ON THE SUMMABILITY OF POWER SERIES AND FOURIER SERIES

GEN-ICHIRÔ SUNOUCHI

(Received January 10, 1955)

1. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}$$

be a function regular for $r = |z| < 1$. If for some $p > 0$, the integral

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta$$

remains bounded when $r \to 1-0$, the function $f(z)$ is said to belong to the class H^p. If $p > 1$, a necessary and sufficient condition for the function $f(z)$ to belong to the class H^p, is that the real part of the series

$$\sum_{n=0}^{\infty} c_n e^{in\theta}$$

is the Fourier series of a function of the class L^p.

Throughout this paper we put

$$s_n(f, \theta) \equiv s_n(\theta) = \sum_{\nu=0}^{n} c_\nu e^{i\nu\theta}, \quad t_n(f, \theta) \equiv t_n(\theta) = nc_n e^{in\theta},$$

$$\sigma_n^\alpha(f, \theta) \equiv \sigma_n^\alpha(\theta) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu(\theta) = \frac{s_n^\alpha(\theta)}{A_n^\alpha} \qquad \text{for } \alpha > -1$$

and

$$\tau_n^\alpha(f, \theta) \equiv \tau_n^\alpha(\theta) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} t_\nu(\theta) \qquad \text{for } \alpha > 0,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

Then we have $\tau_n^\alpha(\theta) = n\{\sigma_n^\alpha(\theta) - \sigma_{n-1}^\alpha(\theta)\} = \alpha\{\sigma_n^{\alpha-1}(\theta) - \sigma_n^\alpha(\theta)\}$.

A. Zygmund [6] has proved the following theorem:

THEOREM A. *) *If $f(z) \in H$,*

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} \frac{|\tau_n^1(\theta)|^2}{n} \right\}^{p/2} d\theta \leq A_p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta, \qquad p > 1,$$

---

*) $A_p, B, C, \ldots\ldots$ denote constants, which are not the same with different occurrence.

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^1(\theta)|^2}{n}\right\}^{1/2} d\theta \leqq B\int_{-\pi}^{\pi} |f(e^{i\theta})|\log^+|f(e^{i\theta})|d\theta + B',$$

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^1(\theta)|^2}{n}\right\}^{\mu/2} d\theta \leqq C_\mu\left\{\int_{-\pi}^{\pi} |f(e^{i\theta})|d\theta\right\}^{\mu}, \quad 0<\mu<1.$$

H. C. Chow [1] has extended this theorem in the following form:

THEOREM *B*. *If $f(z)$ belongs to H^p $(p>0)$, then the series*

$$\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}$$

converges for almost all θ, where $\alpha=1/p$ or $\alpha>1/p$ according as $0<p\leqq 1$ or $1<p\leqq 2$.

In this note we complete Theorem B in the type of Theorem A. We prove firstly

THEOREM 1. *If $f(z)\in H^p$ $(0<p\leqq 2)$, then*

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p/2} d\theta \leqq A_{p,\alpha}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta,$$

where

$$\alpha=(1+\delta)/p \text{ and } \qquad \delta>0.$$

THEOREM 2. *If $f(z)\in H^p(0<p\leqq 1)$,*

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p/2} d\theta \leqq A_p\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \log^+|f(e^{i\theta})|d\theta + B_p$$

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p\mu/2} d\theta \leqq C_{p,\mu}\left\{\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta\right\}^{\mu}, \quad 0<\mu<1,$$

where $\alpha=1/p$.

From these theorems we prove

THEOREM 3. *If $f(z)\in H^p$ $(0<p\leqq 1)$, then*

$$\int_{-\pi}^{\pi}\left\{\sup_{0\leqq n<\infty}|\sigma_n^\alpha(\theta)|\right\}^p d\theta \leqq A_{p,\alpha}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta,$$

$$\int_{-\pi}^{\pi}|\sigma_n^\alpha(\theta)|^p d\theta \leqq A_{p,\alpha}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta,$$

where $\quad\alpha>1/p-1.$

THEOREM 4. *If $f(z)\in H^p$ $(0<p\leqq 1/2)$,*

$$\int_{-\pi}^{\pi}\left\{\sup_{0\leqq n<\infty}|\sigma_n^\alpha(\theta)|\right\}^p d\theta \leqq A_p\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \log^+|f(e^{i\theta})|d\theta + B_p,$$

$$\int_{-\pi}^{\pi}\left\{\sup_{0\leqq n<\infty}|\sigma_n^\alpha(\theta)|\right\}^{p\mu}d\theta\leqq C_{p,\mu}\left\{\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\mu},\quad 0<\mu<1,$$

where $\alpha=1/p-1$.

Theorem 3 is a generalization of results of Hardy-Littlewood [4] and Gwilliam [2]. Theorem 4 settles a conjecture of A. Zygmund [7]. Concerning his another conjecture [7], we prove the following theorem:

THEOREM 5. *If* $f(z)\in H^p(1<p\leqq 2)$, *then*

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2/p}}\right\}^{p/2}d\theta\leqq A_p\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\,d\theta,$$

where $\alpha=1/p$.

THEOREM 6. *If* $f(z)\in H^p\ (1/2<p\leqq 1)$, *then*

$$\int_{-\pi}^{\pi}\left\{\sup_{0\leqq n<\infty}\left|\frac{\sigma_n^\alpha(\theta)}{\{\log(n+2)\}^{1/p}}\right|\right\}^{p}d\theta\leqq B_p\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\,d\theta,$$

where $\alpha=1/p-1$.

But there is discrepancy*) between Theorem 6 and Zygmund's conjecture.

2. For the proof of these theorems, we need the following lemmas.

LEMMA 1. *If* $f(z)\in H^k(k\geqq 1)$, *and*

$$f_k^*(\theta)=\sup_{0<|h|<\pi}\left|\frac{1}{h}\int_0^h|f(e^{i(\theta+u)})|^k\,du\right|^{1/k}$$

then

$$\left|f(re^{i(\theta+u)})\right|\leqq A_k f_k^*(\theta)\left\{1+\frac{|u|}{1-r}\right\}^{1/k}$$

Further if $f(z)\in H^2$ *and* $k<2$, *then*

$$\int_{-\pi}^{\pi}\{f_k^*(\theta)\}^2\,d\theta\leqq B_k\int_{-\pi}^{\pi}|f(e^{i\theta})|^2\,d\theta.$$

This is proved implicitly in the Hardy-Littlewood paper [5].

LEMMA 2. *If* $q/(q-1)\leqq 2\leqq q$ *and* $\mu=(2-q)/2q$, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^q\,d\theta\leqq A_q\left\{\sum_{n=0}^{\infty}(n+1)^{-2\mu}|c_n|^2r^{2n}\right\}^{q/2}.$$

This is a particular case of a Hardy-Littlewood theorem [3].

LEMMA 3. *If* $f(z)\in H$, *and we put*

$$\left\{\sup_{0<r<1}(1-r)\sum_{n=0}^{\infty}|s_n(\theta)|^2r^{2n}\right\}^{1/2}=f_1^*(\theta),$$

then

*) See addendum at the end of the paper.

$$\int_{-\pi}^{\pi} |f_1^*(\theta)|\,d\theta \leqq A\int_{-\pi}^{\pi} |f(e^{i\theta})|\log^+|f(e^{i\theta})|\,d\theta + B,$$

$$\int_{-\pi}^{\pi} |f_1^*(\theta)|^{\mu}d\theta \leqq C_{\mu}\left\{\int_{-\pi}^{\pi} |f(e^{i\theta})|\,d\theta\right\}^{\mu},\ 0<\mu<1.$$

PROOF. Since the function $(1-r)r^{2n}$ has a maximum at $r = 1 - 1/(2n+1)$, we have

$$(1-r)r^{2n} \leqq \frac{1}{2n+1}\left(1-\frac{1}{2n+1}\right)^{2n} \leqq \frac{K}{n}.$$

Hence

$$\left\{\sup_{0<r<1}(1-r)\sum_{n=0}^{\infty}|s_n(\theta)|^2 r^{2n}\right\}^{1/2}$$

$$\leqq \left\{\sup_{0<r<1}(1-r)\sum_{n=0}^{\infty}|s_n(\theta)-\sigma_n^1(\theta)|^2 r^{2n}\right\}^{1/2}$$

$$+\left\{\sup_{0<r<1}(1-r)\sum_{n=0}^{\infty}|\sigma_n^1(\theta)|^2 r^{2n}\right\}^{1/2}$$

$$\leqq A_1\left\{\sum_{n=0}^{\infty}\frac{|s_n(\theta)-\sigma_n^1(\theta)|^2}{n+1}\right\}^{1/2} + A_2\left\{\sup_{0\leqq n<\infty}|\sigma_n^1(\theta)|^2\right\}^{1/2}$$

$$\leqq A_1\left\{\sum_{n=0}^{\infty}\frac{|\tau_n^1(\theta)|^2}{n+1}\right\}^{1/2} + A_2\sup_{0\leqq n<\infty}|\sigma_n^1(\theta)|.$$

From Theorem A and a well-known maximal theorem, we get the lemma.

LEMMA 4. *If $f(z) \in H^2$, and we put*

$$\left\{\sup_{0<r<1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}|s_n^{-1/2}(\theta)|^2 r^{2n}\right\} = \{f_2^*(\theta)\}^2$$

then

$$\int_{-\pi}^{\pi}|f_2^*(\theta)|^2\,d\theta \leqq A\int_{-\pi}^{\pi}|f(e^{i\theta})|^2\,d\theta.$$

PROOF. The proof of this lemma is analogous to that of the above lemma.

Since

$$r^{2n}/|\log(1-r)| \leqq \frac{K}{\log(n+1)},$$

we have

$$\left\{\sup_{0<r<1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}|s_n^{-1/2}(\theta)|^2 r^{2n}\right\}$$

$$\leqq \left\{\sup_{0<r<1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}\frac{|\sigma_n^{-1/2}(\theta)-\sigma_n^{1/2}(\theta)|^2}{n+1}r^{2n}\right\}$$

$$+\left\{\sup_{0<r<1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}\frac{|\sigma_n^{1/2}(\theta)|^2}{n+1}r^{2n}\right\}$$

$$\leqq A_1\left\{\sum_{n=0}^{\infty}\frac{|\tau_n^{1/2}(\theta)|^2}{(n+1)\log(n+2)}\right\}+A_2\left\{\sup_{0\leqq n<\infty}|\sigma_n^{1/2}(\theta)|\right\}^2.$$

By the well-known result [6], we get

$$\int_{-\pi}^{\pi}|f_2^*(\theta)|^2d\theta\leqq A\int_{-\pi}^{\pi}|f(e^{i\theta})|^2d\theta,$$

which is the required.

LEMMA 5. *If $f(z)$ is zeropoint-free and $f(z)=g^2(z)$, then*

$$|\sigma_n^{\alpha}(f,\theta)|\leqq A_1\sum_{\nu=0}^{n}\frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta)-\sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2}{\nu+1}+A_2\left\{\sup_{0\leqq n<\infty}\sigma_n^{(\alpha+1)/2}(g,\theta)\right\}^2,$$

where α is a positive number.

PROOF. Since $f(z)=g^2(z)$, we have

$$\frac{f(z)}{(1-z)^{\alpha+1}}=\sum_{n=0}^{\infty}s_n^{\alpha}(f,\theta)z^n=\left\{\frac{g(z)}{(1-z)^{(\alpha+1)/2}}\right\}^2$$

and then

$$|s_n^{\alpha}(f,\theta)|=\left|\sum_{\nu=0}^{n}s_{n-\nu}^{(\alpha-1)/2}(g,\theta)\,s_\nu^{(\alpha-1)/2}(g,\theta)\right|$$

$$\leqq\sum_{\nu=0}^{n}|s_\nu^{(\alpha-1)/2}(g,\theta)|^2\leqq\sum_{\nu=0}^{n}\frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta)|^2}{(\nu+1)^{1-\alpha}}$$

$$\leqq\sum_{\nu=0}^{n}\frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta)-\sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2}{(\nu+1)^{1-\alpha}}+\sum_{\nu=0}^{n}\frac{|\sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2}{(\nu+1)^{1-\alpha}}.$$

Thus we get

$$|\sigma_n^{\alpha}(f,\theta)|\leqq\sum_{\nu=0}^{n}\frac{|\tau_\nu^{(\alpha+1)/2}(g,\theta)|^2}{\nu+1}$$

$$+\left\{\sup_{0\leqq\nu<\infty}|\sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2\right\}\left\{\frac{1}{n^{\alpha}}\sum_{\nu=0}^{n}(\nu+1)^{\alpha-1}\right\},$$

$$\leqq A_1\sum_{\nu=0}^{n}\frac{|\tau_\nu^{(\alpha+1)/2}(g,\theta)|^2}{\nu+1}+A_2\{\sup|\sigma_\nu^{(\alpha+1)/2}(g,\theta)|\}^2.$$

LEMMA 6. *For any positive λ,α,* and large n,

$$\frac{A}{(\log n)^{\lambda}n^{2\alpha+1}}\leqq\int_0^1\frac{(1-r)^{2\alpha}r^{2n}}{|\log(1-r)|^{\lambda}}dr\leqq\frac{B}{(\log n)^{\lambda}n^{2\alpha+1}}.$$

PROOF. From the change of variables

$$\int_0^1\frac{(1-r)^{2\alpha}r^{2n}}{|\log(1-r)|^{\lambda}}dr=\int_0^1\frac{r^{2\alpha}(1-r)^{2n}}{|\log r|^{\lambda}}dr=\int_0^{1/n}+\int_{1/n}^1=I+J,$$

say. Then

$$I=\int_0^{1/n}\frac{r^{2\alpha}(1-r)^{2n}dr}{|\log r|^{\lambda}}\leqq\frac{(1/n)^{2\alpha}}{(\log n)^{\lambda}}\int_0^{1/n}(1-r)^{2n}\,dr\leqq\frac{K}{n^{2\alpha+1}(\log n)^{\lambda}}$$

and

$$J=\int_{1/n}^{1}\frac{r^{2\alpha}(1-r)^{2n}}{|\log r|^{\lambda}}dr=\int_{1/n}^{1}\frac{(1-r)}{|\log r|^{\lambda}}r^{2\alpha}(1-r)^{2n-1}\,dr$$

$$\leqq\frac{1-1/n}{|\log n|^{\lambda}}\int_0^1 r^{2\alpha}(1-r)^{2n-1}\,dr\leqq\frac{L}{n^{2\alpha+1}(\log n)^{\lambda}}.$$

On the other hand,

$$J\geqq\frac{M}{n^{2\alpha}(\log n)^{\lambda}}\int_{1/n}^{1}(1-r)^{2n}dr=\frac{M}{n^{2\alpha}(\log n)^{\lambda}}\left[\frac{-(1-r)^{2n+1}}{2n+1}\right]_{1/n}^{1}$$

$$\geqq\frac{N}{n^{2\alpha+1}(\log n)^{\lambda}}.$$

3. **Proof of Theorem 1.** Let us put

$$\Phi_\alpha(r,\theta)=\sum_{n=1}^{\infty}(A_n^{\alpha})^2|\tau_n^{\alpha}(\theta)|^2r^{2n}$$

and

$$\Psi_\alpha(\rho,\theta)=\sum_{n=1}^{\infty}\frac{(A_n^{\alpha})^2}{(2n+2\alpha+1)A_{2n}^{2\alpha}}|\tau_n^{\alpha}(\theta)|^2\rho^{2n}\qquad(0<\rho<1)$$

$$=\frac{1}{\rho^{2\alpha+1}}\int_0^{\rho}(\rho-r)^{2\alpha}\Phi_\alpha(r,\theta)d\,r$$

$$=\int_0^1(1-r)^{2\alpha}\Phi_\alpha(r\rho,\theta)dr,$$

then

$$\sum_{n=1}^{\infty}\frac{|\tau_n^{\alpha}(\theta)|^2}{n}\leqq A_1\Psi_\alpha(1,\theta)=A_1\int_0^1(1-r)^{2\alpha}\Phi_\alpha(r,\theta)dr.$$

On the other hand, since

$$\sum_{n=1}^{\infty}A_n^{\alpha}\tau_n^{\alpha}(\theta)z^n=\frac{ze^{i\theta}f'(ze^{i\theta})}{(1-z)^{\alpha}}$$

we have by Parseval's identity,

$$\Phi_\alpha(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|rf'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{2\alpha}}d\varphi.$$

In proving these theorems we can suppose that $f(z)$ has no zeros inside the unite circle. Put

$$F^2(z)=f^p(z)$$

then $F(z)$ belongs to H^2. Let $\beta = (1+\delta)/2$, $\alpha = (1+\delta)/p, \delta > 0$. Since

$$f'(z) = \frac{2}{p}\{F(z)\}^{(2/p-1)} F'(z),$$

we have

$$\frac{ze^{i\theta}f'(ze^{i\theta})}{(1-z)^\alpha} = \frac{2}{p}\left\{\frac{F(ze^{i\theta})}{(1-z)^\beta}\right\}^{(2/p-1)} \frac{ze^{i\theta}F'(ze^{i\theta})}{(1-z)^\beta}$$

and hence

$$\Phi_\alpha(r,\theta) = \frac{2}{p^2\pi}\int_{-\pi}^{\pi}\left\{\frac{|F(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{2\beta}}\right\}^{(2/p-1)} \frac{|rF'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{2\beta}}d\varphi.$$

If we take $k<2$, then by Lemma 1,

$$\Phi_\alpha(r,\theta) \leqq A_1\int_{-\pi}^{\pi}\left\{|F_k^*(\theta)|^2\left(1+\frac{|\varphi|}{1-r}\right)^{2/k}\right\}^{(2/p-1)} \frac{|rF'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{4\beta/p}}d\varphi$$

$$\leqq A_1\{F_k^*(\theta)\}^{2(2/p-1)}\int_{-\pi}^{\pi}\left(1+\frac{|\varphi|}{1-r}\right)^{(2/k)(2/p-1)} \frac{|rF'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{4\beta/p}}d\varphi.$$

Thus we get

$$\Psi_\alpha(\rho,\theta) \leqq A_2\{F_k^*(\theta)\}^{2(2/p-1)}\int_0^1 (1-r)^{2\alpha}dr$$

$$\cdot\int_{-\pi}^{\pi}\left(1+\frac{|\varphi|}{1-r}\right)^{(2/k)(2/p-1)} \frac{|r\rho F'(r\rho e^{i\varphi+i\theta})|^2}{|1-r\rho e^{i\varphi}|^{4\beta/p}}d\varphi$$

and

$$\int_{-\pi}^{\pi}|\Psi_\alpha(1,\theta)|^{p/2}d\theta$$

$$\leqq A_3\int_{-\pi}^{\pi}\{F_k^*(\theta)\}^{p(2/p-1)}\left\{\int_0^1(1-r)^{2\alpha}dr\int_{-\pi}^{\pi}\left(1+\frac{|\varphi|}{1-r}\right)^{(2/k)(2/p-1)}\right.$$

$$\left.\cdot\frac{|rF'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{4\beta/p}}d\varphi\right\}^{p/2}d\theta.$$

By Hölder's inequality for the indices $2/p, 2/(2-p)$, it follows,

$$\int_{-\pi}^{\pi}|\Psi_\alpha(1,\theta)|^{p/2}d\theta$$

$$\leqq A_4\left\{\int_{-\pi}^{\pi}|F_k^*(\theta)|^2d\theta\right\}^{(2-p)/2}\left\{\int_{-\pi}^{\pi}d\theta\int_0^1(1-r)^{2\alpha}dr\right.$$

$$\left.\cdot\int_{-\pi}^{\pi}\left(1+\frac{|\varphi|}{1-r}\right)^{(2/k)(2/p-1)}\frac{|rF'(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|^{4\beta/p}}d\varphi\right\}^{p/2}$$

$$\leqq A_4\,(I_k)^{(2-p)/2}(J_k)^{p/2},$$

say. From Lemma 1,

$$I_k \leqq A_5 \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta.$$

For the sake of estimation of J_k, we consider the integral

$$\int_0^{\pi} \frac{|1 - r + |\varphi||^{(2/k)(2/p-1)}}{\{(1-r)^2 + \varphi^2\}^{2\beta/p}} \, d\varphi = \int_0^{1-r} + \int_{1-r}^{\pi} = K + L,$$

$$K \leqq A_6(1-r)^{(2/k)(2/p-1)-4\beta/p+1} = A_6(1-r)^{(2/k-1)(2/p-1)-2\delta/p}$$

and

$$L \leqq A_7 \int_{1-r}^{\pi} \frac{\varphi^{(2/k)(2/p-1)}}{\varphi^{4\beta/p}} \, d\varphi = A_7 \int_{1-r}^{\pi} \varphi^{(2/k)(2/p-1)-4\beta/p} \, d\varphi$$

$$= A_7 \Big[\varphi^{(2/k-1)(2/p-1)-2\delta/p} \Big]_{1-r}^{\pi}.$$

Since $\delta > 0$, if we take k sufficiently near to 2, then

$$(2/k-1)(2/p-1) - 2\delta/p < 0$$

and

$$L \leqq A_7(1-r)^{(2/k-1)(2/p-1)-2\delta/p}.$$

Hence

$$\int_{-\pi}^{\pi} \frac{|1 - r + |\varphi||^{(2/k)(2/p-1)}}{\{(1-r)^2 + \varphi^2\}^{2\beta/p}} \, d\theta \leqq A_8(1-r)^{(2/k-1)(2/p-1)-2\delta/p},$$

and

$$J_k \leqq \int_0^1 (1-r)^{2\alpha} dr \int_{-\pi}^{\pi} \frac{\left\{1 + \dfrac{|\varphi|}{1-r}\right\}^{(2/k)(2/p-1)}}{|1 - re^{i\varphi}|^{4\beta/p}} \, d\varphi \int_{-\pi}^{\pi} |rF'(re^{i\theta})|^2 d\theta$$

$$\leqq A_8 \int_0^1 (1-r)^{2\alpha}(1-r)^{-(2/k)(2/p-1)}(1-r)^{(2/k-1)(2/p-1)-2\delta/p} \, dr$$

$$\cdot \int_{-\pi}^{\pi} |rF'(re^{i\theta})|^2 \, d\theta$$

$$\leqq A_8 \int_0^1 (1-r) \, dr \int_{-\pi}^{\pi} |F'(re^{i\theta})|^2 \, d\theta$$

$$\leqq A_8 \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta.$$

Collecting these estimations, we get

$$\int_{-\pi}^{\pi} \left\{ \frac{|\tau_n^{\alpha}(\theta)|^2}{n} \right\}^{p/2} d\theta \leqq A_9 \int_{-\pi}^{\pi} |\Psi_\alpha(1, \theta)|^{p/2} \, d\theta$$

$$\leqq A_9 \left\{ \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta \right\}^{(2-p)/2+p/2}$$

$$\leqq A_9 \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta = A_9 \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta.$$

Thus we get the theorem.

4. Proof of Theorem 2. The case $p = 1$ is Theorem A. Suppose that $1/2 < p < 1$. Let

$$\alpha = 1/p, \quad G(z) \equiv \{f(z)\}^p,$$

then $G(z)$ belongs to H. Then

$$f'(z) = \alpha \{G(z)\}^{\alpha-1} G'(z),$$

so that

$$\frac{ze^{i\theta} f'(ze^{i\theta})}{(1-z)^\alpha} = \alpha \left\{ \frac{G(ze^{i\theta})}{1-z} \right\}^{\alpha-1} \frac{ze^{i\theta} \, G'(ze^{i\theta})}{1-z}$$

and hence we have by Parseval's identity

$$\Phi_\alpha(r, \theta) = \frac{\alpha^2}{2\pi} \int_{-\pi}^{\pi} \left\{ \left| \frac{G(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^2 \right\}^{\alpha-1} \left| \frac{rG'(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^2 d\varphi.$$

Since $1/2 < p < 1$, $\alpha - 1 < 1$, and then we have by Hölder's inequality

$$\Phi_\alpha(r, \theta) \leqq A_1 \left\{ \int_{-\pi}^{\pi} \left| \frac{G(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^2 d\varphi \right\}^{\alpha-1} \left\{ \int_{-\pi}^{\pi} \left| \frac{rG'(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^{2/(2-\alpha)} d\varphi \right\}^{2-\alpha}$$

$$\leqq A_1 \left\{ (1-r) \int_{-\pi}^{\pi} \left| \frac{G(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^2 d\varphi \right\}^{\alpha-1} \left\{ (1-r)^{1-\alpha} \cdot \left(\int_{-\pi}^{\pi} \left| \frac{rG'(re^{i\varphi+i\theta})}{1-re^{i\varphi}} \right|^{2/(2-\alpha)} d\varphi \right)^{2-\alpha} \right\}$$

$$\leqq A_1 \{I(\theta)\}^{\alpha-1} J(\theta),$$

say.

Let us put

$$G(z) = \sum_{n=0}^{\infty} c_n^* z^n,$$

$$s_n^*(\theta) = \sum_{\nu=0}^{n} c_\nu^* e^{i\nu\theta}$$

and

$$\tau_n^*(\theta) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu c_\nu^* e^{i\nu\theta}$$

then

$$I(\theta) \leqq A_2 (1-r) \sum_{n=0}^{\infty} |s_n^*(\theta)|^2 r^{2n}$$

$$\leqq A_3\{G_1^*(\theta)\}^2,$$

by Lemma 3.

On the other hand by Lemma 2, we have

$$J(\theta) \leqq A_4(1-r)^{1-\alpha}\left\{\int_{-\pi}^{\pi}\left|\frac{rG'(re^{i\varphi+i\theta})}{1-re^{i\varphi}}\right|^{2/(2-\alpha)} d\varphi\right\}^{2-\alpha}$$

$$\leqq A_5(1-r)^{1-\alpha}\sum_{n=1}^{\infty} n^{\alpha+1}|\tau_n^*(\theta)|^2 r^{2n}.$$

Hence

$$\Psi_\alpha(1,\theta) \leqq A_6\{G_1^*(\theta)\}^{2(\alpha-1)}\int_0^1 (1-r)^{2\alpha}(1-r)^{1-\alpha}$$

$$\cdot\sum_{n=1}^{\infty} n^{\alpha+1}|\tau_n^*(\theta)|^2 r^{2n}\, dr$$

$$\leqq A_7\{G_1^*(\theta)\}^{2(\alpha-1)}\{G_1^{**}(\theta)\}^2 \leqq A_8\{H_1^*(\theta)\}^{2\alpha},$$

where

$$G_1^{**}(\theta) = \left\{\sum_{n=1}^{\infty}\frac{|\tau_n^*(\theta)|^2}{n}\right\}^{1/2}, \quad H_1^*(\theta) = \max\{G_1^*(\theta), G_1^{**}(\theta)\}.$$

Since by Lemma 3 and Theorem A

$$\int_{-\pi}^{\pi}|H_1^*(\theta)|\,d\theta \leqq A\int_{-\pi}^{\pi}|G(e^{i\theta})|\log^+|G(e^{i\theta})|\,d\theta + B$$

$$\int_{-\pi}^{\pi}|H_1^*(\theta)|^\mu d\theta \leqq C_\mu\left\{\int_{-\pi}^{\pi}|G(e^{i\theta})|\,d\theta\right\}^\mu, \quad 0<\mu<1,$$

we get the required

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p/2} d\theta \leqq \int_{-\pi}^{\pi}|\Psi_\alpha(1,\theta)|^{1/2\alpha}\,d\theta$$

$$\leqq A\int_{-\pi}^{\pi}|H^*(\theta)|\,d\theta \leqq A\int_{-\pi}^{\pi}|G(e^{i\theta})|\log^+|G(e^{i\theta})|\,d\theta + B$$

$$\leqq A\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\log^+|f(e^{i\theta})|\,d\theta + B,$$

and

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p\mu/2} d\theta \leqq C_\mu\left\{\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\,d\theta\right\}^\mu, \quad (0<\mu<1)$$

where $\alpha = 1/p$.

For the case $1/(m+1) \leqq p \leqq 1/m$, $(m=2,3,\ldots.)$, we can proceed similarly, cf. H. C. Chow [2].

5. **Proofs of Theorem 3 and 4.** If we put

$$f(z) = g^2(z),$$

then $f(z) \in H^p(0 < p < 1)$ implies $g(z) \in H^{2p}(0 < 2p < 2)$. By Lemma 5, we have

$$|\sigma_n^\alpha(f, \theta)| \leqq A_1 \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(\alpha-1)/2}(g, \theta) - \sigma_\nu^{(\alpha+1)/2}(g, \theta)|^2}{\nu+1} + A_2 \sup_{0 \leqq \nu < \infty} |\sigma_\nu^{(\alpha+1)/2}(g, \theta)|^2$$

and if $(\alpha - 1)/2 > 1/2p - 1$, that is $\alpha > 1/p - 1 > 0$, we get

$$|\sigma_n^\alpha(f, \theta)| \leqq A_1 \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(\alpha-1)/2}(g, \theta) - \sigma_\nu^{(\alpha+1)/2}(g, \theta)|^2}{\nu+1} + A_2 \left\{ \sup_{0 \leqq \nu < \infty} |\sigma_\nu^{(\alpha+1)/2}(g, \theta)| \right\}^2,$$

and

$$\int_{-\pi}^{\pi} \left\{ \sup_{0 \leqq n < \infty} |\sigma_n^\alpha(\theta)| \right\}^{2p/2} d\theta \leqq A_1 \int_{-\pi}^{\pi} \left\{ \sum_{\nu=0}^{\infty} \frac{|\sigma_\nu^{(\alpha-1)/2}(g, \theta) - \sigma_\nu^{(\alpha+1)/2}(g, \theta)|^2}{\nu+1} \right\}^{2p/2} d\theta$$
$$+ A_2 \int_{-\pi}^{\pi} \left\{ \sup |\sigma_\nu^{(\alpha+1)/2}(g, \theta)| \right\}^{2p} d\theta$$
$$\leqq A_3 \int_{-\pi}^{\pi} |g(e^{i\theta})|^{2p} d\theta \leqq A_4 \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta,$$

by Theorem 1.

Thus we get Theorem 3, that is

$$\int_{-\pi}^{\pi} \left\{ \sup_{0 \leqq n < \infty} |\sigma_n^\alpha(\theta)| \right\}^p d\theta \leqq C \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta, \qquad 0 < p < 1,$$

where $\alpha > 1/p - 1$.

If $0 < p \leqq 1/2$, then $g(z) \in H^{2p}(0 < 2p \leqq 1)$, and we can apply Theorem 2. Thus we get for $\alpha = 1/p - 1$

$$\int_{-\pi}^{\pi} \left\{ \sup_{0 \leqq n < \infty} |\sigma_n^\alpha(f, \theta)| \right\}^p d\theta \leqq A \int_{-\pi}^{\pi} \left\{ \sum_{\nu=0}^{\infty} \frac{|\sigma_\nu^{(\alpha-1)/2}(g, \theta) - \sigma_\nu^{(\alpha+1)/2}(g, \theta)|^2}{\nu+1} \right\}^{2p/2} d\theta$$
$$+ \int_{-\pi}^{\pi} \left\{ \sup_{0 \leqq n < \infty} |\sigma_\nu^{(\alpha+1)/2}(g, \theta)| \right\}^{2p} d\theta$$
$$\leqq A \int_{-\pi}^{\pi} |g(e^{i\theta})|^{2p} \log^+ |g(e^{i\theta})| d\theta + B$$
$$\leqq A \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \log^+ |f(e^{i\theta})| d\theta + B.$$

Similarly we can get the remaining inequalities.

6. Proof of Theorem 5 and 6. If we write

$$\Psi_\alpha^*(\rho,\theta)=\sum_{n=1}^{\infty}\frac{(A_n^\alpha)^2}{(2n+2\alpha+1)A_{2n}^{2\alpha}(\log n)^{2/p}}|\tau_n^\alpha(\theta)|^2\rho^{2n}$$

then

$$\Psi_\alpha^*(\rho,\theta)\leqq A_1\frac{1}{\rho^{2\alpha}}\int_0^\rho\frac{(\rho-r)^{2\alpha}}{|\log(\rho-r)|^{2/p}}|\tau_n^\alpha(\theta)|^2\rho^{2n},$$

by Lemma 6. Let

$$F(z)=\{f(z)\}^{p/2},\ \text{and}\ \alpha=1/p,\ \text{then}$$

$$\Phi_\alpha(r,\theta)=\sum_{n=1}^{\infty}(A_n^\alpha)^2|\tau_n^\alpha(\theta)|^2r^{2n}$$
$$\leqq A_2\left\{\int_{-\pi}^{\pi}\frac{|F(re^{i\varphi+i\theta})|^2}{|1-re^{i\varphi}|}d\varphi\right\}^{2/p-1}\left\{\int_{-\pi}^{\pi}\left|\frac{rF'(re^{i\varphi+i\theta})}{1-re^{i\varphi}}\right|^{p'}d\varphi\right\}^{2/p'}$$

by Hölder's inequality for $(2-p)/p<1$, where $p'=p/(p-1)$. The last term is majorated by

$$A_2\{I(\theta)\}^{2/p-1}J(\theta).$$

While

$$I(\theta)=\sum_{n=0}^{\infty}|s_n^{-1/2}(F,\theta)|^2r^{2n}$$
$$\leqq A_2\sup_{0<r<1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}|s_n^{-1/2}(F,\theta)|^2r^{2n}|\log(1-r)|$$
$$\leqq A_3|F_2^*(\theta)|^2\log(1-r),$$

by Lemma 4. Hence

$$\Phi_\alpha(r,\theta)\leqq A_4|F_2^*(\theta)|^{2(2/p-1)}\{|\log(1-r)|\}^{2/p-1}\left\{\int_{-\pi}^{\pi}\left|\frac{rF'(re^{i\varphi+i\theta})}{1-re^{i\varphi}}\right|^{p'}d\varphi\right\}^{2/p'}$$
$$\leqq A_5|F_2^*(\theta)|^{2(2/p-1)}|\log(1-r)|^{2/p-1}\sum_{n=1}^{\infty}n^{2/p}|\tau_n^{1/2}(F,\theta)|^2r^{2n},$$

by Lemma 2. Therefore

$$\Psi_\alpha^*(\rho,\theta)\leqq A_6|F_2^*(\theta)|^{(2/p-1)}\left\{\sum_{n=1}^{\infty}n^{2/p}|\tau_n^{1/2}(F,\theta)|^2\rho^{2n}\cdot\int_0^1\frac{(1-r)^{2\alpha}|\log(1-r)|^{2/p-1}}{|\log(1-r)|^{2/p}}r^{2n}\,dr\right\}$$
$$\leqq A_6|F_2^*(\theta)|^{2(2/p-1)}\left\{\sum_{n=1}^{\infty}n^{2/p}|\tau_n^{1/2}(F,\theta)|^2\rho^{2n}\int_0^1\frac{(1-r)^{2\alpha}r^{2n}}{|\log(1-r)|}dr\right\}$$
$$\leqq A_7|F_2^*(\theta)|^{2(2/p-1)}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^{1/2}(F,\theta)|^2}{n\log n}\right\}$$
$$\leqq A_8|F_2^*(\theta)|^{(2/p-1)}\{F_2^*(\theta)\}^2$$

$$\leqq A_9\{F_2^*(\theta)\}^{4/p}$$

by Lemma 4 and 6. Since

$$\int_{-\pi}^{\pi} \{F_2^*(\theta)\}^2\, d\theta \leqq \int_{-\pi}^{\pi} |F(e^{i\theta})|^2\, d\theta,$$

we get

$$\int_{-\pi}^{\pi} \left\{\sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta)|^2}{(n+1)\{\log(n+2)\}^{2/p}}\right\}^{p/2} d\theta \leqq A_{10}\int_{-\pi}^{\pi} |F(e^{i\theta})|^2\, d\theta$$

$$\leqq A_{11}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p\, d\theta,$$

where $\alpha = 1/p$. This is nothing but Theorem 5.

If we put $f(z) = g^2(z)$ and $\alpha = 1/p - 1 > 0$, then

$$g(z) \in H^{2p} \qquad (1 < 2p < 2),$$

and hence we have by Lemma 5,

$$\sigma_n^\alpha(f,\theta) \leqq A_1 \sum_{v=0}^{n} \frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta) - \sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2}{\nu+1}$$

$$+ A_2 \sup_{0\leqq\nu<\infty} |\sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2$$

$$\leqq A_3 (\log n)^{2/2p} \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta) - \sigma_\nu^{(\alpha+1)/2}(g,\theta)|^2}{(\nu+1)\,(\log(\nu+2))^{2/2p}}$$

$$+ A_4 \{\sup_{0\leqq\nu<\infty} |\sigma_\nu^{(\alpha+1)/2}(g,\theta)|\}^2.$$

From the above theorem,

$$\int_{-\pi}^{\pi} \left\{\sup_{0\leqq n<\infty} \left|\frac{\sigma_n^\alpha(\theta)}{\{\log(n+2)^{1/p}}\right|\right\}^p d\theta$$

$$\leqq A_5 \int_{-\pi}^{\pi} \left\{\sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(\alpha-1)/2}(g,\theta) - \sigma_\nu^{(\alpha-1)/2}(g,\theta)|^2}{(\nu+1)\{\log(\nu+2)\}^{1/p}}\right\}^{2p/2} d\theta$$

$$+ A_6 \int_{-\pi}^{\pi} \left\{\sup_{0\leqq n<\infty} |\sigma_n^{(\alpha+1)/2}(g,\theta)|\right\}^{2p} d\theta$$

$$\leqq A_7 \int_{-\pi}^{\pi} |g(e^{i\theta})|^{2p} d\theta \leqq A_7 \int_{-\pi}^{\pi} |f(e^{i\theta})|^p\, d\theta.$$

Thus we get Theorem 6.

LITERATURES

[1] H. C. CHOW, An extension of a theorem of Zygmund and its application, Journ. London Math. Soc., 29(1954), 189-198.

[2] A. E. GWILLIAM, Cesàro means of power series (I), Journ. London Math. Soc., 10(1935), 248-253.

[3] G. H. HARDY and J. E. Littlewood, Some new properties of Fourier constants,

Math. Ann., 97(1927), 159–209.
[4] G. H. HARDY and J. E. Littlewood, Theorems concerning Cesàro means of power series, Proc. London Math. Soc., 36(1934), 516–531.
[5] G. H. HARDY and J. E. Littlewood, The strong summability of Fourier series, Fund. Math., 25(1935), 162–189.
[6] A. ZYGMUND, On the convergence and summability of power series on the circle of convergence (I), Fund. Math., 30(1938), 170–196.
[7] A. ZYGMUND, On the convergence and summability of power series on the circle of convergence (II), Proc. London Math. Soc., 47(1942), 326–350.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.

Added in the proof. There is a slip in Zygmund's paper [7]. See his correction, Bull. Amer. M. S., 51(1945), p. 446. So his conjecture coincides with our Theorem 6. The detailed argument will be given in another paper.