Salem. But the theorem due to Hardy-Littlewood [3] (Theorem 2) is not contained. We shall now generalize above criteria such that the last theorem is contained.

2. THEOREM 1. *If $f(x)$ is a continuous function such that*

$$\sum_{k=1}^{[n/2]} \frac{n}{k^{1+\alpha}} \int_{\pi/n}^{2\pi/n} |f(x+t+2k\pi/n) - f(x+t+(2k+1)\pi/n)|\,dt = o(1) \tag{1}$$

*uniformly in $x$ for $-1 < \alpha < 1$, then the Fourier series of $f(x)$ is summable $(C, \alpha)$ uniformly.*

PROOF. Let $\sigma_n^\alpha(x)$ be the $n$-th Cesàro mean of the Fourier series of $f(x)$ of order $\alpha$. Then

$$\delta_n(x) = \sigma_n^\alpha(x) - f(x) = \frac{1}{\pi}\int_0^\pi \varphi_x(t) K_n^\alpha(t)\,dt$$

where $K_n^\alpha(t)$ is the Fejér kernel of order $\alpha$. It is well known that

$$K_n^\alpha(t) = \psi_n^\alpha(t) + r_n^\alpha(t)$$

where

$$\psi_n^\alpha(t) = \cos\left(\left(n + \frac{1+\alpha}{2}\right)t - \frac{1-\alpha}{2}\pi\right)\Big/ A_n^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha}, \tag{2}$$

$$r_n^\alpha(t) = O(1/nt^2). \tag{3}$$

We have

$$\delta_n(x) = \frac{1}{\pi}\int_0^\pi = \frac{1}{\pi}\int_0^{\pi/n} + \frac{1}{\pi}\int_{\pi/n}^\pi = I_1 + I_2.$$

By the continuity of $f(x)$, $I_1 = o(1)$ uniformly. Concerning $I_2$, we have

$$I_2 = \frac{1}{\pi}\int_{\pi/n}^\pi \varphi_x(t)\psi_n^\alpha(t)\,dt + \frac{1}{\pi}\int_{\pi/n}^\pi \varphi_x(t) r_n^\alpha(t)\,dt = I_3 + I_4,$$

say. By (3), $I_4 = o(1)$ uniformly.

$$I_3 = \frac{1}{\pi}\int_{\pi/n}^{\pi} \varphi_x(t)\frac{\cos\left(\left(n+\frac{1+\alpha}{2}\right)t - \frac{1+\alpha}{2}\pi\right)}{A_n^\alpha(2\sin t/2)^{1+\alpha}}dt$$

$$= \frac{1}{\pi}\int_{\pi/n}^{\pi} \varphi_x(t)\frac{\cos\left(\left(n+\frac{1+\alpha}{2}\right)t - \frac{1+\alpha}{2}\pi\right)}{A_n^\alpha t^{1+\alpha}}dt + I_6 = I_5 + I_6,$$

say. $I_6 = o(1)$ uniformly by the Riemann-Lebesgue theorem. We have

$$I_5 = \frac{1}{\pi A_n^\alpha}\int_{\pi/n}^{\pi} \varphi_x(t)\cos((1+\alpha)(t-\pi)/2)\frac{\cos nt}{t^{1+\alpha}}dt$$

$$- \frac{1}{\pi A_n^\alpha}\int_{\pi/n}^{\pi} \varphi_x(t)\sin((1+\alpha)(t-\pi)/2)\frac{\sin nt}{t^{1+\alpha}}dt = I_7 - I_8,$$

say. Putting

$$\chi(t) = \varphi_x(t)\sin((1+\alpha)(t-\pi)/2),$$

we get, by the Salem method,

$$I_8 = \frac{1}{\pi A_n^\alpha}\int_{\pi/n}^{\pi} \chi(t)\frac{\sin nt}{t^{1+\alpha}}dt$$

$$(4) \qquad = \frac{1}{\pi A_n^\alpha}\int_{\pi/n}^{2\pi/n}\left\{\sum_{k=0}^{n-1}(-1)^k\frac{\chi(t+k\pi/n)}{(t+k\pi/n)^{1+\alpha}}\right\}\sin nt\,dt$$

$$\leqq C\sum_{k=0}^{[n/2]}\frac{n}{k^{1+\alpha}}\int_{\pi/n}^{2\pi/n}|f(x+t+2k\pi/n) - f(x+t+(2k+1)\pi/n)|dt + I_9,$$

where $I_9 = o(1)$ uniformly by the continuity of $f(x)$. Thus we get the theorem.

From the proof we can see that the absolute sign in (1) may be replaced by the ordinary bracket.

3. We shall now prove that Theorem 1 contains the following theorem due to Hardy and Littlewood.

THEOREM 2. *If $0 < \alpha \leqq 1, \alpha_p > 1$ and $f(x)$ belongs to the $Lip(\alpha, p)$ class, i. e.*

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(t^\alpha),$$

*then the Fourier series of $f(x)$ is almost everywhere $(C, -\alpha+\delta)$ summable for any $\delta > 0$.*

PROOF. Let $\gamma = -\alpha + \delta$. Since the function in the $Lip(\alpha, p)$ class $(\alpha_p > 1)$ is equivalent to a continuous function in the $Lip(\alpha - 1/p)$ class, we can suppose that $f(x)$ is continuous. The left side of (1) equals to, with $\alpha = \gamma$,

$$n\int_{\pi/n}^{2\pi/n}\left\{\sum_{k=1}^{[n/2]}\frac{1}{k^{1+\gamma}}|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)|\right\}dt,$$

which is less than

$$(5)\quad n\int_{\pi/n}^{2\pi/n}\left\{\sum_{k=1}^{[n/2]}|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)|^{p}\right\}^{1/p}dt\left\{\sum_{k=1}^{[n/2]}k^{-(1+\gamma)q}\right\}^{1/q}$$

where $1/p+1/q=1$. Since

$$(1+\gamma)q=(-\alpha+\delta+1)q=(-\alpha+\delta+1)p/(p-1)$$
$$\geqq (p-1+\alpha\delta)/(p-1)>1,$$

the second factor of (5) is bounded and the first factor is less than

$$n^{1-1/q}\left\{\int_{\pi/n}^{2\pi/n}\left(\sum_{k=1}^{[n/2]}|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)|^{p}\right)dt\right\}^{1/p}$$
$$\leqq n^{1-1/q}\left(\int_0^{2\pi}|f(t)-f(t+\pi/n)|^{p}dt\right)^{1/p}$$
$$\leqq Cn^{1/p}/n^{\alpha}=o(1)$$

as $n\to\infty$. Thus the condition (1) is satisfied, and then Theorem 2 is deduced from Theorem 1.

## PART II.

1. The object of this part is to prove the following theorem and to give its application to the theory of approximation.[1)]

THEOREM 1. *Let $f(t)$ be an integrable function with period $2\pi$ and let*

$$\varphi_x(t)=f(x+t)+f(x-t)-2f(x).$$

*If we denote by $\sigma_n^{\delta}(x)$ the n-th Cesàro mean of order $\delta$ of the Fourier series of $f(t)$ at $t=x$, then*

$$(1)\qquad \sum_{n=1}^{\infty}n^{-r}|n^{\delta}(\sigma_n^{\delta}(x)-f(x))|^{p}\leqq (A_r p)^{p}\int_0^{\pi}\left|\frac{\varphi_x(t)}{t^{\delta}}\right|^{p}t^{r-2}dt,$$

*where $p>1, r>1, 0\leqq\delta\leqq 1$ and $A_r$ is a constant depending only on $r$.*

For the proof we use a lemma due to Hardy and Littlewood [5].

LEMMA 1. *If $\varphi(t)\geqq 0, p>1, r>1$ and*

$$\varphi_1(t)=\int_0^{\tau}f(u)du,\quad \varphi_2(t)=\int_t^{\infty}u^{-1}\varphi(u)du,$$

1) The proof depends on the idea of Hardy-Littlewood [4], which the author learned from G. Sunouchi. The author expresses his hearty thanks to G. Sunouchi, who gave him important criticisms and remarks.

*then we have*

$$\int_0^\infty t^{-r}(\varphi_1(t))^p dt \leqq \left(\frac{p}{r-1}\right)^p \int_0^\infty t^{-r}(\varphi(t))^p dt, \tag{2}$$

$$\int_0^\infty t^{r-2}(\varphi_2(t))^p dt \leqq \left(\frac{p}{r-1}\right)^p \int_0^\infty t^{r-2}(\varphi(t))^p dt. \tag{3}$$

Let us consider first the case $\delta = 1$. We have

$$|\sigma_n(x) - f(x)| \leqq An\int_0 |\varphi(t)|dt + \frac{A}{n}\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt, \tag{4}$$

where $\varphi(t) = \varphi_x(t)$ and $A$ is an absolute constant.

$$\sum_{n=1}^{\infty} n^\alpha |\sigma_n(x) - f(x)|^p \leqq A^p \sum_{n=1}^{\infty} n^\alpha \left(n\int_0^{1/n} |\varphi(t)|dt\right)^p + A^p \sum_{n=1}^{\infty} n^\alpha \left(\frac{1}{n}\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt\right)^p$$
$$= I + J,$$

say. By (2), if $p > 1, \alpha + p + 2 > 1$, then

$$I \leqq \sum_{n=1}^{\infty} n^{\alpha+p}(\varphi_1(1/n))^p \leqq A \sum_{n=1}^{\infty} \int_n^{n+1} t^{\alpha+p}(\varphi_1(1/t))^p dt$$

$$\leqq A\int_0^\infty t^{\alpha+p}(\varphi_1(1/t))^p dt = A\int_0^\infty \frac{(\varphi_1(t))^p}{t^{\alpha+p+2}} dt$$

$$\leqq A\left(\frac{p}{r-1}\right)^p \int_0^\infty \frac{t^p|\varphi(t)|^p}{t^{\alpha+p+2}} dt = A\left(\frac{p}{r-1}\right)^p \int_0^\infty \frac{|\varphi(t)|^p}{t^{\alpha+2}} dt$$

$$J \leqq \sum_{n=1}^{\infty} n^{\alpha-p}\left(\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt\right)^p \leqq \sum_{n=1}^{\infty} n^{\alpha-p}(\varphi_2(1/n))^p,$$

where $\varphi_2(t) = \int_t^\infty \frac{\varphi(\mu)}{u^2} du$. By (3), if $p > 1$, $p - \alpha = r > 1$, then

$$J \leqq A\int_0^\infty t^{\alpha-p}(\varphi_2(1/t))^p dt \leqq A\int_0^\infty t^{p-\alpha-2}(\varphi_2(t))^p dt$$

$$\leqq A\left(\frac{p}{r-1}\right)^p \int_0^\infty t^{p-\alpha-2}\left(\frac{|\varphi(t)|}{t}\right) dt$$

$$= A\left(\frac{p}{r-1}\right)^p \int_0^\infty \frac{|\varphi(t)|^p}{t^{\alpha+2}} dt.$$

Hence we get, by $p - \alpha = r$,

$$\sum \frac{1}{n^r} |n(\sigma_n(x) - f(x))|^p = \sum n^\alpha |\sigma_n(x) - f(x)|^p \leqq A^p \left(\frac{p}{r-1}\right)^p \int_0^\infty \frac{|\varphi(t)|^p}{t^{\alpha+2}} dt$$

$$\leqq A_r^p p^p \int_0^\pi \frac{|\varphi(t)|^p}{t^p} t^{r-2} \, dt.$$

Thus we have proved the case $\delta = 1$ of (1).

For the case $0 < \delta < 1$, we start from

$$|\sigma_n^\delta(x) - f(x)| \leqq An \int_0^{1/n} |\varphi_x(t)| dt + \frac{A}{n^\delta} \int_{1/n}^\pi \frac{|\varphi_x(t)|}{t^{1+\delta}} dt$$

instead of (4). Then we get the proof of (1), following the similar line as the former case. The case $\delta = 0$ is also easy.

2. As an application, we can prove a classical approximation theorem. From (1) we get

$$n^\delta |\sigma_n^\delta(x) - f(x)| \leqq A_r p n^{r/p} \left( \int_0^\pi \frac{|\varphi_x(t)|^p}{t^{p\delta}} t^{r-2} dt \right)^{1/p}$$

for $p > 1, r > 1, 0 < \delta \leqq 1$. Taking $p = r \log n$, we get

$$(5) \qquad |\sigma_n^\delta(x) - f(x)| \leqq A \frac{\log n}{n^\delta} \left( \int_0^\pi \left| \frac{\varphi_x(t)}{t^\delta} \right|^p t^{r-2} dt \right)^{1/p}.$$

Thus if $f(t)$ belongs to the *Lip* $\alpha$ class $(0 \leqq \alpha \leqq 1)$, then

$$(6) \qquad \sigma_n^\alpha(x) - f(x) = O(\log n / n^\alpha).$$

## REFERENCES


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