ON INTEGRAL INEQUALITIES AND CERTAIN APPLICATIONS TO FOURIER SERIES

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1. Introduction. Let $\varphi(z)$ be a function regular in the unit circle, then Littlewood-Paley [4] introduced the following functions:

$$g(\theta,\varphi)=g(\theta)=\left\{\int_0^1(1-\rho)|\varphi'(z)|^2d\rho\right\}^{1/2},\quad z=\rho e^{i\theta},\tag{1.01}$$

$$g_p(\theta,\varphi)=g_p(\theta)=\left\{\int_0^1(1-\rho)^{p-1}|\varphi'(z)|^p\,d\rho\right\}^{1/p},\quad \text{for } p>1\tag{1.02}$$

and

$$g^*(\theta,\varphi)=g^*(\theta)=\left\{\int_0^1(1-\rho)\,\chi^2(\rho,\theta)\,d\rho\right\}^{1/2},\tag{1.03}$$

where

$$\chi(\rho,\theta)=\left\{\frac{1}{\pi}\int_0^{2\pi}|\varphi'(\rho e^{i\theta+it})|^2P(\rho,t)\,dt\right\}^{1/2},\tag{1.04}$$

and

$$P(\rho,t)=(1-\rho^2)/(1-2\rho\cos t+\rho^2).\tag{1.05}$$

They established important theorems in the theory of Fourier series by the use of the above integrals. These auxiliarly functions have been researched by other authors, A. Zygmund [7, 8], J. Marcinkiewicz-A. Zygmund [5], and G. Sunouchi [6], and they have given complete generalized forms and simple proofs.

If for some $p>0$, the integral

$$\int_{-\pi}^{\pi}|\varphi(\rho e^{i\theta})|^p\,d\theta$$

remains bounded when $\rho\to 1-0$, the function $\varphi(z)$ is said to belong to the class H^p, then their theorems read as follows:

THEOREM A. *Let $\varphi(z)\in H^r$, $r>0$, then we have*

$$\left\{\int_0^{2\pi}g^r(\theta)\,d\theta\right\}^{1/r}\le A_r\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r\,d\theta\right\}^{1/r},\tag{1.06}$$

where

$$A=O(r)\text{ as } r\to\infty\tag{1.07}$$

and for $q > 2$

$$(1.08)\qquad \left\{\int_0^{2\pi} g_q^r(\theta)\, d\theta\right\}^{1/r} \leqq A_{r,q}\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r\, d\theta\right\}^{1/r}.$$

THEOREM B. *Let* $\varphi(z) \in H^r, r > 1,$ *and* $\varphi(0) = 0$ *then we have*

$$(1.09)\qquad B_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r\, d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g^r(\theta)\, d\theta\right\}^{1/r},$$

and for $1 < p < 2$

$$(1.10)\qquad B_{r,p}\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r\, d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g_p^r(\theta)\, d\theta\right\}^{1/r}.$$

Further,

THEOREM C. *Let* $\varphi(z) \in H^r, r > 1,$ *then we have*

$$(1.11)\qquad \left\{\int_0^{2\pi} (g^*(\theta))^r\, d\theta\right\}^{1/r} \leqq A_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}$$

$$(1.12)\qquad B_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r\, d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} (g^*(\theta))^r\, d\theta\right\}^{1/r}, \quad (\varphi(0) = 0)$$

and

$$(1.13)\qquad g(\theta) \leqq 2\, g^*(\theta).$$

Now, we define the following function analogous to $g_p(\theta)$:

$$(1.14)\qquad g_p^*(\theta) = \left\{\int_0^1 (1-\rho)^{p-1}\chi_p^p(\rho, \theta) d\rho\right\}^{1/p}, \quad p > 1$$

where

$$(1.15)\qquad \chi_p(\rho, \theta) = \left\{\frac{1}{\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta+it})|^p P(\rho, t) dt\right\}^{1/p}.$$

and $P(\rho, t)$ is the Poisson kernel.
Then we may expect analogous inequalities to the second parts of Theorem A and B and the proof of which is the main purpose of this paper. In §3 we may apply these theorems to the theory of Fourier series.

THEOREM D. *Let* $\varphi(z) \in H^r,$ $r > 1,$ then we have for $q > 2$

$$(1.16)\qquad \left\{\int_0^{2\pi} (g_q^*(\theta))^r\, d\theta\right\}^{1/r} \leqq A_{r,q}\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r\, de\right\}^{1/r},$$

and for $1 < p < 2$

$$(1.17)\qquad B_{r,p}\left\{\int^{2\pi} |\varphi(e^{i\theta})|^r\, d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} (g_p^*(\theta))^r d\theta\right\}^{1/r}, \quad (\varphi(0) = 0).$$

In all cases we denote that $A_r, A_{r,p}, \ldots.$ are constants depending only on indicating parameters and $A, B, \ldots.$ are absolute constants, not always the same from one occurrence to another.

2. Proof of the Theorem D.

We shall give a simple proof of this theorem according to the method of G. Sunouchi [6]. First we need two lemmas

LEMMA A. *Let* $\varphi(z) \in H^r$, $r > 1$,

$$\varphi^*(\theta) = \sup_{0<|h|\leqq\pi} \left| \frac{1}{h} \int_0^h |\varphi(e^{i\theta+it})|\, dt \right|, \tag{2.01}$$

then we have

$$|\varphi'(\rho e^{i\theta+it})| \leqq C\varphi^*(\theta)/(1-\rho). \tag{2.02}$$

This result is contained implicitly in the Lemma 7 of Hardy and Littlewood [2], but for the sake of completeness we prove it here.

Proof of Lemma A. For $r > 1$, it is well known that a necessary and sufficient condition for the function $\varphi(z)$ to belong to the class H^r, is that the real part of the boundary function of $\varphi(z)$ belongs to the class L^r. Therefore, if we put

$$\Re\varphi(\rho e^{i\theta}) = u(\rho, \theta), \quad \Re\varphi(e^{i\theta}) = f(\theta), \tag{2.03}$$

$$f^*(\theta) = \sup_{0<|h|\leqq\pi} \left| \frac{1}{h} \int_0^h |f(\theta+t)|\, dt \right|, \tag{2.04}$$

and

$$\frac{\partial u(\rho,\theta)}{\partial\rho} = u_\rho(\rho,\theta), \quad \frac{\partial u(\rho,\theta)}{\partial\theta} = u_\theta(\rho,\theta), \tag{2.05}$$

then it is sufficient to prove the following two inequalities:

$$\begin{aligned} &|u_\rho(\rho, \theta+t)| \leqq Cf^*(\theta)/(1-\rho), \\ &|u_\theta(\rho, \theta+t)| \leqq Cf^*(\theta)/(1-\rho), \quad \textit{for } \rho > \tfrac{1}{2}. \end{aligned} \tag{2.06}$$

For the first part, we have

$$u_\rho(\rho, \theta+t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta+s) \frac{\partial}{\partial\rho} \left(\frac{1-\rho^2}{1-2\rho\cos(s-t)+\rho^2} \right) ds$$

and

$$\left| \frac{\partial}{\partial\rho} \left(\frac{1-\rho^2}{1-2\rho\cos(s-t)+\rho^2} \right) \right| = \frac{|2(1-\rho^2) - 4(1+\rho^2)\sin^2\frac{1}{2}(s-t)|}{\{(1-\rho)^2 + 4\rho\sin^2\frac{1}{2}(s-t)\}^2}$$

$$\leqq C\frac{\delta^2 + (s-t)^2}{\{\delta^2 + (s-t)^2\}^2} \leqq \frac{C}{\delta^2 + (s-t)^2},$$

where $\delta = 1 - \rho$. Thus we get

$$|u_\rho(\rho, \theta + t)| \leqq C\int_{-\pi}^{\pi} \frac{|f(\theta+s)|}{\delta^2 + (s-t)^2}\, ds$$

$$\leqq Cf^*(\theta)\int_0^{\infty} \frac{ds}{\delta^2 + (s-t)^2} \leqq Cf^*(\theta)/\delta.$$

For the second part we may estimate it by the same way. We have

$$u_\theta(\rho, \theta + t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + s)\frac{\partial}{\partial t}\left(\frac{1-\rho^2}{1 - 2\rho\cos(s-t) + \rho^2}\right) ds,$$

and since

$$\left|\frac{\partial}{\partial t}\left(\frac{1-\rho^2}{1-2\cos(s-t)+\rho^2}\right)\right| = \frac{|2\rho(1-\rho^2)\sin(s-t)|}{\{1-2\rho\cos(s-t)+\rho^2\}^2}$$

$$\leqq C\frac{\delta|s-t|}{\{\delta^2+(s-t)^2\}^2} \leqq \frac{C}{\delta^2+(s-t)^2},$$

we get easily the required result. Thus the inequalities (2.06) are established.

LEMMA B. *Let $f(\theta) \in L^r$, $r > 1$, then for $f^*(\theta)$ of (2.04), we have*

$$\left\{\int_0^{2\pi} (f^*(\theta))^r\, d\theta\right\}^{1/r} \leqq A_r\left\{\int_0^{2\pi} |f(\theta)|^r\, d\theta\right\}^{1/r}, \tag{2.07}$$

where

$$A_r = O(r/(1-r)) \text{ as } r \to 1. \tag{2.08}$$

This maximal theorem is also due to Hardy and Littlewood [3]. Now we prove theorem D.

Proof of (1.16). From (1.03)-(1.05), (1.14)-(1.15), and Lemma A we have

$$(g_q^*(\theta))^q = \int_0^1 (1-\rho)^{q-1}\, d\rho\left\{\frac{1}{\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta+it})|^q P(\rho, t)\, dt\right\}$$

$$\leqq C(\varphi^*(\theta))^{q-2}\int_0^1 (1-\rho)\, d\rho\left(\frac{1}{\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta+it})|^2 P(\rho, t)\, dt\right)$$

$$\leqq C(\varphi^*(\theta))^{q-2}(g^*(\theta))^2.$$

Therefore

$$\int_0^{2\pi} (g_q^*(\theta))^r d\theta \leqq C\int_0^{2\pi} \{(\varphi^*(\theta))^{q-2}(g^*(\theta))^2\}^{r/q}\, d\theta$$

$$= C\int_0^{2\pi} (\varphi^*(\theta))^{(1-2/q)r}(g^*(\theta))^{2r/q}\, d\theta.$$

Since $0 < 2/q < 1$, if we apply the Hölder's inequality and using (1.11) and Lemma B, we obtain

$$\int_0^{2\pi} (g_q^*(\theta))^r \quad d\theta \leqq C\left\{\int_0^{2\pi} (\varphi^*(\theta))^r \, d\theta\right\}^{1-2/q} \left\{\int_0^{2\pi} (g^*(\theta))^r \, d\theta\right\}^{2/q}$$

$$\leqq A_{r,q}\int_0^{2\pi} |\varphi(e^{i\theta})|^r \, d\theta$$

which is the inequality (1. 16).

Proof of (1. 17). Similarly as in the proof of (1. 16) we can reduce the proof of (1. 17) to the second part of Theorem C, or we can reduce it to the second part of Theorem B, by the inequality

(2. 09) $$g_p(\theta) \leqq C g_p^*(\theta), \qquad p > 1$$

which is deduced as follows:

$$g_p^p(\theta) = \int_0^1 (1-\rho)^{p-1} |\varphi'(\rho e^{i\theta})|^p d\theta = \int_0^1 (1-\rho^2)^{p-1} |\varphi'(\rho^2 e^{i\theta})|^2 2\rho \, d\rho$$

$$\leqq 2^{p+1} \int_0^1 (1-\rho)^{p-1} \; |\varphi'(\rho^2 e^{i\theta})|^p \, d\rho$$

and by the fact that $\varphi'(\rho^2 e^{i\theta})$ is the Poisson integral of $\varphi'(\rho\, e^{i\theta})$ and $p > 1$, we can apply Gauss' theorem and Jenssen's inequality. And then the last side of the above formula does not exceed the following expression:

$$2^{p+1}\int_0^1 (1-\rho)^{p-1} \left| \frac{1}{\pi} \int_0^{2\pi} \varphi'(\rho e^{i\theta+it}) P(\rho, t) \, dt \right| d\rho$$

$$\leqq 2^{p+1}\int_0^1 (1-\rho)^{p-1} \left(\frac{1}{\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta+it})|^p \, P(\rho, t) \, dt\right) d\rho = 2^{p+1}(g_p^*(\theta))^p.$$

Thus we obtained (2. 09) and so (1. 17).

3. Applications. In this section, we show some applications of above theorems to the strong summability of the power series on the circle of convergence. Let

(3. 01) $$\varphi(z) = \sum_{n=1}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n r^n \, e^{in\theta},$$

and we write

(3. 02) $$s_n(\theta) = \sum_{\nu=1}^{n} c_\nu e^{i\nu\theta},$$

(3. 03) $$\sigma_n(\theta) = \sum_{\nu=1}^{n} \left(1 - \frac{\nu}{n}\right) c_\nu e^{i\nu\theta},$$

and

$$t_n(\theta) = s_n(\theta) - \sigma_n(\theta), \tag{3.04}$$

where $c_0 = 0$. This condition is not essential. Then A. Zygmund [9] stated the following theorem.

THEOREM E. *Let $\varphi(z) \in H^p, p > 1$, then we have*

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|t_n(\theta)|^2}{n} \right\}^{p/2} d\theta \leqq A_p \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta, \tag{3.05}$$

and

$$B_p \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p \, d\theta \leqq \int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|t_n(\theta)|^2}{n} \right\}^{p/2} d\theta. \tag{3.06}$$

Further, G. Sunouchi [6] extended above theorem, that is

THEOREM F. *Let $\varphi(z) \in H^p$, $p > 1$, then for $q > 2$ we have*

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|t^n(\theta)|^q}{n} \right\}^{p/q} d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p \, d\theta. \tag{3.07}$$

These theorems are well known, but their proofs due to many complicated lemmas. Here we may give very simple proofs.

Proof of theorem E. First we prove (3.05). Let us put by definition

$$\Phi(r, \theta) = \sum_{n=1}^{\infty} n^2 |t^n(\theta)|^2 r^{2n} \tag{3.08}$$

and

$$\Psi(\rho, \theta) = \sum_{n=1}^{\infty} \frac{n^2}{(2n+3)(2n^2)} |t_n(\theta)|^2 \rho^{2n} \tag{3.09}$$

$$= \frac{1}{\rho^3} \int_0^{\rho} (\rho - r)^2 \Phi(r, \theta) \, dr = C \int_0^1 (1 - r)^2 \Phi(r\rho, \theta) \, dr$$

then we have

$$\sum_{n=1}^{\infty} \frac{|t_n(\theta)|^2}{n} = C\Psi(1, \theta) = C \int_0^1 (1 - r)^2 \Phi(r, \theta) \, dr. \tag{3.10}$$

These formula appeared in the paper of H. C. Chow [1].

On the other hand, (3.01)-(3.04), since

$$\sum_{n=1}^{\infty} n t_n(\theta) z^n = \frac{z e^{i\theta} \varphi'(z e^{i\theta})}{(1 - z)}, \tag{3.11}$$

we have by Parseval's theorem

$$\Phi(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|r\varphi'(re^{i\theta + it})|^2}{|1 - re^{it}|^2} dt. \tag{3.12}$$

Therefore from Theorem C we have

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|t_n(\theta)|^2}{n}\right\}^{p/2} d\theta = A_p\int_{-\pi}^{\pi} d\theta\left\{\int_0^1 (1-r)^2\Phi(r,\theta)\,dr\right\}^{p/2} \tag{3.13}$$

$$\leqq A_p\int_{-\pi}^{\pi} d\theta\left\{\int_0^1 (1-r)\,dr\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(re^{i\theta+it})|^2\frac{(1-r^2)}{|1-re^{it}|^2}dt\right)\right\}^{p/2}$$

$$= A_p\int_{-\pi}^{\pi}(g^*(\theta))^p\,d\theta \leqq A_p\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|^p\,d\theta.$$

Thus we obtain (3.05).

Secondly we prove (3.06). We have

$$\int_{-\pi}^{\pi}(g^*(\theta))^p\,d\theta \tag{3.14}$$

$$= \int_{-\pi}^{\pi} d\theta\left\{\left(\int_0^{1/4}+\int_{1/4}^1\right)(1-r)\,dr\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(r^{i\theta+it})|^2P(r,t)\,dt\right\}^{p/2}$$

$$= \int_{-\pi}^{\pi}\{I_1(\theta)+I_2(\theta)\}^{p/2}\,d\theta$$

say, then we have

$$I_1(\theta) = \int_0^{1/4}(1-r)\,dr\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(re^{i\theta+it})|^2P(r,t)\,dt \tag{3.15}$$

$$\leqq C\int_0^{1/4} dr\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(re^{i\theta+it})|^2\,dt$$

$$\leqq C\int_{1/4}^{1/2} dr\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(re^{i\theta+it})|^2\,dt$$

$$\leqq C\int_{1/4}^{1/2}(1-r)\,dr\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi'(re^{i\theta+it})|^2\,P(r,t)\,dt$$

$$\leqq CI_2(\theta).$$

Here we use the property that $M_\lambda(r,\varphi)$ is a continuous increasing function of r for any function $\varphi(z)$ regular in $|z|<1$, where

$$M_\lambda(r,\varphi) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(re^{i\theta})|^\lambda d\theta\right)^{1/\lambda}, \quad (\lambda>0). \tag{3.16}$$

And that we have

$$I_2(\theta) \leqq C\int_{1/4}^1(1-r)^2dr\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|r\varphi'(re^{i\theta+it})|^2}{|1-re^{it}|^2}dt. \tag{3.17}$$

Therefore we obtain from above formulas

$$(3.18)\qquad \int_{-\pi}^{\pi} (g^*(\theta))^{p/2}\, d\theta \leqq A_p \int_{-\pi}^{\pi} \{I_2(\theta)\}^{p/2}\, d\theta$$

$$\leqq A_p \int_{-\pi}^{\pi} d\theta \left\{\int_{1/4}^{1} (1-r)^2\, dr \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|r\varphi'(re^{i\theta+it})|^2}{|1-re^{it}|^2} dt\right\}^{p/2}$$

$$\leqq A_p \int_{-\pi}^{\pi} d\theta \left\{\int_{0}^{1} (1-r)^2 \Phi(r,\theta)\, dr\right\}^{p/2} \leqq A_p \int_{-\pi}^{\pi} \left\{\sum_{n=1}^{\infty} \frac{|t_n(\theta)|^2}{n}\right\}^{p/2} d\theta.$$

Applying second part of Theorem C to (3.18), we prove

$$(3.19)\qquad \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p\, d\theta \leqq A_p \int_{-\pi}^{\pi} \left\{\sum_{n=1}^{\infty} \frac{|t_n(\theta)|^2}{n}\right\}^{p/2} d\theta.$$

Proof of theorem F. In this case, if we apply Hardy-Littlewood's theorem (cf. A. Zygmund [10], Chap IX Theorem 9.51) instead of Parseval's theorem in the above argument we obtain

$$(3.20)\qquad \Phi_q(r,\theta) = \sum_{n=1}^{\infty} n^q |t_n(\theta)|^q r^{qn}$$

$$\leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|r\varphi'(re^{i\theta+it})|^q}{|1-re^{it}|^q}\, t^{q-2}\, dt$$

and

$$(3.21)\qquad \Psi_q(\rho,\theta)) = \sum_{n=1}^{\infty} \frac{|t_n(\theta)|^q}{n} \rho^{nq}$$

$$\leqq \frac{A_q}{\rho^{q+1}} \int_0^{\rho} (\rho-r)^q \Phi_q(r,\theta) dr = A_q \int_0^1 (1-r)^q \Phi_q(r\rho,\theta) dr.$$

Therefore from (3.07), (3.20) and (3.21) we have

$$(3.22)\qquad \int_{-\pi}^{\pi} \left\{\sum_{n=1}^{\infty} \frac{|t_n(\theta)|^q}{n}\right\}^{p/q} d\theta \leqq \int_{-\pi}^{\pi} \{\Psi_q(1,\theta)\}^{p/q}\, d\theta$$

$$\leqq A_q \int_{-\pi}^{\pi} d\theta \left\{\int_0^1 (1-r)^q \Phi_q(r,\theta)\, dr\right\}^{p/q}$$

$$\leqq A_q \int_{-\pi}^{\pi} d\theta \left\{\int_0^1 (1-r)^q\, dr \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |\varphi'(re^{i\theta+it})|^q \cdot \frac{t^{q-2}}{|1-re^{it}|^q}\, dt\right)\right\}^{p/q}.$$

By an elementary calculation, it is easily proved

$$(3.23)\qquad \left|\frac{t}{1-re^{it}}\right| \leqq C\frac{t/2}{\sin t/2} \leqq C, \qquad |t| \leqq \pi,\ 0 \leqq r \leqq 1.$$

And so, by the use of (3.23) and Theorem D, we have

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|t_n(\theta)|^q}{n} \right\}^{p/q} d\theta \tag{3.24}$$

$$\leqq A_q \int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 (1-r)^{q-1} dr \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |\varphi'(re^{i\theta+it})|^q \frac{(1-r^2)}{|1-re^{it}|^2} dt \right) \right\}^{p/q}$$

$$\leqq A_q \int_{-\pi}^{\pi} (g_q^*(\theta))^p \, d\theta \leqq A_{p,q} \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p \, d\theta.$$

This is our disired result. Thus the theorem is completely proved.

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