ON ABSOLUTE LOGARITHMIC SUMMABILITY OF A SEQUENCE RELATED TO A FOURIER SERIES

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1. DEFINITION A. Let $\lambda(\omega)$ be continuous, differentiable and monotone increasing in (A, ∞), where A is some positive number, and let $\lambda(\omega)\to\infty$ as $\omega \to \infty$. Suppose Σu_n is a given infinite series and let

$$c(\omega) = \sum_{n\leqq\omega} \{\lambda(\omega) - \lambda(n)\} u_n.$$

The series Σu_n is said to be summable $|R, \lambda(n), 1|$ if

$$\int_A^\infty \left| d\left[\frac{c(\omega)}{\lambda(\omega)}\right]\right| < \infty,$$

i.e. if

$$\int_A^\infty \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^2} \left|\sum_{n\leqq\omega} \lambda(n)u_n\right| d\omega < \infty.$$

DEFINITION B. Suppose $\{t_n\}$ is a given sequence and let $\tau_n = \left(t_1 + \frac{1}{2}t_2 + \dots + \frac{1}{n}t_n\right)/\log n$. If $\tau_n \to t$ as $n\to\infty$, then the sequence $\{t_n\}$ is said to be summable $(R, \log n, 1)$ to t. If the sequence $\{\tau_n\}$ is of bounded variation, i.e., if $\sum^\infty |\tau_n - \tau_{n+1}| < \infty$, the sequence is said to be summable $|R, \log n, 1|$.

2. Let $\varphi(t)$ be an even function integrable in the sense of Lebesgue in $(0, \pi)$ and defined outside $(-\pi, \pi)$ by periodicity. We assume that the constant term in the Fourier series of $\varphi(t)$ is zero and that the special point to be considered is the origin. In these circumstances

$$\varphi(t)\sim\sum_1^\infty a_n \cos nt, \tag{2.1}$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi \varphi(t) \cos nt\, dt, \tag{2.2}$$

and we are to consider the series $\sum_1^\infty a_n$. It is well-known that these formal simplifications do not impair the generality of the problem. We write s_n for $\sum_1^n a_k$ and use the following notations.

(2.3) $$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)du\ (t>0)\text{ and }\quad (0<\alpha<1),$$

(2.4) $$\Phi_0(t)=\varphi(t),$$

(2.5) $$\varphi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t)\qquad (0\leqq\alpha<1),$$

(2.6) $$p(\omega)=\sum_{n\leqq\omega} e^{(\log n)^\Delta}(\log n)^{-1}a_n\qquad (\Delta>0),$$

(2.7) $$\xi(\omega,t)=\sum_{n\leqq\omega} e^{(\log n)^\Delta}(\log n)^{-1}\cos nt,$$

(2.8) $$\eta(\omega,t)=\sum_{n\leqq\omega} e^{(\log n)^\Delta}(\log n)^{-1}n^{-1}\sin nt,$$

(2.9) $$g(\omega,u)=\frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\xi(\omega,t)\,dt\qquad (0\leqq u\leqq\pi),$$

(2.10) $$G(\omega,u)=\frac{1}{\Gamma(\alpha+1)}\int_0^u v^\alpha\frac{d}{dv}g(\omega,v)\,dv\qquad (0\leqq u\leqq\pi).$$

3. The following result is well-known:

(a) If $\varphi(t)=o(1)$, then $s_n=o(\log n)$.

The statement (a) is equivalent to

(b) If $\varphi(t)=o(1)$, then the sequence $\{na_n\}$ is summable $(R,\log n,1)$ to 0. It is reasonable to expect that the analoge of (b) for absolute summability would be

(c) If $\varphi(t)$ is of bounded variation, the sequence $\{na_n\}$ is summable $|R,\log n,1|$.

We shall however show that the statement (c) is false. We first prove the following

LEMMA. *If the series Σu_n is summable $|R,\log n,1|$, the necessary and sufficient condition that it is absolutely convergent is that the sequence $\{(n\log n)u_n\}$ is summable $|R,\log n,1|$.*

PROOF OF LEMMA. We have on writing $\sigma_n=\sum_1^n u_k\log k$, the identity

(3.1) $$\frac{\sigma_n}{\log n}-\frac{\sigma_{n+1}}{\log(n+1)}=\sigma_n\frac{\log\left(1+\frac{1}{n}\right)}{\log n\log(n+1)}-u_{n+1},\quad\text{for } n\geqq 2.$$

Hence we have the following inequalities

(3.2) $$\sum_2^\infty\left|\frac{\sigma_n}{\log n}-\frac{\sigma_{n+1}}{\log(n+1)}\right|<A\sum_2^\infty\frac{|\sigma_n|}{n(\log n)^2}+\sum_2^\infty|u_{n+1}|$$

and

(3.3) $$\sum_{2}^{\infty} |u_{n+1}| < A\sum_{2}^{\infty}\frac{|\sigma_n|}{n(\log n)^2} + \sum_{2}^{\infty}\left|\frac{\sigma_n}{\log n} - \frac{\sigma_{n+1}}{\log(n+1)}\right|.$$

Since the series Σu_n is summable $|R, \log n, 1|$, we have, by Definition A,

(3.4) $$\int_{2}^{\infty}\frac{1}{\omega(\log\omega)^2}\left|\sum_{n\leqq\omega}\log n\, u_n\right|d\omega < \infty,$$

from which it follows that

(3.5) $$\sum_{2}^{\infty}\frac{|\sigma_n|}{n(\log n)^2} < \infty.$$

The proof of the Lemma then follows from Definition B and (3.2), (3.3) and (3.5).

In order to prove that the statement (c) is false we observe that the series $\sum_{2}^{\infty} a_n/\log n$ is summable $|R, \log n, 1|$ if

(3.6) $$\int_{2}^{\infty}\frac{|s_{[\omega]}|}{\omega(\log\omega)^2}d\omega < \infty$$

The above condition is obviously satisfied when $\varphi(t)$ is of bounded variation in $(0, \pi)$ and indeed when a continuity condition of the type $\varphi(t) = O\left\{\left(\log\frac{1}{t}\right)^{-\eta}\right\}$ $(0 < \eta < 1)$ is satisfied; since with the latter condition we can assert that $s_n = O\left\{(\log n)^{1-\eta}\right\}$[1]. But bounded variation of $\varphi(t)$ in $(0, \pi)$ is not sufficient to ensure absolute convergence of the series $\sum_{2}^{\infty} a_n/\log n$. [3] Hence writing $a_n/\log n$ for u_n in the Lemma proved above, we can easily see that bounded variation of $\varphi(t)$ alone is not sufficient ensure summability $|R, \log n, 1|$ of the sequence $\{na_n\}$.

4. We now proceed to establish some tests for the absolute convergence of the series $\sum_{2}^{\infty} a_n/\log n$. In the first instance we prove the

THEOREM. *If $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the series $\sum_{2}^{\infty} a_n/\log n$ is summable $|R, e^{(\log n)^\Delta}, 1|$, where*

$$0 < \alpha < 1 \text{ and } \Delta = 1 + \frac{1}{\alpha}.$$

We first establish the following inequalities:

1) This can be proved with a slight modification of the proof of the theorem "$s_n = o(\log n)$, when $\varphi(t) = o(1)$" as given in Titchmarsh [4].

(4.1) $$\xi(\omega, t) = O\{e^{(\log\omega)^\Delta}\omega(\log\omega)^{-\Delta}\}$$

(4.2) $$\xi(\omega, t) = O\{e^{(\log\omega)^\Delta}t^{-1}(\log\omega)^{-1}\}$$

(4.3) $$\eta(\omega, t) = O\{e^{(\log\omega)^\Delta}(\log\omega)^{-\Delta}\}$$

(4.4) $$\eta(\omega, t) = O\{e^{(\log\omega)^\Delta}t^{-1}(\omega\log\omega)^{-1}\}$$

(4.5) $$g(\omega, u) = O\{e^{(\log\omega)^\Delta}\omega^\alpha(\log\omega)^{-\Delta}\}$$

(4.6) $$g(\omega, u) = O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}u^{-1}(\log\omega)^{-1}\}$$

(4.7) $$G(\omega, u) = O\{e^{(\log\omega)^\Delta}\omega^\alpha u^\alpha(\log\omega)^{-\Delta}\}$$

(4.8) $$G(\omega, u) = O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}u^{-1+\alpha}(\log\omega)^{-1}\}$$

The inequalities (4.1) and (4.3) can be proved exactly in the same manner as in [3]. The inequalities (4.2) and (4.4) can be proved by using Abel's Lemma.

PROOF OF (4.5) AND (4.6). For $u + \omega^{-1} < \pi$[2], we write

$$\Gamma(1-\alpha)g(\omega, u) = \int_u^{u+\omega^{-1}} + \int_{u+\omega^{-1}}^{\pi} = I_1 + I_2$$

$$|I_1| < Ae^{(\log\omega)^\Delta}\int_u^{u+\omega^{-1}} (t-u)^{-\alpha}\min[\omega(\log\omega)^{-\Delta},\ t^{-1}(\log\omega)^{-1}]\,dt$$

$$< Ae^{(\log\omega)^\Delta}\min[\omega(\log\omega)^{-\Delta},\ u^{-1}(\log\omega)^{-1}]\int_u^{u+\omega^{-1}} (t-u)^{-\alpha}dt$$

$$= O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}\}\min[\omega(\log\omega)^{-\Delta},\ u^{-1}(\log\omega)^{-1}]$$

$$I_2 = \left(\frac{1}{\omega}\right)^{-\alpha}\int_{u+\omega^{-1}}^{\rho} \xi(\omega, t)\,dt \qquad (u + \omega^{-1} < \rho < \pi)$$

$$= \omega^\alpha[\eta(\omega, t)]_{u+\omega^{-1}}^{\rho}$$

$$= O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}\}\min[\omega(\log\omega)^{-\Delta},\ u^{-1}(\log\omega)^{-1}]$$

PROOF OF (4.7). We have

$$\Gamma(\alpha+1)G(\omega, u) = \int_0^u v^\alpha\frac{d}{dv}g(\omega, v)\,dv$$

$$= u^\alpha g(\omega, u) - \alpha\int_0^u v^{\alpha-1}g(\omega, v)\,dv$$

$$= O\{e^{(\log\omega)^\Delta}\omega^\alpha u^\alpha(\log\omega)^{-\Delta}\},\ \text{by (4.6)}.$$

PROOF OF (4.8). It is easy to see that

(4.9) $$g(\omega, \pi) = O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}(\log\omega)^{-1}\}.$$

Further

$$\Gamma(\alpha+1)G(\omega, \pi) = \left[v^\alpha g(\omega, v)\right]_0^{\pi} - \alpha\int_0^{\pi} v^{\alpha-1}g(\omega, v)\,dv.$$

But

2) For $u+\omega^{-1} \geqq \pi$, the integral need not be split up and the arguments for I_1 will hold for the integral.

$$\Gamma(1-\alpha)\int_0^\pi v^{\alpha-1}g(\omega,v)\,dv = \int_0^\pi v^{\alpha-1}\left(\int_v^\tau (t-v)^{-\alpha}\xi(\omega,t)\,dt\right)dv$$

$$= \int_0^\pi \xi(\omega,t)\left(\int_0^t v^{\alpha-1}(t-v)^{-\alpha}\,dv\right)dt$$

$$= \int_0^\pi \xi(\omega,t)\left(\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\,dx\right)dt$$

$$= 0.$$

Hence

(4.10) $$G(\omega,\pi) = O\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}(\log\omega)^{-1}\}, \text{ by (4.9).}$$

To prove (4.8), we have

$$\Gamma(\alpha+1)\left\{G(\omega,\ \pi) - G(\omega,\ u)\right\} = \Big[v^\alpha g(\omega,v)\Big]_u^\pi - \alpha\int_u^\pi v^{\alpha-1}g(\omega,v)\,dv$$

$$= O\left\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}(\log\omega)^{-1}\right\} + O\left\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}u^{-1+\alpha}(\log\omega)^{-1}\right\}$$

$$-\alpha\int_u^\pi v^{\alpha-1}\,O\left\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}v^{-1}(\log\omega)^{-1}\right\}dv.$$

Since $\alpha < 1$, using (4.10)

$$G(\omega,u) = O\left\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}u^{-1+\alpha}(\log\omega)^{-1}\right\}.$$

PROOF OF THEOREM. To prove the theorem, we have to show that when $\Delta = 1 + \dfrac{1}{\alpha}$

$$I = \int_2^\infty \Delta\omega^{-1}(\log\omega)^{\Delta-1}e^{-(\log\omega)\Delta}\left|p(\omega)\right|d\omega < \infty.$$

We have

$$a_n = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^\pi \cos nt\int_0^t (t-u)^{-\alpha}\,d\Phi_\alpha(u)$$

$$= \frac{2}{\pi}\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^\pi d\Phi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\cos nt\ dt,\ [1]$$

Hence

$$\frac{1}{2}\pi\,p(\omega) = \frac{1}{\Gamma(1-\alpha)}\int_0^\pi d\Phi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\sum_{n\leqq\omega} e^{(\log n)^\Delta}(\log n)^{-1}\cos nt\ dt$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^\pi d\Phi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\xi(\omega,t)\ dt$$

$$= \int_0^{\pi} g(\omega, u)\, d\Phi_\alpha(u)$$

$$= \Big[g(\omega, u)\Phi_\alpha(u) \Big]_0^{\pi} - \int_0^{\pi} \Phi_\alpha(u) \frac{d}{du} g(\omega, u)\, du.$$

Further since $\varphi_\alpha(+0)$ is finite

$$\int_0^{\pi} \Phi_\alpha(u) \frac{d}{du} g(\omega, u) du = \frac{1}{\Gamma(\alpha+1)} \int_0^{\pi} \varphi_\alpha(u)\, u^\alpha \frac{d}{du} g(\omega, u)\, du$$

$$= \frac{1}{\Gamma(\alpha+1)} \Big[\varphi_\alpha(u) \int_0^{u} v^\alpha \frac{d}{dv} g(\omega, v)\, dv \Big]_0^{\pi} - \frac{1}{\Gamma(\alpha+1)} \int_0^{\pi} d\varphi_\alpha(u) \Big\} \int_0^{u} v^\alpha \frac{d}{dv} g(\omega, v)\, dv$$

$$= \varphi_\alpha(\pi) G(\omega, \pi) - \int_0^{\pi} G(\omega, u)\, d\varphi_\alpha(u).$$

So we have finally

$$\frac{1}{2}\pi p(\omega) = \Big[g(\omega, u)\Phi_\alpha(u) \Big]_0^{\pi} - \varphi_\alpha(\pi) G(\omega, \pi) + \int_0^{\pi} G(\omega, u)\, d\varphi_\alpha(u)$$

$$= O\Big\{ e^{(\log \omega)^\Delta} \omega^{-1+\alpha} (\log \omega)^{-1} \Big\} + \int_0^{\pi} G(\omega, u)\, d\varphi_\alpha(u), \text{ by (4.9) and (4.10).}$$

Hence

$$I < A \int_2^{\infty} \frac{(\log \omega)^{\Delta-2}}{\omega^{2-\alpha}} d\omega + \frac{2}{\pi} \int_2^{\infty} \frac{\Delta}{\omega} (\log \omega)^{\Delta-1} e^{-(\log \omega)^\Delta} \left| \int_0^{\pi} G(\omega, u)\, d\, \varphi_\alpha(u) \right| d\omega$$

The integral

$$\int_2^{\infty} \frac{(\log \omega)^{\Delta-2}}{\omega^{2-\alpha}} d\omega < \infty,$$

and the integral

$$\int_2^{\infty} \frac{\Delta}{\omega} (\log \omega)^{\Delta-1} e^{(\log \omega)^\Delta} \left| \int_0^{\pi} G(\omega, u)\, d\, \varphi_\alpha(u) \right| d\omega$$

$$\leqq \int_0^{\pi} |d\, \varphi_\alpha(u)| \int_2^{\infty} \Delta \omega^{-1} (\log \omega)^{\Delta-1} e^{-(\log \omega)^\Delta} |G(\omega, u)|\, d\omega$$

Since $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, to prove the theorem it will be sufficient to show that

$$J = \int^{\infty} \Delta \omega^{-1} (\log \omega)^{\Delta-1} e^{-(\log \omega)^\Delta} |G(\omega, u)|\, d\omega < \infty.$$

Writing

$$J = \int_2^{\tau} + \int_{\tau}^{\infty} = J_1 + J_2, \quad \text{where } \tau = \frac{k}{u}\Big(\log \frac{k}{u}\Big)^{\frac{1}{\alpha}}, \qquad (k > e\pi),$$

we have

$$J_1 = \int_2^T \Delta\omega^{-1}(\log\omega)^{\Delta-1}e^{-(\log\omega)^\Delta}O\left\{e^{(\log\omega)^\Delta}\omega^\alpha u^\alpha(\log\omega)^{-\Delta}\right\}d\omega, \text{ by (4.7)}$$

$$= O\left\{u^\alpha\int_2^T \frac{\omega^{-1+\alpha}}{\log\omega}d\omega\right\} = O(1),$$

and

$$_2 = \int_T^\infty \Delta\omega^{-1}(\log\omega)^{\Delta-1}e^{-(\log\omega)^\Delta}O\left\{e^{(\log\omega)^\Delta}\omega^{-1+\alpha}u^{-1+\alpha}(\log\omega)^{-1}\right\}d\omega, \text{ by (4.8)}$$

$$= O\left\{u^{-1+\alpha}\int_T^\infty \frac{(\log\omega)^{\Delta-2}}{\omega^{2-\alpha}}d\omega\right\} = O(1), \text{ if } \Delta = 1+\frac{1}{\alpha}.$$

Hence the theorem is proved.

5. The following theorems have been proved elsewhere [2], [3].

THEOREM A. *If* (i) *the sequence* $\left\{\frac{u_n\lambda(n)}{\lambda(n)-\lambda(n-1)}\right\}$ *is of bounded variation,* (ii) *the sequence* $\left\{\frac{\lambda(n)}{\lambda(n+1)}\right\}$ *is of bounded variation and* (iii) *the series* Σu_n *is summable* $|R, \lambda(n), 1|$, *then the series is absolutely convergent.*

THEOREM B. *If* $\varphi(t)$ *is of bounded variation in* $(0,\pi)$ *then the series* $\Sigma a_n/\log n$ *is summable* $|R, e^{n^\alpha}, 1|$, *where* $0<\alpha<1$.

Combining theorems A and B on the one hand and Theorem A and the Theorem proved above on the other, we have the following criteria for the absolute convergence of the series $\Sigma a_n/\log n$.

(I) If $\varphi(t)$ is of bounded variation in $(0,\pi)$ and the seqnence $\{n^\delta(\log n)^{-1}a_n\}$ is of bounded variation for $\delta>0$ then the series $\Sigma a_n/\log n$ is absolutely convergent.

(II) If $\varphi_\alpha(t)$ is of bounded variation in $(0,\pi)$ for $0<\alpha<1$ and the sequence $\left\{\frac{n}{(\log n)^{1+\frac{1}{\alpha}}}a_n\right\}$ is of bounded variation, then the series $\Sigma a_n/\log n$ is absolutely convergent.

We have already remarked that if $\varphi(t) = O\left\{\left(\log\frac{1}{t}\right)^{-\eta}\right\}$ $(0<\eta<1)$, then the series $\Sigma a_n/\log n$ is summable $|R, \log n, 1|$. Combining this with Theorem A, we have

(III) If $\varphi(t) = O\left\{\left(\log\frac{1}{t}\right)^{-\eta}\right\}$ $(0<\eta<1)$ and the sequence $\{na_n\}$ is of bounded variation, then the series $\Sigma a_n/\log n$ is absolutely convergent.

Reverting to the original problem ot summability $|R, \log n, 1|$ of the sequence $\{na_n\}$, we have the following results, which by virtue of the Lemma of this paper, which by victure of the lemma of this paper are practically restatements of (I)-(III) above.

(d) If $\varphi(t)$ is of bounded variation in $(0, \pi)$ and the sequence $\{n^{\delta}(\log n)^{-1}a_n\}$ is of bounded variation for $\delta > 0$, then the sequence $\{na_n\}$ is summable $|R, \log n, 1|$:

(e) If $\varphi_{\alpha}(t)$ is of bounded variation in $(0, \pi)$ for $0 < \alpha < 1$ and the sequence $\left\{\frac{na_n}{(\log n)^{1+\frac{1}{\alpha}}}\right\}$ is of bounded variation, then the sequence $\{na_n\}$ is summable $|R, \log n, 1|$; and

(f) If $\varphi(t) = O\left\{\left(\log\frac{1}{t}\right)^{-\eta}\right\}$ $(0 < \eta < 1)$ and the sequence $\{na_n\}$ is of bounded variation, then the sequence is summable $|R, \log n, 1|$.

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