ON ASYMPTOTES IN A METRIC SPACE WITH NON-POSITIVE CURVATURE*

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Introduction. The concept of parallelism for Euclidean straight lines was extended by E. Cartan [1][1)] to that of parallelism for geodesics in a simply connected Riemannian space with non-positive curvature. A further extension was given by H. Busemann [2], [3]. He developed in terms of corays a theory of parallelism for rays in a metric space called by him an E-space (or a G-space) and introduced the concept of asymptotes.

In a Euclidean space a coray from a point p to a ray $\mathfrak{l}$ is identical with the half straight line $\mathfrak{x}$ parallel to $\mathfrak{l}$ from p and the asymptote through p with the straight line which contains $\mathfrak{x}$ as a subray. In an E-space the union of all corays which contain a coray to a ray $\mathfrak{l}$ is called an asymptote. In [3] he defined a G-space with non-positive curvature and showed that, if such a space is simply connected, the concept of asymptotes is symmetric and transitive.

The initial point of an asymptote is called an asymptotic conjugate point if it exists. In the previous paper [7] we showed some properties of the set of asymptotic conjugate points for a ray. H. Busemann [5] studied a G-surface which is a 2-dimensional G-space and homeomorphic to the Euclidean sphere S punctured at finite number of points $a_0, a_1, \ldots, a_k$. In the present paper we deal with asymptotes on such a G-surface $\mathfrak{R}$ which has non-positive curvature. Throughout this paper we denote by $K(\mathfrak{l})$ the set of asymptotic conjugate points for a ray $\mathfrak{l}$. The main results of this paper are summed up as follows:

1. For a ray $\mathfrak{l}$ there exist at least two asymptotes from every point of the set $K(\mathfrak{l})$ and the number of these asymptotes does not exceed $k+1$ [§2, §4].

2. If $\mathfrak{R}$ is a Finsler space of class C^r $(r \geqq 4)$ (or a Riemann space of class C^r $(r \geqq 3)$) and a point p does not belong to the set $K(\mathfrak{l})$, then in a suitable neighborhood of p the limit circle through p is an arc of class C^1 at least [§3].

If any point of p has locally differentiable circles in the sense of H. Busemann [THE GEOMETRY OF GEODESICS, ACAD. PRESS INC.] we have the following 3, 4 and 5.

3. The union of all rays which contain a coray as a subray coincides with an asymptote [§3].

* We wish to thank Prof. H Busemann for his kind advice in the investigation.

1) Numbers in brackets refer to the references cited at the end of the paper.

4. The relation between ray and coray is symmetric and transitive [§3].

5. The set $K(\mathfrak{l})$ consists of finite number of unbounded and continuous curves no two of which has common points. If the set $K(\mathfrak{l})$ has branch points, then the number of these points is finite. At each branch point the number of branch curves is equal to that of asymptotes issuing from this point [§4, §5].

1. In this paragraph we explain some preliminary concepts.

In a metric space the distance between two points x and y will be denoted by $\rho(x,y)$. The axioms for a metric space $\mathfrak{R}$ to be an E-space are follows:

A. $\mathfrak{R}$ is metric with distance $\rho(x,y)$ not necessarily symmetric.

B. $\mathfrak{R}$ is finitely compact.

C. $\mathfrak{R}$ is convex metric.

D. Every point x has a spherical neighborhood $S(x,\alpha(x))$ ($=\{y\,|\,\rho(x,y)<\alpha(x), \rho(y,x)<\alpha(x)\}(\alpha(x)>0)$) such that for any two distinct points a, b in $S(x, \alpha(x))$ and any positive number ε there exist positive numbers $\delta_k(a,b)$ ($k=1,2$) not greater than ε for which a point a_1 with $\rho(a_1,a)+\rho(a,b)=\rho(a_1,b)$ and another point b_1 with $\rho(a,b)+\rho(b,b_1)=\rho(a,b_1)$ exist and are unique.

If the metric is symmetric, then $\mathfrak{R}$ is said a G-space. If further R has dimension 2 in the sense of Menger-Uryson, $\mathfrak{R}$ is said a G-surface and is topologically a connected manifold.

The axioms A, B and C guarantee the existence of a segment $T(p,q)$ from p to q (or $T(q,p)$ from q to p) whose length equals the distance $\rho(p,q)$ (or $\rho(q,p)$). The prolongation of a segment is locally possible and unique under the axiom D. The whole prologation of a segment is said a geodesic.

A geodesic $\mathfrak{G}$ has a parametric representation $x(\tau)$, $-\infty<\tau<+\infty$, such that for any real number τ_0 a positive number $\varepsilon(\tau_0)$ exists such that $\rho(x(\tau_1), x(\tau_2))=\tau_2-\tau_1(\tau_2\geqq\tau_1)$ for $|\tau_i-\tau_0|\leqq\varepsilon(\tau_0)$ ($i=1,2$). If for any two real numbers τ_1 and $\tau_2(\tau_2\geqq\tau_1)$ $\rho(x(\tau_1), x(\tau_2))=\tau_2-\tau_1$, then we say $\mathfrak{G}$ is a straight line. A half straight line is said a ray.

In [2; §4] the number $\eta_\lambda(x)$ and the term "direction" were introduced. The number $\eta_\lambda(x)$ is defined as the least upper bound of those β's for which any segment with end points in $S(x,\beta)$ is a cocentral subsegment of length $\lambda\beta$. $\eta_\lambda(x)$ is positive for any point x and any number λ not less than 2. The number η_x is defined as min $(\eta_6(x), 1)$. A segment $T(a,b)$ with the length η_a is said a direction.

Let D_1 and D_2 be two directions. Let a_1 and a_2 be the initial points of D_1 and D_2 and b_1 and b_2 the end points of D_1 and D_2. Following H. Busemann [2; §7], the distance of D_1 and D_2 is defined as

$$\zeta(D_1,D_2)=\frac{1}{2}(\rho(a_1,a_2)+\rho(b_1,b_2)).$$

The set of all directions on $\mathfrak{R}$ is finitely compact under the above metric. The distance of two half geodesics (or two geodesic subarcs) is defined as that of their initial directions.

Let $x(\tau)$, $[0\leqq\tau<+\infty$, be a parametric representation of a ray $\mathfrak{l}$, and let

$\{p_n\}$ be a sequence of points which converges to a point p and $\{\tau_n\}$ a sequence of positive numbers which diverges to infinity. Then the sequence of segments $\{T(p_n, x(\tau_n))\}$ always contains a subsequence $\{T(p_{n_i}, x(\tau_{n_i}))\}$ which converges to a ray $\mathfrak{x}$ under the above metric [2; §11]. This limit is equivalent to the closed limit introduced by Hausdorff [2; §7]. We denote this by $Fl_{n_i\to+\infty} T(p_{n_i}, x(\tau_{n_i})) = \mathfrak{x}$. $\mathfrak{x}$ is said a coray from p to $\mathfrak{l}$. The union $\mathfrak{U}$ of all corays which contains $\mathfrak{x}$ as a subray is said an asymptote to $\mathfrak{l}$. If $\mathfrak{U}$ is not a straight line, its initial point exists and is called the asymptotic conjugate point of $\mathfrak{U}$.

From the result of H. Busemann [2; (11.6)] we deduce:

(1.1) Let a ray $\mathfrak{l}$ be given. If a point p does not belong to the set $K(\mathfrak{l})$, then there exists only one coray from p to $\mathfrak{l}$.

If $\mathfrak{R}$ is a G-space and every point p has a spherical neighborhood $S(p, \varepsilon_p)$ such that a side be of a geodesic triangle abc in $S(p, \varepsilon_p)$ is at least twice as long as the segment connecting midpoints b' and c' of the others, i.e., $\rho(b, c) \geqq 2\rho(b', c')$, then $\mathfrak{R}$ is said to be with non-positive curvature [3][2].

If $\mathfrak{R}$ is a G-space with non-positive curvature and simply connected, then all geodesics are straight lines. From this and the result of H. Busemann [2: (11.9)] we immediately see that

(1.2) Under the above condition for any $\mathfrak{l}$ the set $K(\mathfrak{l})$ is vacuous.

We further see from his results [3; (4.3), (4.4)] that the following (1.3) and (1, 4) hold.

(1.3) Under the same condition let $x_1(\tau)$, $0 \leqq \tau \leqq +\infty$, and $x_2(\tau)$, $0 \leqq \tau < +\infty$ be parametric representations of rays $\mathfrak{x}_1$ and $\mathfrak{x}_2$ respectively. The following two conditions are equivalent and necessary and sufficient for these rays to be corays each other.

(a) $\rho(x_1(\tau), x_2(\tau)) < +\infty$ for $0 \leqq \tau < +\infty$.

(b) $\rho(x_1(\tau), \mathfrak{x}_2)$[3] $< +\infty$ (or $\rho(x_2(\tau), \mathfrak{x}_1) < +\infty$) for $0 \leqq \tau < +\infty$.

(1.4) Under the same condition, the relation between ray and coray is symmetric and transive.

In what follows we restrict ourselves to the case where $\mathfrak{R}$ is a G-surface with non-positive curvature which is homeomorphic to the sphere S punctured at the finite number of points a, a_1, ..., a_k. Suppose that C_0, C_1, ..., C_k are simple closed geodesic polygons on $\mathfrak{R}$, no two of which have common point, such that each C_i bounds a tube U_i homeomorphic to a circular disk punctured at center. Then $\mathfrak{R} - \bigcup_{i=0}^k U_i$ is a bounded open set [4]. A tube U_i is said non-expanding or expanding according as any minimal sequence of closed curves homotopic to the boundary C_i is bounded or not.

Let $\widetilde{\mathfrak{R}}$ be a universal covering G-surface of $\mathfrak{R}$ and Φ a covering trans-

2) A Riemann space of class $C^r (r \geqq 4)$ is with non-positive curvature in the usual sense, if and only if this condition is fulfilled [3].

3) The distance between a point p and a set E is defined as $\inf_{x\in E} \rho(p, x)$ and denoted by $\rho(p, E)$.

formation of $\widetilde{\Re}$ onto $\Re$. The image $\widetilde{A}\Phi$ of a set $\widetilde{A}$ on $\widetilde{\Re}$ will be called A and $\widetilde{A}$ is said to lie over A. For every point p on $\Re$ there exists a positive number α_p such that under Φ an open circular disk $S(\widetilde{p}, \alpha_p)$ is isometrically mapped onto $S(p, \alpha_p)$. For a ray $\mathfrak{l}$ there exists only one tube which contains $\mathfrak{l}$ or a subray of $\mathfrak{l}$. To fix the ideas, in what follows this tube will be denoted by U_0. By replacing the ray $\mathfrak{l}$ by its suitable subray or the boundary C_0 by another suitable closed geodesic polygon homotopic to it, if necessary, we suppose that the initial point r of $\mathfrak{l}$ coincides with a vertex of C_0 but $\mathfrak{l}$ has no other common point with C_0. For a point $\widetilde{r}$ there exist only one ray $\widetilde{\mathfrak{l}}$ and only one simple geodesic polygon $\widetilde{C}_0$ with the common initial point $\widetilde{r}$ which lie over $\mathfrak{l}$ and C_0 respectively. The other end point $\widetilde{r}'$ of $\widetilde{C}_0$ lies over the point r. Hence there exists only one ray $\widetilde{\mathfrak{l}}'$ with the initial point $\widetilde{r}'$ which lies over $\mathfrak{l}$. Under Φ a part $\widetilde{U}_0$ of $\widetilde{\Re}$ bounded by $\widetilde{\mathfrak{l}}$, $\widetilde{\mathfrak{l}}'$ and $\widetilde{C}_0$ are mapped onto the tube U_0, and the contraction $\Phi|(\widetilde{U}_0 - \widetilde{\mathfrak{l}} - \widetilde{\mathfrak{l}}')$ of Φ is univalent. From (1.3) it is easy to see that, if and only if the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ are not corays each other, the tube U_0 is expanding.

2. We begin by showing the following important theorem.

(2.1) THEOREM. *Let $\mathfrak{l}$ be a ray on $\Re$. Then the number of corays from any point p to $\mathfrak{l}$ is finite and does not exceed $k+1$.*

PROOF. We prove only the first part. The second part will be proved in [§4].

By choosing suitably the closed polygon C_0, if necessary, we further suppose that the point p does not belong to the tube U_0. Let ε be a positive number and r the initial point of the ray $\mathfrak{l}$. Further let l_0 denote the length of the closed geodesic polygon C_0 and m_0 l. u. b.$_{x \in C_0}$ $\rho(p, x)$. We show at first that the number of the homotopic classes containing the closed geodesic polygons

$$T(p, r) + T(r, q) + T(q, x) + T(x, p),$$

where x is any point of $S(p, \varepsilon)$ and q any point on $\mathfrak{l}$ such that $\rho(p, q) > l_0 + m_0 + \varepsilon$, is finite.

To do this, suppose that this is false. Since $\Re$ is with non-positive curvature, there exists a unique geodesic arc L from p to q homotopic to the geodesic polygon $T(x, p)^{-1} + T(q, x)^{-1}$, and L is continuously deformable to a geodesic arc M which connects p to r when q varies on $\mathfrak{l}$ from q to r. Let $\widetilde{p}$ be a point on $\widetilde{\Re}$ which lies over the point p and $\widetilde{r}$ the end point of the segment $\widetilde{M}$ issuing from $\widetilde{p}$ which lies over M. The number of such points $\widetilde{r}$ contained in $S(\widetilde{p}, l_0 + m_0 + \varepsilon)$ is finite. For if this is not so, in virtue of the finite compactness of $\widetilde{\Re}$, the set of such points $\widetilde{r}$ has an accumulation point. But this contradicts that $\widetilde{\Re}$ is a universal covering G-surface of $\Re$. From this we see that there exist in $S(p, \varepsilon)$ a point $\widetilde{r}$ and on $\mathfrak{l}$ a point q $(\rho(p, q) > l_0 + m_0 +$

ε) such that the point $\tilde{r}$ obtained in the above way does not belong to $S(p, l_0 + m_0 + \varepsilon)$. Next we show that this contradicts to the above supposition.

On $\tilde{\mathfrak{R}}$ let $\tilde{\mathfrak{l}}$ be the ray issuing from $\tilde{r}$ which lies over the ray $\mathfrak{l}$. Then the end point q of the geodesic polygon $T(x,p)^{-1} + T(q,x)^{-1}$, is on the ray $\mathfrak{l}$. Let $\tilde{y}$ be the point on the segment $T(\tilde{x},\tilde{q})^{-1}$ such that $\rho(\tilde{x},\tilde{y}) = m_0 + \varepsilon$. Further let the segment $T(x,q)^{-1}$ intersect at first the closed geodesic polygon C_0 at a point s. Then there exist on $T(\tilde{x},\tilde{y})$ the point $\tilde{s}$ which lies over s and on $\tilde{\mathfrak{R}}$ the geodesic polygon $\tilde{C}_0'$ which lies over C_0 and contains the point $\tilde{s}$. We denote by $\tilde{r}'$ the initial point of $\tilde{C}_0'$ and the subarc C_0'' of C_0 from r to s. Let $\tilde{\mathfrak{l}}'$ be the ray issuing from $\tilde{r}'$ which lies over $\mathfrak{l}$ and, suppose that the subarc $\tilde{C}_0''$ of $\tilde{C}_0'$ from $\tilde{r}'$ to $\tilde{s}$ lies over C_0' and $\tilde{q}'$ is the point on $\tilde{\mathfrak{l}}'$ which lies over the point q. It is not hard to see that the closed curve $C_0''^{-1} + T(r,q) + T(q,s)$ is continuously contractible to a point. Hence the segment $T(\tilde{q}',\tilde{s})$ lies over the segment $T(q,s)$ and there exists on the prolongation of $T(\tilde{q}',\tilde{s})$ the point $\tilde{x}'$ such that $\rho(\tilde{q}',\tilde{x}') = \rho(q,x)$ and $T(\tilde{q}',\tilde{x}')$ lies over $T(q,x)$. Under Φ the points $\tilde{x}$ and $\tilde{x}'$ are mapped onto the point x.

On the other hand the rays $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{l}}'$ are distinct since $\tilde{r}' \in S(p, l_0 + m_0 + \varepsilon)$ but $\tilde{r} \notin S(p, l_0 + m_0 + \varepsilon)$. Hence the segments $T(\tilde{s},\tilde{x})\Phi$ and $T(\tilde{s},\tilde{x}')\Phi$ are distinct, but this contradicts that $T(\tilde{q},\tilde{x})\Phi$ and $T(\tilde{q}',\tilde{x}')\Phi$ are segments. For the segments $T(\tilde{s},\tilde{x})\Phi$ and $T(\tilde{s},\tilde{x}')\Phi$ are proper subsegments of $T(\tilde{q},\tilde{x})\Phi$ and $T(\tilde{q}',\tilde{x}')\Phi$ respectively. Thus we end the proof of the above. By use of this the first part is proved as follows:

Let $\{p_n\}$ be a sequence of points which converges to the point p and $\{q_n\}$ the sequence of points on $\mathfrak{l}$ which diverges to infinity. Without loss of generality we suppose that each p_n is contained in $S(p,\varepsilon)$ and the distance from p to each q_n is greater than $l_0 + m_0 + \varepsilon$. Let us denote by $\Gamma_0, \Gamma_1, \ldots.$ Γ_K the homotopy classes which contain the closed geodesic polygons L_n:

$$T(p,r) + T(r,q_n) + T(q_n,p_n) + T(p_n,p) \ (n = 1, 2, \ldots.).$$

For each $\Gamma_i (0 \leqq i \leqq K)$ there exists a geodesic arc T_i, which connects p to r, such that the closed geodesic polygon $T(p,r) + T_i^{-1}$ is contained in the homotopy class Γ_i.

Let $\tilde{T}_i$ be the segment issuing from p which lies over each T_i. The end point $\tilde{r}_i$ of each segment $\tilde{T}_i$ lies over the initial point r of $\mathfrak{l}$. From the above proof we see that the points r_i $(0 \leqq i \leqq K)$ are contained in $S(p, l_0 + m_0 + \varepsilon)$. Let $\tilde{\mathfrak{l}}_i$ be the ray issuing from each $\tilde{r}_i$ which lies over the ray $\mathfrak{l}$. Then it is easy to see that the end point of the geodesic polygon, which lies over each $T(p_n,p)^{-1} + T(q_n,p_n)^{-1}$, is on one of the rays $\tilde{\mathfrak{l}}_0, \tilde{\mathfrak{l}}_1, \ldots.$ and $\tilde{\mathfrak{l}}_K$. Let $\{T(p_{n_i}, q_{n_i})\}$ be any subsequence of $\{T(p_n,q_n)\}$ which converges to a coray $\mathfrak{x}$ from p to $\mathfrak{l}$. From the above it is easy to see that the ray $\mathfrak{x}$ is the image of the coray $\widetilde{\mathfrak{x}}(=Fl_{n_i\to\infty}\,T(\widetilde{p}_{n_i},\widetilde{q}_{n_i}))$ from $\widetilde{p}$ to one of the rays $\widetilde{\mathfrak{l}}_0, \widetilde{\mathfrak{l}}_1, \ldots$ and $\widetilde{\mathfrak{l}}_K$ under Φ, which proves the first part.

From the above proof we have the following

(2.2) *Suppose that for a ray* $\mathfrak{l}$ *there exist* m *corays* $\mathfrak{U}_1, \mathfrak{U}_2, \ldots \mathfrak{U}_m$ *from* p, *and let* $\widetilde{\mathfrak{U}}_j$ *be the coray from a poiut* $\widetilde{p}$ *to a ray* $\widetilde{\mathfrak{l}}_{i_j}$ $(\widetilde{\mathfrak{l}}_{i_j}\Phi=\mathfrak{l})$ *and lie over each* $\mathfrak{U}_j$. *Then there exists a positive number* $\beta_p(<\alpha_p)$ *such that a coray from every point* x *of* $S(p,\beta_p)$ *to* $\mathfrak{l}$ *coincides with the image of the coray from the point* $\widetilde{x}(\in S(\widetilde{p},\beta_p))$ *to one of the rays* $\widetilde{\mathfrak{l}}_{i_1}, \widetilde{\mathfrak{l}}_{i_2}, \ldots, \mathfrak{l}_{i_m}$ *under* Φ.

If the proposition does not hold, there exists a sequence of points $\{p_n\}$, which converges to p, such that a coray $\mathfrak{x}_n$ from each p_n to $\mathfrak{l}$ does not coincide with the image of the coray from $\widetilde{p}_n$ to any of the rays $\widetilde{\mathfrak{l}}_{i_1}, \widetilde{\mathfrak{l}}_{i_2}, \ldots,$ $\widetilde{\mathfrak{l}}_{i_m}$. Then the sequence of the rays $\{\mathfrak{x}_n\}$ contains a subsequence which converges to a coray $\mathfrak{x}$ from p to $\mathfrak{l}$, since the set of corays to $\mathfrak{l}$ forms a closed subset of all half geodesics of $\mathfrak{R}$ under the metric ζ [2; §9]. Then the ray $\mathfrak{x}$ issuing from $\widetilde{p}$ which lies over $\mathfrak{x}$ is not a coray from $\widetilde{p}$ to any of the rays $\widetilde{\mathfrak{l}}_{i_1}, \widetilde{\mathfrak{l}}_{i_2}, \ldots$ $\widetilde{\mathfrak{l}}_{i_m}$. Thus we arrive at a contradiction.

(2.3) THEOREM. *For a ray* $\mathfrak{l}$ *there exist at least two asymptotes from every point* p *of the set* $K(\mathfrak{l})$ *to* $\mathfrak{l}$.

PROOF. Let $\mathfrak{U}$ be an asymptote from p to $\mathfrak{l}$ and $\mathfrak{G}$ the geodesic which contains $\mathfrak{U}$ as a subray. Let $\{p_n\}$ be a sequence of points, which converges to p, such that each p_n lies on $\mathfrak{G}$ but not on $\mathfrak{U}$. By virtue of the definition of asymptotes, a coray $\mathfrak{x}_n$ from each p_n to $\mathfrak{l}$ is disjoint from $\mathfrak{U}$.

Suppose that there exists only one asymptote $\mathfrak{U}$ from p to $\mathfrak{l}$. Then there exists on $\widetilde{\mathfrak{R}}$ a point $\widetilde{p}$ and a ray $\widetilde{\mathfrak{l}}$, which lie over p and $\mathfrak{l}$ respectively, such that a coray from every point x $(\in S(p,\beta_p))$ to $\mathfrak{l}$ coincides with the image of the coray from $\widetilde{x}$ $(\in S(\widetilde{p},\beta_p))$ to $\widetilde{\mathfrak{l}}$ under Φ. Take a positive integer N so large that

$$S(p,\beta_p)\ni p_n \text{ for every } n\geqq N.$$

Then the ray $\widetilde{\mathfrak{x}}_n$ which lies over each $\mathfrak{x}_n$ and issues from p_n $(n\geqq N)$ is a coray to $\widetilde{\mathfrak{l}}$. Hence each $\widetilde{\mathfrak{x}}_n$ contains a ray $\widetilde{\mathfrak{U}}$ which lies over $\mathfrak{U}$. From this it follows that each $\mathfrak{x}_n$ contains $\mathfrak{U}$ as a subray, which contradicts to the fact mentioned above. Thus the theorem is proved.

(2.4) THEOREM. *For a ray* $\mathfrak{l}$ *the set* $K(\mathfrak{l})$ *is closed.*

PROOF. Let p be any point which does not belong to the set $K(\mathfrak{l})$. Then there exists on $\widetilde{\mathfrak{R}}$ a point $\widetilde{p}$ and a ray $\widetilde{\mathfrak{l}}$ as described in the proof of (2.3). Let $\widetilde{\mathfrak{l}}'$ be any ray which lies over $\widetilde{\mathfrak{l}}$ and is not a coray to $\widetilde{\mathfrak{l}}$. Then the coray $\widetilde{\mathfrak{x}}$ from every point $\widetilde{x}$ of $S(\widetilde{p},\beta_p)$ to $\widetilde{\mathfrak{l}}'$ does not lie over a coray from x to $\mathfrak{l}$. For the coray from every point of $\widetilde{\mathfrak{x}}\cap S(\widetilde{p},\beta_p)$ to $\widetilde{\mathfrak{l}}$ lies over a coray to $\mathfrak{l}$. From this we see $S(p,\beta_p)\cap K(\mathfrak{l})=\phi$. Thus the theorem is proved.

(2.5) THEOREM. *Let* $\mathfrak{l}$ *be a ray on* $\mathfrak{R}$. *Then the set* $K(\mathfrak{l})$ *does not contain an isolated point.*

PROOF. Suppose that the set $K(\mathfrak{l})$ contains an isolated point p. The initial point of every asymptote coincides with the point p and the set $K(\mathfrak{l})$ consists of only one point p [7]. Hence $\mathfrak{R}$ is simply covered by the system of asymptotes issuing from p. From this it follows that $\mathfrak{R}$ is simply connected and the set $K(\mathfrak{l})$ is vacuous, which contradicts to the asssmption. Thus the theorem is proved.

As we see in the above, Theorems (2.3), (2.4) and (2.5) follow from the first part of (2.1) and (2.2). In this paper Theorem (2.1) and Proposition (2.2) play an important role. Next we show by the following example that the finite connectivity of $\mathfrak{R}$ is essential for (2.1) and (2.2).

EXAMPLE 1°. In a 3-dimentional Euclidean space referred to the rectangular coordinate system (x, y, z), consider the sequence of points $\{p_{\pm n}\}$ $(n = \pm 1, \pm 2, \ldots.)$ where for each n the point p_{+n} and p_{-n} are given by $(2^n, 1, 0)$ and $(-2^n, 1, 0)$ respectively, and replace the circular disks with the centers $P_{\pm n}$ and the radii $1/4$ by the half cylinders

$$Z_{+n}:\ (x-2^n)^2+(y-1)^2=1/16,\ \ 0 \leqq z < +\infty, \text{ and}$$

$$Z_{-n}:\ (x+2^n)^2+(y-1)^2=1/16,\ \ 0 \leqq z < +\infty \qquad (n = 1, 2, \ldots.).$$

Then we have a surface S' instead of the xy-plane. To smooth the joint parts we use an arc C of the algebraic curve represented by rectangular coordinate system (ξ, η) as follows:

$$(\xi - 1/4)^5 + (\eta - 1/4)^5 + 1/4^5 = 0.$$

As shown in Figure [1], for the joint part of each Z_{+n}(or Z_{-n}) we smooth the section by every plane through the axis. Thus we have the surface S

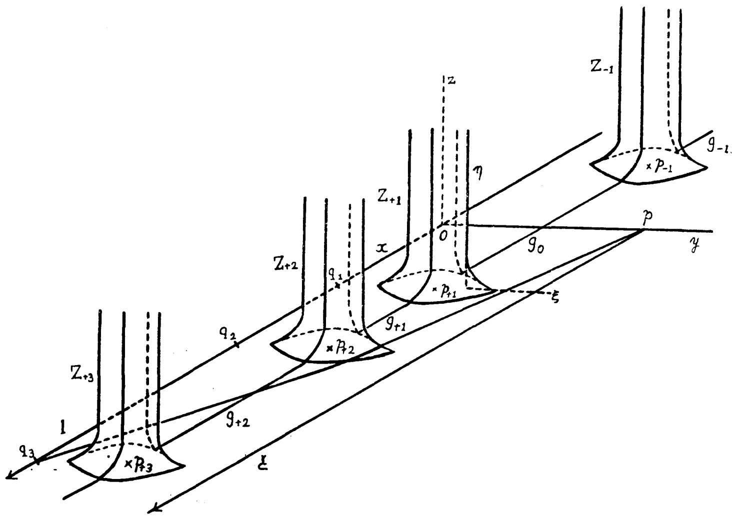

Fig. 1

replaced the circular disks with the centers $p_{\pm n}$ and the radii 1/2 by the surfaces of revolution.

Obviously S is a differentiable surface of the fourth order and with nonpositive curvature. Let $\mathfrak{l}$ be the half straight line

$$0 \leqq x < +\infty, \ y = 0, z = 0$$

and $\mathfrak{g}_0$ and $\mathfrak{g}_{+n}$ (or $\mathfrak{g}_{-n}$) the geodesics which contain the segments:

$$-3/2 \leqq x \leqq 3/2, \ y = 1, z = 0 \text{ and}$$

$$2^n + 1/2 \leqq x \leqq 2^{n+1} - 1/2, \ y = 1, \ z = 0$$

$$(\text{or } -2^{n+1} + 1/2 \leqq x \leqq -2^n - 1/2, \ y = 1, \ z = 0) \ (n = 1, 2, \ldots.).$$

Further let $\{q_n\}$ be the sequence of the points where each q_n is given by $(2^n, 0, 0)$. It is easy to see that for $n \geqq 3$ there exists a unique segment from p ($= (0, 2, 0)$) to q_n which intersects the geodesic $\mathfrak{g}_{n-1}$, Obviously $Fl_{n \to +\infty} T(p, q_n)$ coincides with the half straight line $\mathfrak{r}$:

$$0 \leqq x < +\infty, \ y = 2, \ z = 0.$$

Let $\widetilde{S}$ be a universal covering surface of S and $\widetilde{\mathfrak{r}}$ a ray which lies over the ray $\mathfrak{r}$. Next we show that the ray $\widetilde{\mathfrak{r}}$ is not a coray to any of the rays which lie over the ray $\mathfrak{l}$.

Suppose that $\widetilde{\mathfrak{r}}$ is a coray to a ray $\widetilde{\mathfrak{l}}$ which lies over the ray $\mathfrak{l}$. We denote by $\widetilde{p}$ and $\widetilde{r}$ the initial points of $\widetilde{\mathfrak{r}}$ and $\widetilde{\mathfrak{l}}$ respectively. Since all geodesics on $\widetilde{S}$ are straight lines, for any sequence of points $\{\widetilde{\mathfrak{r}}_n\}$ on $\widetilde{\mathfrak{l}}$ which diverges to infinity $Fl_{\ n \to +\infty} T(\widetilde{p}, \widetilde{r}_n)$ coincids with the ray $\widetilde{\mathfrak{r}}$. Under a covering transformation Φ the closed geodesic polygons

$$T(\widetilde{p}, \widetilde{r}) + T(\widetilde{r}, \widetilde{r}_n) + T(\widetilde{r}_n, \widetilde{p}) \qquad (n = 1, 2, \ldots.)$$

are mapped onto a system of closed curves any two of which are homotopic.

Under Φ each $T(\widetilde{p}, \widetilde{r}_n)$ is mapped onto a geodesic arc from p to the point r_n. Let s_n be the first common point of each $T(\widetilde{p}, \widetilde{r}_n)$ with the system of the geodesics $\mathfrak{g}_0$ and $\mathfrak{g}_{\pm n}(n = 1, 2, \ldots.)$. Obviously all points s_n lie on one of these geodesics. For a positive number $\lambda(\leqq 1)$ let $M_{n,\lambda}$ be the geodesic arc where the length λ is laid off each $T(\widetilde{p}, \widetilde{r}_n)\Phi$ from p. Then it follows that $Fl_{n \to +\infty} M_{n,\lambda}$ coincides with the segment:

$$0 \leqq x \leqq \lambda, \ y = 2, \ z = 0.$$

But this is impossible, since the subarc of $T(\widetilde{p}, \widetilde{r}_n)\Phi$ from p to each s_n is a segment.

3. In this paragraph, we deal with the relation between ray and coray. To do this, we begin by showing some properties of the limit circles with respect to a ray $\mathfrak{l}$. Let $x(\tau)$, $0 \leqq x < +\infty$, be a parametric representation of the ray $\mathfrak{l}$. In [2] the function $\alpha(p, \mathfrak{l})$ was defined as follows:

$$\alpha(p, \mathfrak{l}) = \lim_{\tau \to +\infty} (\rho(p, x(\tau)) - \tau).$$

This limit exists for every point p, since $\rho(p, x(\tau)) - \tau$ is a bounded and non-increasing function of τ. By making use of this function the limit circle $L(p, \mathfrak{l})$

through a point p is defined as the set of points x which satisfy the relation

(3.1) $$\alpha(p, \mathfrak{l}) = \alpha(x, \mathfrak{l}).$$

As can be easily seen from the definition $|\alpha(p, \mathfrak{l}) - \alpha(q, \mathfrak{l})| \leqq \rho(p, q)$ holds for any two points p and q. Hence $\alpha(p, \mathfrak{l})$ is a continuous function of a point p. From this it follows that limit circles are closed sets. The following (i) and (ii) were proved by H. Busemann [2; §10, §11].

(i) If a limit circle $L(p, \mathfrak{l})$ intersects the ray $\mathfrak{l}$ at a point $x(\tau_0)$, then the closed limit of the circles $K(x(\tau), \tau - \tau_0)$ $(= \{x | \rho(x(\tau), x) = \tau - \tau_0(\tau > \tau_0))$ as $\tau \to +\infty$ coincides with the limit circle $L(p, \mathfrak{l})$.

(ii) A half geodesic $\mathfrak{x}$ is a coray to the ray $\mathfrak{l}$, if and only if its parametric representation $y(\tau), 0 \leqq \tau < +\infty$, fulfills the following relation:

$$\alpha(y(\tau_1), \mathfrak{l}) - \alpha((y)(\tau_2), \mathfrak{l}) = \tau_2 - \tau_1 \text{ for } \tau_1, \tau_2 \geqq 0.$$

From (ii) we see that, if $\mathfrak{U}$ is an asymptote to the ray $\mathfrak{l}$ and a point a is on $\mathfrak{U}$ but not the initial point of $\mathfrak{U}$, then any point z of $\mathfrak{U}$ is the unique foot of a on $L(z, \mathfrak{l})$. We further see from (2.3) that if a is the initial point of $\mathfrak{U}$ then any point z of $\mathfrak{U} - a$ is a foot of a on $L(z, \mathfrak{l})$ but not unique. For a limit circle $L(p, \mathfrak{l})$ the set $E_1(= \{x | \alpha(p, \mathfrak{l}) > \alpha(x, \mathfrak{l})\})$ is said the interior of $L(p, \mathfrak{l})$ and the set E_2 $(= \{x | \alpha(p, \mathfrak{l}) < \alpha(x, \mathfrak{l})\})$ the exterior of $L(p, \mathfrak{l})$ [2; §10].

Generally in an E-space the limit spheres with respect to a ray simply cover the whole space. If a G-space with non-positive curvature is simply connected, all geodesics are straight lines and the interior of any sphere is convex. Hence the interior of a limit sphere $L(p, \mathfrak{l})$ is convex. For if $x(\tau)$, $-\infty < \tau < +\infty$, represents the geodesic carrying $\mathfrak{l}$, say $p = x(\tau_0)$, then the interior of $L(p, \mathfrak{l})$ is the union of the spheres $S(x(\tau_\nu), \tau_\nu - \tau_0)$ where $\tau_0 < \tau_1 < \tau_2 < \dots < \tau_n \to +\infty$.

If a G-space with negative curvature is simply connected and of finite dimension, then the interior of a limit sphere $L(p, \mathfrak{l})$ is strictly convex. For suppose that a segment $T(a, b)$ lies on $L(p, \mathfrak{l})$, and x' on the asymptote to $\mathfrak{l}$ through a variable point $x (\in T(a, b))$ lay off from x in the interior of $L(p, \mathfrak{l})$ a fixed distance $\alpha > 0$ obtaining a curve $w(x')$. If the space is two dimensional, the curve $w(x')$ is strictly convex turning its concave side toward $T(a, b)$. On the other hand $w(x') \in L(w(a'), \mathfrak{l})$ because $L(p, \mathfrak{l})$ and $L(w(a'), \mathfrak{l})$ is equidistant at distance α. But $L(w(a'), \mathfrak{l})$ is convex and turns its concavity away from the segment $T(a, b)$ which is impossible [3; §4]. This can be extended to higher dimensions.

(3.2) THEOREM. *If $\mathfrak{R}$ is a Finsler space of class C^r $(r \geqq 4)$ (or a Riemann space of class C^r $(r \geqq 3)$ and simply connected, then any limit circle is an arc of class C^1 at least.*

PROOF. Let $x(\tau)$, $-\infty < \tau < +\infty$, represents the geodesic $\mathfrak{G}$ carrying a ray $\mathfrak{l}$. A limit circle $L(x(\tau_0), \mathfrak{l})$ is a convex curve turning its concavity toward with $\tau(> \tau_0)$. As such, it has one sided geodesic tangent. Consider any point $s \in L(x(\tau_0), \mathfrak{l})$ and on the asymptote $\mathfrak{U}$ through s a point x in the interior of the limit circle $L(x(\tau_0), \mathfrak{l})$. The circle $K(x, \rho(x, s))$ is convex. If it has a unique supporting line or tangent at s, then $L(x(\tau_0), \mathfrak{l})$ can have only one supporting line at s (because it touches $K(x, \rho(x, s))$) and $K(x, \rho(x, s))$ lies on the side of $L(x(\tau_0), \mathfrak{l})$ toward which it is concave. On $\mathfrak{R}$ small circles are differentiable both in the sense of the local coordinates and geodesically. Hence $K(x, \rho(x, s))$ has for small $\rho(x, s)$ a unique supporting line at x. In virtue of the concavity of $L(x(\tau_0), \mathfrak{l})$, we can see that the limit circle $L(x(\tau_0), \mathfrak{l})$ is of class C^1 at least.

The above argument can clearly be extended to higher dimensions.

(3.3) THEOREM. *Let* $\mathfrak{l}$ *be a Finsler space of class* C^r $(r \geqq 4)$ *(or a Riemann space of class* C^r $(r \geqq 3)$*). Let a ray* $\mathfrak{l}$ *be given and* p *any point on* $\mathfrak{R}$ *which does not belong to the set* $K(\mathfrak{l})$. *In* $S(p, \beta_p)$ *the limit circle* $L(p, \mathfrak{l})$ *is an arc of class* C^1 *at least.*

PROOF. Let $\mathfrak{y}$ be a coray from p to $\mathfrak{l}$ and $\widetilde{\mathfrak{y}}$ a ray on $\widetilde{\mathfrak{R}}$ which lies over $\mathfrak{y}$. Then $\widetilde{\mathfrak{y}}$ is a coray to a ray $\widetilde{\mathfrak{l}}$ which lies over $\mathfrak{l}$. We denote by $\widetilde{p}$ and $\widetilde{r}$ the initial points of $\widetilde{\mathfrak{y}}$ and $\widetilde{\mathfrak{l}}$ respectively. By (2.3) a coray from every point x of $S(p, \beta_p)$ to $\mathfrak{l}$ coincides with the image of the coray from the point $\widetilde{x}$ of $S(\widetilde{p}, \beta_p)$ to $\widetilde{\mathfrak{l}}$ under Φ.

Let $\{q_n\}$ be a sequence of points on $\mathfrak{l}$ such that $\lim_{n\to+\infty}\rho(r, q_n) = +\infty$. Then for a point x of $S(p, \beta_p)$ and a sequence of segment $\{T(x, q_n)\}$, there exists a positive integer N such that the end points $\widetilde{q}_n$ of the segments $T(\widetilde{x}, \widetilde{q}_n)$ $(x \in S(\widetilde{p}, \beta_p))$, which lie over the segments $T(x, q_n)$ $(n \geqq N)$, are on the ray $\widetilde{\mathfrak{l}}$ or other rays (finite in number) which are corays to $\widetilde{\mathfrak{l}}$ and lie over $\widetilde{\mathfrak{l}}$. The closed limit of the sequence of the segments $\{T(\widetilde{x}, \widetilde{q}_n)\}$ coincides with the coray $\widetilde{\mathfrak{y}}$ from $\widetilde{x}$ to $\widetilde{\mathfrak{l}}$, and $\widetilde{\mathfrak{y}}$ lies over a coray $\mathfrak{y}$ from x to $\mathfrak{l}$. Obviously

$$\rho(\widetilde{x}, \widetilde{q}_n) = \rho(x, q_n) \qquad \text{for each } n.$$

Hence, by virtue of the definition of the function $\alpha(x, \mathfrak{l})$, we have

$$\alpha(x, \mathfrak{l}) = \alpha(\widetilde{x}, \widetilde{\mathfrak{l}})$$

for every point x of $S(p, \beta_p)$. From this it follows that under Φ the subarc of the limit circle $L(\widetilde{p}, \widetilde{\mathfrak{l}})$ in $S(\widetilde{p}, \beta_p)$ is isometrically mapped onto $S(p, \beta_p) \cap L(p, \mathfrak{l})$. By (3.2) the arc $S(\widetilde{p}, \beta_p) \cap L(\widetilde{p}, \widetilde{\mathfrak{l}})$ is of class C^1, and hence $S(p, \beta_p) \cap L(p, \mathfrak{l})$ is also of class C^1. Thus the theorem is proved.

In the rest of the paper we assume that every point p of $\mathfrak{R}$ has locally differentiable circles $K(p, \tau)$ with p as center $(0 < \tau < \min(\eta_p, \varepsilon_p))$, i.e., the circles $K(p, \tau)$ have unique supporting direction at every point.

(3.4) *Let a ray* $\mathfrak{l}$ *be given and* $\mathfrak{U}_1$ *and* $\mathfrak{U}_2$ *two distinct asymptotes from a point* p *of the set* $K(\mathfrak{l})$ *to* $\mathfrak{l}$. *Then, for a ray* $\widetilde{\mathfrak{l}}$ *on* $\widetilde{\mathfrak{R}}$ *which lies over* $\mathfrak{l}$, *there exist on* $\widetilde{\mathfrak{R}}$ *two corays* $\widetilde{\mathfrak{U}}_1$ *and* $\widetilde{\mathfrak{U}}_2$ *to* $\widetilde{\mathfrak{l}}$, *which lie over* $\mathfrak{U}_1$ *and* $\mathfrak{U}_2$ *respectively and the initial points of* $\widetilde{\mathfrak{U}}_1$ *and* $\widetilde{\mathfrak{U}}_2$ *lie on a limit circle with respect to the ray* $\widetilde{\mathfrak{l}}$.

PROOF. Let $\widetilde{p}_1$ be a point on $\widetilde{\mathfrak{R}}$ which lies over p and $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2'$ the rays issuing from $\widetilde{p}_1$ which lie over the rays $\mathfrak{U}_1$ and $\mathfrak{U}_2$ respectively. Then $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2'$ are corays to rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ respectively which lie over the ray $\mathfrak{l}$. We denote by $\widetilde{r}$ and $\widetilde{r}'$ the initial points of $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ respectively. The points $\widetilde{r}$ and $\widetilde{r}'$ lie over the initial point r of $\mathfrak{l}$. Hence under Φ the segment $T(\widetilde{r}, \widetilde{r}')$ is mapped onto a closed curve C not homotopic to zero. For the curve C there exists the free homotopy class Λ containing C for which there exists on $\widetilde{\mathfrak{R}}$ a motion $\widetilde{\Lambda}$ such that $\widetilde{r}'\widetilde{\Lambda} = \widetilde{r}$ and $\widetilde{\Lambda}\Phi$ is a covering transformation of $\widetilde{\mathfrak{R}}$ onto $\mathfrak{R}$ [2; §12]. The motion $\widetilde{\Lambda}$ carries $\widetilde{\mathfrak{l}}'$ onto $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{U}}_2'$ onto a ray $\mathfrak{U}_2$ which is a coray to $\widetilde{\mathfrak{l}}$ and lies over the ray $\mathfrak{U}_2$. Thus the first part is proved. Next we prove the second part.

Let $\widetilde{E}_1$ and $\widetilde{E}_2$ be the interior and the exterior of limit circle $L(\widetilde{p}_1, \widetilde{\mathfrak{l}})$ respectively. It is sufficient to prove that $\widetilde{E}_1$ and $\widetilde{E}_2$ do not contain the initial point $\widetilde{p}_2$ of $\widetilde{\mathfrak{U}}_2$. Suppose now $\widetilde{p}_2 \in \widetilde{E}_2$. Then the ray $\widetilde{\mathfrak{U}}_2$ intersects $L(\widetilde{p}_1, \widetilde{\mathfrak{l}})$ at a point $\widetilde{p}'$ distinct from $\widetilde{p}_2$. The rays $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2$ are corays each other. Hence it follows from (ii) that for any two points $\widetilde{x}$ and $\widetilde{x}'$ on an asymptote to $\widetilde{\mathfrak{l}}$ the following relation holds.

$$\alpha(\widetilde{x}, \widetilde{\mathfrak{U}}_2) - \alpha(\widetilde{x}', \widetilde{\mathfrak{U}}_2) = \alpha(\widetilde{x}, \widetilde{\mathfrak{l}}) - \alpha(\widetilde{x}', \widetilde{\mathfrak{l}}) \; (= \pm\rho(\widetilde{x}, \widetilde{x}')).$$

Therefore

$$\alpha(\widetilde{x}, \widetilde{\mathfrak{l}}) = \alpha(\widetilde{x}, \widetilde{\mathfrak{U}}_2) + \text{const. for any point } \widetilde{x} \text{ on } \widetilde{\mathfrak{R}}.$$

From this it is easy to see

$$L(\widetilde{p}_1, \widetilde{\mathfrak{l}}) = L(\widetilde{p}_1, \widetilde{\mathfrak{U}}_2).$$

Let $\{q_n\}$ be a sequence of points on $\widetilde{\mathfrak{U}}_2$ such that $\lim_{n\to+\infty}\rho(p_2, q_n) = +\infty$. Then we have

$$\alpha(\widetilde{p}', \widetilde{\mathfrak{U}}_2) - \alpha(\widetilde{p}_2, \widetilde{\mathfrak{U}}_2) = -\rho(\widetilde{p}_2, \widetilde{p}') < 0,$$

and hence we see

$$\lim_{n\to+\infty}\{\rho(\widetilde{p}_2, \widetilde{q}_n) - \rho(\widetilde{p}_1, \widetilde{q}_n)\} > 0,$$

since $\alpha(\widetilde{p}_1, \widetilde{\mathfrak{U}}_2) = \alpha(\widetilde{p}', \widetilde{\mathfrak{U}}_2)$. From this it follows that there exists a positive integer N such that

$$\rho(\widetilde{p}_1, \widetilde{q}_n) < \rho(\widetilde{p}_2, \widetilde{q}_n) \qquad \text{for every } n \geqq N.$$

Hence the segments $T(\widetilde{p}_2, \widetilde{q}_n)$ $(n \geqq N)$ do not lie over segments on $\mathfrak{R}$, which contradicts that $\widetilde{\mathfrak{U}}_2$ lies over the ray $\mathfrak{U}_2$.

If $\widetilde{p}_2 \in \widetilde{E}_1$, then $\widetilde{p}_1$ is contained in the exterior of the limit circle $L(\widetilde{p}_2, \widetilde{\mathfrak{l}})$. Hence the developments are entirely parallel to the above, i.e., we again arrive at a contradiction. Thus the second part is proved.

From (3.4) we immediately see that

(3.5) *For a ray* $\mathfrak{l}$ *there exist* m *corays* $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_m$ *from a point* p *to a ray* $\mathfrak{l}$, *and let* $\widetilde{\mathfrak{U}}_i$ *lie over each* $\mathfrak{U}_i$ *and be the coray from a point* $\widetilde{p}$ *to a* *ray $\widetilde{\mathfrak{l}}_i$ which lies over $\mathfrak{l}$. Further suppose that the straight line, which contains each $\widetilde{\mathfrak{l}}_i$, intersects the limit circle $L(\widetilde{p},\widetilde{\mathfrak{l}}_i)$ at a point $\widetilde{q}_i$. Then the points $\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_m$ lie over a point q on $\mathfrak{R}$ which lies on the geodesic containing $\mathfrak{l}$ as a subray.*

(3.6) THEOREM. *Let $\mathfrak{U}$ be an asymptote to a ray $\mathfrak{l}$. Then there does not exist a ray which contains $\mathfrak{U}$ as a proper subray.*

PROOF. Suppose that a ray $\mathfrak{B}$ contains the asymptote $\mathfrak{U}$ as a proper subray. Let a be the initial point of $\mathfrak{U}$ and b a point on $\mathfrak{B}$ but not on $\mathfrak{U}$. Since $a \in K(\mathfrak{l})$, there exists an asymptote $\mathfrak{U}'$ from a to $\mathfrak{l}$ which is distinct from $\mathfrak{U}$.

By (3.4) there exist on $\widetilde{\mathfrak{R}}$ two corays $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathfrak{U}}'$ to a ray $\widetilde{\mathfrak{l}}(\widetilde{\mathfrak{l}}\Phi = \mathfrak{l})$ which lie over $\mathfrak{U}$ and $\mathfrak{U}'$ respectively. We denote by $\widetilde{a}$ and $\widetilde{a}'$ the initial points of $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathfrak{U}}'$ respectively. Then a lies on the limit circle $L(\widetilde{a}',\widetilde{\mathfrak{l}})$. Hence the point b which lies on the ray $\widetilde{\mathfrak{B}}$ $(\widetilde{\mathfrak{B}}\Phi = \mathfrak{B})$ containing $\widetilde{\mathfrak{U}}$ as a subray is contained in the exterior of $\widetilde{L}(\widetilde{a}',\mathfrak{l})$. From the proof of (3.4) it is not hard to show that $\widetilde{\mathfrak{B}}$ does not lie over a ray on $\mathfrak{R}$, which contradicts the assumption. Thus the theorem is proved.

From Theorem (3.6) we see that the union of all rays which contains a coray coincides with an asymptote. That $\mathfrak{R}$ has non-positive curvature is essential for Theorem (3.6). To see this we show the following

EXAMPLE 2°. Let (x,y,z) be rectangular coordinates in a 3-dimensional Euclidean space. We denote by D' the convex part of the xy-plane bounded by the segment:

$$1/2 \leqq x \leqq 3/2, \ y = 0, \ z = 0$$

and the half straight lines

$$x = 3/2, \ 0 \leqq y < +\infty, \ z = 0 \text{ and } x = 1/2, \ 0 \leqq y < +\infty, \ z = 0.$$

We replace the segments

$$x = 3/2, \ 0 \leqq y \leqq 1/4, \ z = 0 \text{ and } 5/4 \leqq x \leqq 3/2, \ y = 0, \ z = 0$$

by an arc C of the algebraic curve expressed by rectangular coordinates (ξ, η) as follows:

$$(\xi - 1/4)^5 + (\eta - 1/4)^5 + 1/4^5 = 0,$$

and further by applying the same procedure to the segments

$$x = 1/2, \ 0 \leqq y \leqq 1/4, \ z = 0 \text{ and } 1/2 \leqq x \leqq 3/4, \ y = 0, z = 0,$$

we smooth the boundary of D'. Thus we have a convex part D of the xy-plane instead of D'.

Now cut off D from the xy-plane and join the surface generated by half open segments:

$$x = \alpha, \ y = \beta, \ 0 \leqq z < \lambda \ (3/4 \geqq \lambda \geqq 1/2),$$

where $(\alpha, \beta, 0) \in$ the boundary of D, and further join the part D_λ of the

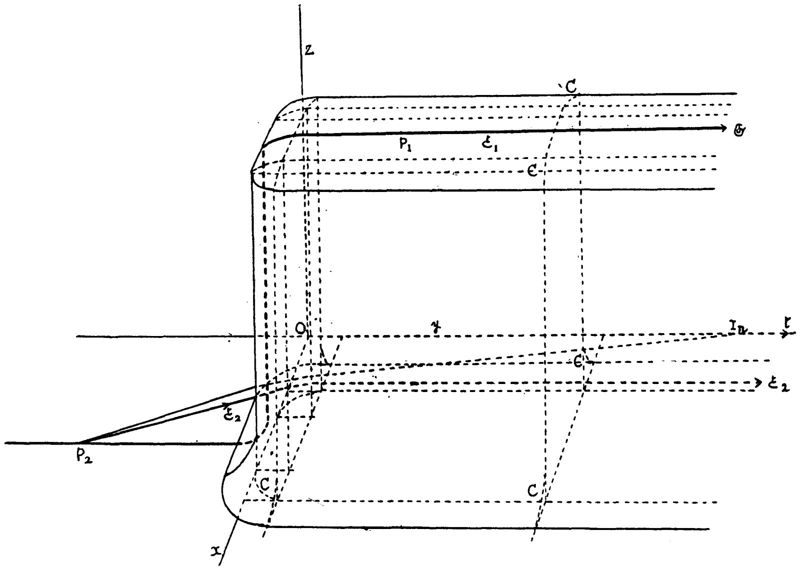

Fig. 2

plane $z=\lambda$:

$$\{(x,y,z)|(x,y,0)\in D, z=\lambda\}.$$

Thus we have a surface S' instead of the xy-plane. To smooth S', we again use an arc of the algebraic curve. As shown in Fig. 2 we smooth the section of S' by every plane parallel to z-axis and perpendicular to the boundary of D. The surface S thus obtained is clearly of class C^4.

On this surface S let $\mathfrak{l}$ be the half straight line:

$$x=0,\ 0\leqq y<+\infty,\ z=0$$

and $\mathfrak{G}$ be the intersection of the surface S with the plane $x=1$. We denote by p_1 a point $(1,\mu,\lambda)$ $(\mu>1)$. The part of S:

$$\{(x,y,z)|(x,y,z)\in S, \mu\leqq y\}$$

is a surface of curvature zero. From this it is easy to see that for a sequence $\{r_n\}$ $(r_n=(0,\mu_n,0))$ which diverges to infinity $Fl_{n\to+\infty} T(p,r_n)$ coincides with the ray

$$\mathfrak{x}_1:\ x=1,\ \mu\leqq y<+\infty,\ z=\lambda.$$

Next we consider the point p_2 $(=(1,\mu',0), \mu'<-1)$. If on S a curve C_n connecting p_2 to each r_n passes through a point whose z-coordinate is equal to λ, then the length of C_n is greater than $|\mu'|+\mu+1$. From this it is easily seen that a coray $\mathfrak{x}_2$ from p_2 to $\mathfrak{l}$ does not coincide with a subray of $\mathfrak{G}$.

From this the geodesic $\mathfrak{G}$, which is a straight line on S, contains a coray to $\mathfrak{l}$ but is not an as mptote.

(3.7) *Let $\tilde{\mathfrak{l}}$ be a ray on $\tilde{\mathfrak{R}}$ which lies over a ray $\mathfrak{l}$ and $\tilde{\mathfrak{U}}$ an asymptote to*

$\tilde{\mathfrak{l}}$. *If under* Φ *a subray* $\tilde{\mathfrak{x}}$ *of* $\tilde{\mathfrak{U}}$ *is mapped onto a ray on* $\mathfrak{R}$, *then* $\mathfrak{x}$ *is a coray to the ray* $\mathfrak{l}$. *Under* Φ *the union of all subrays of* $\tilde{\mathfrak{U}}$ *with this property is mapped onto an asymptote.*

PROOF. We only prove the first part, since the second part is an immediate consequence of the first part.

Suppose that the ray $\mathfrak{l}$, the tube U_0 and a part $\tilde{U}_0$ of $\tilde{\mathfrak{R}}$ which lies over U_0 has been chosen so as described at the end of §1. If we put $\alpha = \inf_{x\in C_0} \alpha(x, \mathfrak{l})$, then α is finite, since C_0 is compact. For the number α there exists on $\mathfrak{l}$ a point s_0 such that

$$\alpha > \alpha(s, \mathfrak{l})$$

for any point s on $\mathfrak{l}$ which follows s_0. Let $\tilde{s}_0$ be the point on $\tilde{\mathfrak{l}}$ which lies over the point s_0 and $L(\tilde{s}_0, \tilde{\mathfrak{l}})$ intersects $\tilde{\mathfrak{U}}$ at a point $\tilde{p}$. By choosing a suitable subray of $\mathfrak{x}$ or further a suitable point s_0, if necessary, the point p is supposed to be the initial point of the ray $\mathfrak{x}$.

At first we consider the case where the tube U_0 is expanding.

Suppose that $\tilde{\mathfrak{x}}'$ is a ray with the initial point $\tilde{p}$ which lies over a coray $\tilde{\mathfrak{x}}'$ from p to $\mathfrak{l}$. If $\tilde{\mathfrak{x}}'$ coincides with the ray $\tilde{\mathfrak{x}}$, then there is nothing to prove. Suppose that $\tilde{\mathfrak{x}}$ does not coincide with $\tilde{\mathfrak{x}}$. We prove $\tilde{\mathfrak{x}}'$ is a coray from $\tilde{p}$ to the ray $\tilde{\mathfrak{l}}'$. To do this we prove $\tilde{\mathfrak{x}}'$ is entirely contained in $\tilde{U}_0$.

This being so, we immediately conclude that $\tilde{\mathfrak{x}}'$ is a coray to $\tilde{\mathfrak{l}}'$. For if this is not so, $\tilde{\mathfrak{x}}'$ is the coray from $\tilde{p}$ to a ray $\tilde{\mathfrak{l}}''$ which lies over $\mathfrak{l}$ and is distinct from $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{l}}'$. Then it follows from (1. 3) that there exists a subray of $\tilde{\mathfrak{l}}''$ contained in $\tilde{U}_0$. But this contradicts that, under the contraction $\Phi | \tilde{U}_0$ of Φ, $\tilde{U}_0$ is univalently mapped onto the tube U_0 except points on $\tilde{\mathfrak{l}}$ and $\mathfrak{l}'$.

If there exists a subray of $\tilde{\mathfrak{x}}'$ which is not contained in $\tilde{U}_0$, then $\tilde{\mathfrak{x}}'$ intersects the boundary of $\tilde{U}_0$. Since $\tilde{\mathfrak{x}}'$ is disjoint from $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{l}}'$, $\tilde{\mathfrak{x}}'$ intersects $\tilde{C}_0$ at a point $\tilde{q}$. Then we have

$$\begin{aligned} \alpha &\leqq \alpha(q, \mathfrak{l}) \\ &= \alpha(\tilde{q}, \tilde{\mathfrak{l}}'') < \alpha(\tilde{p}, \tilde{\mathfrak{l}}), \end{aligned}$$

since $\tilde{q}$ follows $\tilde{p}$. On the other hand we have

$$\begin{aligned} \alpha(\tilde{p}, \tilde{\mathfrak{l}}'') &= \alpha(p, \mathfrak{l}) \\ &\leqq \alpha(\tilde{p}, \tilde{\mathfrak{l}}) = \alpha(\tilde{s}_0, \tilde{\mathfrak{l}}) \\ &= \alpha(s_0, \mathfrak{l}) \leqq \alpha, \end{aligned}$$

which contradicts the above inequalities. Thus we see $\tilde{\mathfrak{x}}' \subset \tilde{U}_0$.

Further we prove

$$(3.8) \qquad \alpha(\tilde{p}, \tilde{\mathfrak{l}}) = \alpha(\tilde{p}, \tilde{\mathfrak{l}}').$$

From the proof of (3. 4) it is easy to see $\alpha(\tilde{p}, \tilde{\mathfrak{l}}') \leqq \alpha(\tilde{p}, \tilde{\mathfrak{l}})$. Suppose $\alpha(\tilde{p}, \tilde{\mathfrak{l}}') < \alpha(\tilde{p}, \tilde{\mathfrak{l}})$. By (3. 4) there exists a coray $\tilde{\mathfrak{x}}''$ to $\tilde{\mathfrak{l}}$ which lies over the coray $\mathfrak{x}'$.

Let $\widetilde{p}''$ be the initial point of $\widetilde{\mathfrak{r}}''$. Then $\widetilde{p}''$ lies over the point p and $\widetilde{p}$ is contained in the exterior of the limit circle $L(\widetilde{p}'', \widetilde{\mathfrak{l}}'')$. Hence it follows from the proof of (3.4) that $\widetilde{\mathfrak{r}}$ does not lie over a ray on $\mathfrak{R}$, which contradicts to the assumption in the theorem. Thus (3.8) is proved.

Let r be the initial point of $\mathfrak{l}$ and $\{s_n\}$ the sequence of points on $\mathfrak{l}$ such that $\lim_{n\to+\infty}\rho(r, s_n) = +\infty$. Further let $\{\widetilde{s}_n\}$ and $\{\widetilde{s}'_n\}$ be the sequences of the points on $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ respectively such that $\widetilde{s}_n\Phi = \widetilde{s}'_n\Phi = s_n$ for each n. The limit circle $L(\widetilde{p}, \widetilde{\mathfrak{l}})$ does not coincide with $L(\widetilde{p}, \widetilde{\mathfrak{l}}')$. For if this is not so, the ray $\widetilde{\mathfrak{r}}'$ coincides with $\widetilde{\mathfrak{r}}$ and $\widetilde{\mathfrak{l}}'$ and $\widetilde{\mathfrak{l}}'$ are corays each other, which contradicts that the tube U_0 is expanding. From this it follows that there exists a sequence of points $\{\widetilde{p}_n\}$, which converges to $\widetilde{p}$, such that each $\widetilde{p}_n$ is commonly contained in the exterior of $L(\widetilde{p}, \widetilde{\mathfrak{l}}')$ and the interior of $L(\widetilde{p}, \widetilde{\mathfrak{l}})$. Since

$$\alpha(\widetilde{p}_n, \widetilde{\mathfrak{l}}) < \alpha(\widetilde{p}_n, \widetilde{\mathfrak{l}}') \qquad \text{for each } n,$$

it is easily seen that for each n there exists a positive integer $m(n)$ such that the segments $T(\widetilde{p}_n, \widetilde{s}_m)$ $(m \geqq m(n))$ lie over segments on $\mathfrak{R}$. For if we take a sufficiently large positive integer M, the inequality

$$\rho(\widetilde{p}_n, \widetilde{s}_m) < \rho(\widetilde{p}_n, \widetilde{s}'_m)$$

holds for every $m \geqq M$. Let N be a positive integer such that

$$S(\widetilde{p}, \beta_p) \ni \widetilde{p}_n \qquad \text{for each } n \geqq N.$$

Then it is easily seen that, even if for each $n(\geqq N)$ there exists a subsequence $\{s_{m_i}\}$ of $\{s_m\}$ such that each $T(\widetilde{p}_n, \widetilde{s}_{m_i})$ does not lie over a segment on $\mathfrak{R}$, $\{s_{m_i}\}$ consists of points finite in number, since a coray from every point x of $S(p, \beta_p)$ to $\mathfrak{l}$ coincides with the image of one of the corays from $\widetilde{x}(\in S(\widetilde{p}, \beta_p))$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ under Φ. This shows that for each n $(\geqq N)$ there exists a positive integer $l(n)$ such that the segment $T(\widetilde{p}_n, \widetilde{s_l(n)})$ lie over a segment on $\mathfrak{R}$.

Obviously the sequence of positive integers $\{l(n)\}$ can be selected so as to diverge to infinity. Since $Fl_{n\to+\infty}T(\widetilde{p}_n, \widetilde{s}_{l(n)})$ coincides with the ray $\widetilde{\mathfrak{r}}$, we see that $\widetilde{\mathfrak{r}}$ also lies over a coray from p to $\mathfrak{l}$.

Next we consider the case where the tube U_0 is non-expanding.

In this case the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ are corays each other. As can easily be seen from the above proof, if there exists a coray $\widetilde{\mathfrak{r}}'$ with the initial point $\widetilde{p}$ which lies over a coray from p to $\mathfrak{l}$, then $\widetilde{\mathfrak{r}}'$ is entirely contained in $\widetilde{U}_0$. From this it follows that $\widetilde{\mathfrak{r}}'$ is a coray to $\widetilde{\mathfrak{l}}$. Hence the ray $\widetilde{\mathfrak{r}}'$, i.e., $\widetilde{\mathfrak{r}}$ lies over a coray from p to $\mathfrak{l}$. Thus we complete the proof.

From (3.7) we have the following

(3.9) THEOREM. *On $\mathfrak{R}$ the relation between ray and coray is symmetric and transitive.*

Suppose that a ray $\mathfrak{r}$ is a coray to a ray $\mathfrak{l}$. Let $\widetilde{\mathfrak{l}}$ be a ray which lies over $\mathfrak{l}$. Then there exists a ray $\widetilde{\mathfrak{x}}$ which lies over $\mathfrak{x}$ and is a coray to $\widetilde{\mathfrak{l}}$. In virtue of (1.3) the ray $\widetilde{\mathfrak{l}}$ is a coray to $\mathfrak{x}$. Hence it follows from (1.3), (1.4) and (3.7).

4. In this paragraph we study the set of asymptotic conjugate points for a given ray $\mathfrak{l}$. We begin by showing the following important theorem.

(4.1) THEOREM. *Let a ray* $\mathfrak{l}$ *be given and* p be *any point of the set* $K(\mathfrak{l})$, *and suppose that there exist* m *asymptotes* $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_m$ *from* p *to* $\mathfrak{l}$. *Then there exists a neighborhood* $W(p)$ *such that in* $\overline{W(p)}$ *the set* $K(\mathfrak{l})$ *is composed of* m *arcs* $M_1, M_2, \ldots, M_m$ *issuing from* p *and each* M_i *lies between two consecutive of these asymptotes.*

PROOF. Let $\widetilde{\mathfrak{U}}_1, \widetilde{\mathfrak{U}}_2, \ldots, \widetilde{\mathfrak{U}}_m$ be m rays issuing from a point $\widetilde{p}$ such that each $\widetilde{\mathfrak{U}}_i$ lies over the asymptote $\mathfrak{U}_i$ and is the coray from $\widetilde{p}$ to a ray $\widetilde{\mathfrak{l}}_i$ which lies over the ray $\mathfrak{l}$. Suppose that $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2$ are consecutive, and take on $\widetilde{\mathfrak{U}}_1$ a point $\widetilde{q}_1$ which follows $\widetilde{p}$. Let us denote by L_1 the arc of the limit circle $L(q_1, \mathfrak{l})$ in $S(q_1, \beta_q)$. Since $L_1 \cap K(\mathfrak{l}) = \phi$, there exists a unique coray from every point of L_1 to $\mathfrak{l}$. If we take a sufficiently small subarc L_1' of L_1 which contains q_1 as an inner point, then the asymptote through every point of L_1' is not a straight line. For if this is not so, there exists on L_1' a sequence of points $\{r_n\}$, which converges to q_1, such that the asymptote $\mathfrak{B}_n$ through each point r_n is a straight line. For a positive number $\kappa(>\rho(p, q_1))$ let r_n' be the point on each $\mathfrak{B}_n$ such that $\kappa = \rho(r_n, r_n')$. Let $\mathfrak{x}_n$ be the subray of each $\mathfrak{B}$ with r_n' as its initial point. Then $Fl_{n \to +\infty}\mathfrak{x}_n$ coincides with a coray $\mathfrak{x}$ to $\mathfrak{l}$ which contains $\mathfrak{U}_1$ as a proper subray. But this contradicts $p \in K(\mathfrak{l})$.

Let x be any point of L_1' and x' the asymptotic conjugate point of the asymptote through x. Taking into account that the asymptote through every point x of L_1' coincides with the image of the coray through $\widetilde{x}(\in \widetilde{S}(p, \beta_p))$ to $\widetilde{\mathfrak{x}}_1$ under Φ and the set $K(\mathfrak{x})$ is closed, the correspondence $x \to x'$ is one-to-one and bicontinuous. We denote by H the image of L_1' in this correspondence. Then H is an arc through p and divided by p into two arcs H_1 and H_2. One of these arcs lies between $\mathfrak{U}_1$ and $\mathfrak{U}_2$. We assume that H_1 lies between $\mathfrak{U}_1$ and $\mathfrak{U}_2$.

Since $H_1 \subset K(\mathfrak{l})$, there exist at least two asymptotes from every point of H_1 to $\mathfrak{l}$. Take a positive number γ_1 so small that H_1 intersects $K(p, \gamma_1)$ at a point a_1. Let a_1 be the first common point of H_1 with $K(p, \gamma_1)$ and M_1 the subarc of H_1 from p to a_1. Further let $\widetilde{M}_1$ be the arc issuing from $\widetilde{p}$ which lies over M_1. If further γ_1 is sufficiently small, there exist exactly two asymptotes from every point x of M_1 to $\mathfrak{l}$ and these asymptotes coincide with the images of the corays from $\widetilde{x}(\in \widetilde{M}_1)$ to $\widetilde{\mathfrak{l}}_1$ and $\widetilde{\mathfrak{l}}_2$ under Φ. We show this.

If this is not so, there exists on M_1 a sequence of points $\{p_n\}$, which converges to p, such that an asymptote $\mathfrak{C}_n$ from each p_n to $\mathfrak{l}$ coincides with the image of the coray $\widetilde{\mathfrak{C}}_n$ from $\widetilde{p}_n$ $(\in \widetilde{M}_1)$ to one of the rays $\widetilde{\mathfrak{l}}_3$, $\widetilde{\mathfrak{l}}_4, \ldots,$ $\mathfrak{l}_m$ under Φ. Then the sequence of the corays $\{\widetilde{\mathfrak{C}}_n\}$ contains a subsequence $\{\widetilde{\mathfrak{C}}_{n_i}\}$ whose limit coincides with a ray $\widetilde{\mathfrak{C}}$ issuing from $\widetilde{p}$. The ray $\widetilde{\mathfrak{C}}$ lies over an asymptote from p to $\mathfrak{l}$ and coincides with one of the rays $\widetilde{\mathfrak{U}}_3, \widetilde{\mathfrak{U}}_4, \ldots,$ $\widetilde{\mathfrak{U}}_m$. Hence there exists a positive integer N such that each $\widetilde{\mathfrak{C}}_{n_i}$ $(n_i \geqq N)$ intersects either $\widetilde{\mathfrak{U}}_1$ or $\widetilde{\mathfrak{U}}_2$, which is a contradiction.

In such a way m arcs $M_1, M_2, \ldots, M_m$ issuing from p are determined. From this we see that there exists a neighborhood $W(p)$ of p such that

$$K(\mathfrak{l}) \cap \overline{W(p)} = \cup_{i=1}^{m} M_i.$$

Thus the theorem is proved.

In what follows we suppose that for a point $p(\in K(\mathfrak{l}))$ the neighborhood $W(p)$ is contained in $S(p, \alpha_p)$. To make clear the circumstances in the above we show the following .

EXAMPLES 3°. Let (x, y, z) be rectangular coordinates in a 3-dimensional Euclidean space and L the curve composed of the semi-circular arc:

$$x^2 + y^2 = 1, \quad -1 \leqq y \leqq 0, \quad z = 0$$

and two half straight lines:

$$x = 1, \quad 0 \leqq y < +\infty, \quad z = 0 \text{ and } x = -1, \quad 0 \leqq y < +\infty, \quad z = 0.$$

We construct a surface S' in accordance with the following steps:

i) Consider the cylindrical surface Z_1 generated by half straight lines:

$$x = \alpha, y = \beta, \quad 0 \leqq z < +\infty,$$

where $(\alpha, \beta, 0) \in L$;

ii) Join the half of the semi-cylinder

Z_2: $\quad x^2 + z^2 = 1, \quad 0 \leqq y < +\infty, \quad -1 \leqq z \leqq 0$

to the surface Z_1;

iii) Further join the half cylinder

Z_3: $\quad x^2 + y^2 = 1, \quad -\infty < z \leqq 0$

to the above surface;

iv) Omit the parts of Z_2 and Z_3 cut off from the other.

Next we smooth the surface S' thus obtained.

The section of the surface S' by a half plane $x = \lambda (|\lambda| < 1)$, $0 \leqq y < +\infty$, consists of the half straight lines

$\mathfrak{g}_1$: $\quad x = \lambda, \sqrt{1-\lambda^2} \leqq y < +\infty, \quad z = -\sqrt{1-\lambda^2}$ and

$\mathfrak{g}_2$: $\quad x = \lambda, \ y = \sqrt{1-\lambda^2}, \ -2\sqrt{1-\lambda^2} \leqq z \leqq -\sqrt{1-\lambda^2}$.

To smooth this section, as shown in Fig. 3, we replace the subsegments of $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$x = \lambda, \quad \sqrt{1-\lambda^2} \leqq y \leqq 2\sqrt{1-\lambda^2}, \quad z = -\sqrt{1-\lambda^2} \quad \text{and}$$

$$x = \lambda, \quad y = \sqrt{1-\lambda^2}, \ -2\sqrt{1-\lambda^2} \leqq z \leqq -\sqrt{1-\lambda^2}$$

by an arc C of the algebraic curve expressed by rectangular coordinates

(ξ, η) as follows:

$$(\xi - \kappa)^4 + (\eta - \kappa)^4 = \kappa^4,$$

where $\kappa = \sqrt{1 - \lambda^2}$. Thus we have a surface S_1 instead of S'.

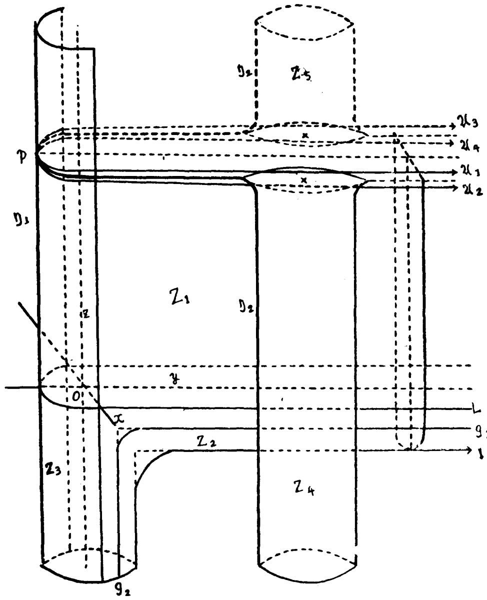

Fig. 3

Next cut off from the surface S_1 the circular disks whose centers are $(1, 5, 5)$ and $(-1, 5, 5)$ and radii 1, and join the half cylinders

Z_4: $\quad 1 \leqq x < +\infty,\ (y-5)^2 + (z-5)^2 = 1$ and

Z_5: $\quad -\infty < x \leqq -1\ (y-5)^2 + (z-5)^2 = 1.$

To smooth the joint part, in the same way as in Example 1° we use an arc of the algebraic curve expressed as follows:

$$(\xi - 1/4)^4 + (\eta - 1/4)^4 = 1/4^4,$$

where (ξ, η) are rectangular coordinates. We denote by S_2 the surface thus obtained.

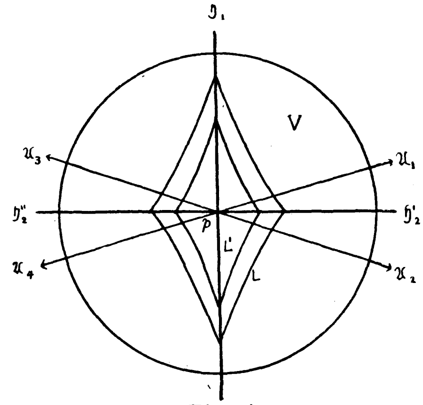

Fig. 4

As we see from the construction of the surfaces S_1 and S_2, these surfaces are of class C^1 and regarded as G-spaces defined by H. Busemann [2; §4].

Let $\mathfrak{l}$ be the half straight line:

$$x = 0,\ 2 \leqq y < +\infty,\ z = -1,$$

and $\mathfrak{y}_1$ and $\mathfrak{y}_2$ be the intersections of S_2 with the half planes:

$$x = 0, \ -\infty < y \leqq 0 \text{ and } -\infty < y \leqq 5, z = 5$$

respectively. Then $\mathfrak{y}_1$ and $\mathfrak{y}_2$ intersect at the point $p(=(0, -1, 5))$.

At first we consider the surface S_1. It is easily seen that the set $K_1(\mathfrak{l})$ of asymptotic conjugate points is identical with the straight line $\mathfrak{y}_1$. The set $K_1(\mathfrak{l})$ has no branch point. There exist exactly two asymptotes from every point of $K_1(\mathfrak{l})$ to $\mathfrak{l}$.

Now we consider the surface S_2. The set $K_2(\mathfrak{l})$ of asymptotic conjugate points is identical with the union of $\mathfrak{y}_1$ and $\mathfrak{y}_2$. It is easy to see that there exists four asymptotes issuing from p to $\mathfrak{l}$. As shown in Fig. 3, we denote by $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3$ and $\mathfrak{U}_4$ these aymptotes. Fig. 4 shows the behaviors of these asymptotes at the point p and limit circles L and L' in a sufficiently small neighborhood V of p. It is easily seen that in V limit circles not through p are simple closed curves and the limit circle $L(p, \mathfrak{l})$ consists of only one point p.

From the above we see that on S_2 a limit circle is not necessarily differentiable at an asymptotically conjugate point.

Next we prove by use of (4. 1) the second part of Theorem (2. 1).

Let p be a point of the set $K(\mathfrak{l})$. Suppose that $\overline{W(p)} \cap K(\mathfrak{l})$ consists of m arcs $M_1, M_2, \ldots, M_m$ and $W(p)$ is divided by these arcs into m domains $D_1, D_2, \ldots, D_m$. Then each D_i is simply covered by a system of corays. For convenience's sake, suppose further that each D_i contains a subsegment of the asymptote $\mathfrak{U}_i$ and has place between M_i and M_{i+1}, where $M_{m+1} = M_1$. Then we see

$$D_i \cap \mathfrak{U}_j = \phi \text{ for each } j(\neq i) \text{ and } D_i \cap K(\mathfrak{l}) = \phi \text{ for each } i.$$

On $\mathfrak{R}$, a neighborhood $W(\tilde{p})$ $(W(\tilde{p})\Phi = W(p), W(\tilde{p}) \subset S(\tilde{p}, \alpha_p))$ is similarly divided by m arcs $\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_m$ into m domains $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_m$, where each $\tilde{M}_i$ lies over the arc M_i. We assume that each $\tilde{D}_i$ lies over D_i. The ray $\tilde{\mathfrak{U}}_i$ issuing from $\tilde{p}$ which lies over each $\mathfrak{U}_i$ is the coray from $\tilde{p}$ to a ray $\tilde{\mathfrak{l}}_i$ which lies over the ray $\mathfrak{l}$.

Let $\{q_n\}$ be a sequence of points on $\mathfrak{l}$ which diverges to infinity and p_i a point of each $D_i \cap V(p)$, where $V(p)$ is a coordinate neighborhood of p. $V(p)$ is supposed to be convex [3]. Then there exists a positive integer N' such that the end point $\tilde{q}_n$ of the segment $T(\tilde{p}_1, \tilde{q}_n)$ $(\tilde{p}_1 \in W(\tilde{p}))$, which lies over each $T(\tilde{p}_1, \tilde{q}_n)$ $(n \geqq N')$, is on $\tilde{\mathfrak{l}}_1$ or other rays (finite in number) which lie over $\mathfrak{l}$ are corays to $\tilde{\mathfrak{l}}_1$. Hence by choosing one of such rays instead of $\tilde{\mathfrak{l}}_1$, if necessary, we have a subsequence $\{q_{n_1}\}$ of $\{q_n\}$ such that the end point $\tilde{q}_{n_1}$ of each $T(\tilde{p}_1, \tilde{q}_{n_1})$ $(T(\tilde{p}_1, \tilde{q}_{n_1})\Phi = T(p_1, q_{n_1}))$ is on the ray $\mathfrak{l}_1$.

Next we consider the sequence of segments $\{T(p_2, q_{n_1})\}$.

Under the same consideration as in the above, we see that there exists a subsequence $\{q_{n_2}\}$ of $\{q_{n_1}\}$ such that the end point $\tilde{q}_{n_2}$ of each $T(\tilde{p}_2, \tilde{q}_{n_2})$ $(T(p_2, q_{n_2})\Phi = T(p_2, q_{n_2}))$ is on the ray $\tilde{\mathfrak{l}}_2$. We continue this. Then after m steps we have a subsequence $\{q_{n_m}\}$ of $\{q_{n_{m-1}}\}$ such that for each i the end point $\widetilde{q}_{n_m}$ of each $T(p_i, q_{n_m})$ $(T(\widetilde{p}_i, \widetilde{q}_{n_m})\Phi = T(p_i, q_{n_m}))$ is on the ray $\widetilde{\mathfrak{l}}_i$.

Let Ψ be a topological transformation of the punctured sphere S onto $\mathfrak{R}$. To fix the idea, we keep a positive integer n_m fixed and put $N = n_m$. To shorten the notation, we denote by T_i each geodesic polygon $T(p, p_i) + T(p_i, q_N)$. Each of these geodesic polygons is not homotopic to the others. It is clear that, if one of these geodesic polygons intersects another at a point x, then x is contained in $V(p)$. Hence we can choose on S m arcs $L_1', L_2', \ldots,$ and L_m', which connect $p'(=p\Psi^{-1})$ to $q'(=q_N\Psi^{-1})$, such that each L_i' is homotopic to $T_i\Psi^{-1}$ and has no common points with the others except p' and q_N'.

Each of the closed curves $L_i' + L_{i+1}'^{-1}$ $(0 \leqq i \leqq m,\ L_{m+1}' = L_1')$ is homotopic to a simple closed curve and not homotopic to zero. The sphere S is divided by these arcs into m domains $E_1', E_2', \ldots$ and E_m'. In this case each E_i' is supposed to be bounded by the simple closed curve $L_i' + L_{i+1}'^{-1}$. Then

$$L_i' + L_{i+1-1}' \sim T_i\Psi^{-1} + T_{i+1}^{-1}\Psi^{-1},$$

and each E_i contains at least one of the holes $a_0, a_1, \ldots, a_k$. From this we see $m \leqq k+1$, which proves the second part of (2.1).

From the above proof it is easily seen that

$$\alpha(p, \mathfrak{l}) = \alpha(\widetilde{p}, \widetilde{\mathfrak{l}}_1) = \ldots = \alpha(\widetilde{p}, \widetilde{\mathfrak{l}}_m). \tag{4.2}$$

It is also clear that

$$\alpha(x, \mathfrak{l}) = \alpha(\widetilde{x}, \widetilde{\mathfrak{l}}_i) \tag{4.3}$$

for any point $\widetilde{x}$ of each $\widetilde{D}_i$ and

$$\alpha(x, \mathfrak{l}) = \alpha(x, \widetilde{\mathfrak{l}}_i) = \alpha(\widetilde{x}, \widetilde{\mathfrak{l}}_{i+1}) \tag{4.4}$$

for any point $\widetilde{x}$ of each $\widetilde{M}_i$.

From (3.6) and (3.7) we have the following

(4.5) THEOREM. *On $\mathfrak{R}$ let $\mathfrak{r}$ and $\mathfrak{l}$ be corays each other. Then the set $K(\mathfrak{l})$ coincides with the set $K(\mathfrak{r})$.*

5. In this paragraph we show in the large some properties of the set $K(\mathfrak{l})$ for a given ray $\mathfrak{l}$. We prove at first the following

(5.1) *For a ray $\mathfrak{l}$ a continuous curve contained in the set $K(\mathfrak{l})$ is unbounded and if the set $K(\mathfrak{l})$ has branch points, then the set of the branch points is discrete.*

PROOF. The second part is clear from Theorem (4.1). Hence we prove only the first part. To do this, for a point p $(\in K(\mathfrak{l}))$ let γ_p be the least upper bound of those γ's for which the circular neighborhood $S(p, \gamma)$ is contained in $W(p)$. As we see easily from Theorem (4.1), every arc contained in the set $K(\mathfrak{l})$ is prolongable. Let K be the whole prolongation of an arc contained in $K(\mathfrak{l})$. Then K has no end point. Suppose now that K is contained in a bounded domain D. Then we show at first that, if K has branch points, the number of these points is finite.

arc has no other common points with C_0 and has not a branch point. For if u is a branch point of K, there exists on $\widetilde{B}$ at least three rays issuing from $u(\in \widetilde{B})$ which lie over asymptotes from u to $\mathfrak{l}$. But this is impossible, since any of these rays is a coray to either $\widetilde{\mathfrak{l}}$ or $\widetilde{\mathfrak{l}}'$. Hence K is an unbounded simple arc. It remains to show $K(\mathfrak{l}) - K = \phi$.

Let p' be any point of the set $K(\mathfrak{l})$ distinct from p. We show that p' is on K. Since the corays from $\widetilde{p}'(\in \widetilde{B})$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ lie over asymptotes from p' to $\mathfrak{l}$, we have

$$\alpha(p', \mathfrak{l}) = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}) = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}').$$

Let the limit circle $L(\widetilde{p}', \widetilde{\mathfrak{l}})$ intersects the straight line $\widetilde{\mathfrak{G}}$ at a point $\widetilde{s}$ and the limit circle $L(\widetilde{p}', \widetilde{\mathfrak{l}}')$ the straight line $\widetilde{\mathfrak{G}}'$ at apoint $\widetilde{s}'$. Then by (3.5) $\widetilde{s}$ and $\widetilde{s}'$ lie over a point on $\mathfrak{G}$. We denote by $\widetilde{M}$ the arc of $\widetilde{L(p', \mathfrak{l})}$ from $\widetilde{s}$ to $\widetilde{p}'$ and by $\widetilde{M}'$ the arc of $L(\widetilde{p}', \widetilde{\mathfrak{l}})$ from $\widetilde{s}'$ to $\widetilde{p}'$. By the same reason as in the above there exists on $T(\widetilde{s}, \widetilde{s}')$ a unique point $\widetilde{q}$ such that $\widetilde{q}$ belongs to the set $K(\mathfrak{l})$. Next we show that q is on K.

Let $\widetilde{x}$ be any point on the segment $T(\widetilde{r}, \widetilde{s})$. Then there exists on $T(\widetilde{r}', \widetilde{s}')$ the point $\widetilde{x}'$ which lies over x. On $T(\widetilde{x}, \widetilde{x}')$ there exists a unique point $\widetilde{y}$ such that $\widetilde{y}$ belongs to $K(\mathfrak{l})$. When $\widetilde{x}$ varies on $T(\widetilde{r}, \widetilde{s})$ from $\widetilde{r}$ to $\widetilde{s}$, $\widetilde{y}$ varies on the arc $\widetilde{K}(= K\Phi^{-1} \cap \widetilde{B})$ from $\widetilde{p}$ to $\widetilde{q}$. This proves $q \in K$.

Since $\widetilde{K}$ is unbounded and has no common points with $T(\widetilde{r}, \widetilde{r}')$ except $\widetilde{p}$, we see from the above that $\widetilde{K}$ intersects $\widetilde{M}$ or $\widetilde{M}'$. Suppose that $\widetilde{K}$ intersects M at a point $\widetilde{v}$ distinct from $\widetilde{p}'$. Then we have

$$\begin{aligned}\alpha(\widetilde{v}, \widetilde{\mathfrak{l}}) &= \alpha(\widetilde{v}, \widetilde{\mathfrak{l}}') \\ &= \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}') = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}).\end{aligned}$$

Hence v lies on the limit circle $K(\widetilde{p}', \widetilde{\mathfrak{l}}')$. Let $\widetilde{\mathfrak{U}}_1$ be the coray from $\widetilde{p}'$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{U}}_2$ the coray from $\widetilde{v}$ to $\widetilde{\mathfrak{l}}'$. Then $\widetilde{\mathfrak{U}}_2$ has no common points with $L(\widetilde{p}', \widetilde{\mathfrak{l}}')$ except $\widetilde{v}$. Hence by virtue of (1.3) $\widetilde{\mathfrak{U}}_2$ intersects $\widetilde{\mathfrak{U}}_1$ at a point. On the other hand, the rays $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2$ lie over asymptotes to $\mathfrak{l}$. Hence $\mathfrak{U}_2\Phi$ is disjoint from $\widetilde{\mathfrak{U}}_1\Phi$. But this is a contradiction. Similarly $\widetilde{K}$ does not intersect $\widetilde{M}'$ at a point distinct from $\widetilde{p}'$. Hence $\widetilde{p}'$ lies on $\widetilde{K}$, which proves $K(\mathfrak{l}) - K = \phi$.

If the tube U_0 is non-expanding, the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ are corays each other. Let $\widetilde{x}$ be any point on $\widetilde{\mathfrak{R}}$ and $\widetilde{\mathfrak{x}}$ the ray issuing from the point $\widetilde{x}\,(\in \widetilde{B})$ which lies over a coray $\mathfrak{x}$ from x to $\mathfrak{l}$. Then $\widetilde{\mathfrak{x}}$ is a coray to $\widetilde{\mathfrak{l}}$. For if $\widetilde{\mathfrak{x}}$ is a coray to a ray $\widetilde{\mathfrak{l}}''$ which lies over $\mathfrak{l}$, by virtue of (1.3) and (1.4) the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}''$ are corays each other. Hence there exists only one coray from x to $\mathfrak{l}$ which proves $K(\mathfrak{l}) = \phi$. Thus we complete the proof.

Let $\mathfrak{R}$ be a Finsler space of class $C^r(r \geqq 4)$ (or a Riemann space of class C^r $(r \geqq 3)$). In the above proof it is easily seen that $\widetilde{M} \cup \widetilde{M}'$ lies over the limit

If this is not so, then the set of the branch points of K contains a sequence of points $\{p_n\}$ which converges to a point p. Since $K(\mathfrak{l})$ is closed, the point p belongs to $K(\mathfrak{l})$. For the sequence $\{p_n\}$ there exists a positive integer N such that

$$S(p, \gamma_p) \ni p_n \text{ for every } n \geqq N.$$

Since $K(\mathfrak{l}) \cap \overline{W(p)}$ consists of arcs issuing from p which is finite in number, each $p_n(n \geqq N)$ is identical with the point p. But this is a contradiction.

From the above we see that K contains a half open simple curve K'. Without loss of generality we assume that the initial point a of K' is not a branch point. By the assumption K' is contained in the domain D. We show that this is a contradiction. To do this we put $\bar{\gamma} = \inf_{x \in K'} \gamma_x$. Then the number $\bar{\gamma}$ is non-negative. If γ is infinite, $\mathfrak{R}$ is simply connected. Hence the set $K(\mathfrak{l})$ is vacuous, and there is nothing to say.

Suppose $\bar{\gamma} = 0$. There exists a sequence of points $\{q_n\}$ $(q_n \in K' \cap D)$ such that $\lim_{n \to +\infty} \gamma_{q_n} = 0$. Since D is bounded, the sequence $\{q_n\}$ contains a subsequence $\{q_{n_i}\}$ which converges to a point q. Then $q \in K(\mathfrak{l})$ and γ_q is positive. Hence there exists a positive integer N' such that

$$S(q, \gamma_q) \ni q_{n_i} \text{ for every } n_i \geqq N'.$$

From this it follows that each q_{n_i} $(n_i \geqq N')$ lies on an arc of $\overline{W(p)} \cap K(\mathfrak{l})$. If q is a branch point, then K' contains the branch point q which contradicts the assumption. If q is not a branch point, it is easy to see $\tilde{\gamma} \neq 0$, which contradicts to $\lim_{n \to +\infty} \gamma_{q_n} = 0$. Thus we see $\tilde{\gamma} \neq 0$ and hence $\tilde{\gamma}$ is positive.

We denote by r_1 the first common point of the curve K' with $K(a, \tilde{\gamma})$ and by L_1 the subarc of K' from a to r_1. Similarly we denote by r_2 the first common point of K' with $K(r_1, \tilde{\gamma})$ which follows r_1 and by L_2 the subarc K' from r_1 to r_2. We continue this process. Then the sequence of the points $\{r_n\}$ contains a subsequence $\{r_{n_i}\}$ which converges to a point r. Since $r \in K(\mathfrak{l})$, $\gamma_r \neq 0$. Hence there exists a positive integer N'' such that

$$S(r, \gamma_r) \ni r_{n_i} \text{ for every } n_i \geqq N''.$$

On the other hand, any two of the arcs L_{n_i} $(n_i \geqq N'')$ are non-overlapping. But this is a contradiction.

From the above it follows that K' is not contained in a bounded domain, and hence K is unbounded. Thus we complete the proof.

(5.2) THEOREM. *Suppose that $\mathfrak{R}$ is homeomorphic to a cylinder. Then for a ray $\mathfrak{l}$ the set $K(\mathfrak{l})$ is an unbounded simple arc or vacuous according as the tube U_0, which contains $\mathfrak{l}$ or a subray of $\mathfrak{l}$, is expanding or non-expanding.*

PROOF. Without loss of generality we assume that $\mathfrak{l}$ is a subray of a straight line $\mathfrak{G}$ whose opposite contains a subray belonging to the other tube U_1. For if $\mathfrak{l}$ is not such a ray, let $\{q_n\}$ be a sequence of points on U_1 which diverges to infinity and $\mathfrak{x}_n$ a coray from each q_n to $\mathfrak{l}$. Then each $\mathfrak{x}_n$ intersects the boundary C_0 at a point r_n. Let $\mathfrak{x}'_n$ be the subray of each $\mathfrak{x}_n$ whose initial arc has no other common points with C_0 and has not a branch point. For if u is a branch point of K, there exists on $\widetilde{B}$ at least three rays issuing from $u(\in \widetilde{B})$ which lie over asymptotes from u to $\mathfrak{l}$. But this is impossible, since any of these rays is a coray to either $\widetilde{\mathfrak{l}}$ or $\widetilde{\mathfrak{l}}'$. Hence K is an unbounded simple arc. It remains to show $K(\mathfrak{l}) - K = \phi$.

Let p' be any point of the set $K(\mathfrak{l})$ distinct from p. We show that p' is on K. Since the corays from $\widetilde{p}'(\in \widetilde{B})$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ lie over asymptotes from p' to $\mathfrak{l}$, we have

$$\alpha(p', \mathfrak{l}) = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}) = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}').$$

Let the limit circle $L(\widetilde{p}', \widetilde{\mathfrak{l}})$ intersects the straight line $\widetilde{\mathfrak{G}}$ at a point $\widetilde{s}$ and the limit circle $L(\widetilde{p}', \widetilde{\mathfrak{l}}')$ the straight line $\widetilde{\mathfrak{G}}'$ at apoint $\widetilde{s}'$. Then by (3.5) $\widetilde{s}$ and $\widetilde{s}'$ lie over a point on $\mathfrak{G}$. We denote by $\widetilde{M}$ the arc of $L(\widetilde{p}', \widetilde{\mathfrak{l}})$ from $\widetilde{s}$ to $\widetilde{p}'$ and by $\widetilde{M}'$ the arc of $L(\widetilde{p}', \widetilde{\mathfrak{l}})$ from $\widetilde{s}'$ to $\widetilde{p}'$. By the same reason as in the above there exists on $T(\widetilde{s}, \widetilde{s}')$ a unique point $\widetilde{q}$ such that $\widetilde{q}$ belongs to the set $K(\mathfrak{l})$. Next we show that q is on K.

Let $\widetilde{x}$ be any point on the segment $T(\widetilde{r}, \widetilde{s})$. Then there exists on $T(\widetilde{r}', \widetilde{s}')$ the point $\widetilde{x}'$ which lies over x. On $T(\widetilde{x}, \widetilde{x}')$ there exists a unique point $\widetilde{y}$ such that $\widetilde{y}$ belongs to $K(\mathfrak{l})$. When $\widetilde{x}$ varies on $T(\widetilde{r}, \widetilde{s})$ from $\widetilde{r}$ to $\widetilde{s}$, $\widetilde{y}$ varies on the arc $\widetilde{K}(= K\Phi^{-1} \cap \widetilde{B})$ from $\widetilde{p}$ to $\widetilde{q}$. This proves $q \in K$.

Since $\widetilde{K}$ is unbounded and has no common points with $T(\widetilde{r}, \widetilde{r}')$ except $\widetilde{p}$, we see from the above that $\widetilde{K}$ intersects $\widetilde{M}$ or $\widetilde{M}'$. Suppose that $\widetilde{K}$ intersects M at a point $\widetilde{v}$ distinct from $\widetilde{p}'$. Then we have

$$\begin{aligned} \alpha(\widetilde{v}, \widetilde{\mathfrak{l}}) &= \alpha(\widetilde{v}, \widetilde{\mathfrak{l}}') \\ &= \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}') = \alpha(\widetilde{p}', \widetilde{\mathfrak{l}}). \end{aligned}$$

Hence v lies on the limit circle $K(\widetilde{p}', \widetilde{\mathfrak{l}}')$. Let $\widetilde{\mathfrak{U}}_1$ be the coray from $\widetilde{p}'$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{U}}_2$ the coray from $\widetilde{v}$ to $\widetilde{\mathfrak{l}}'$. Then $\widetilde{\mathfrak{U}}_2$ has no common points with $L(\widetilde{p}', \widetilde{\mathfrak{l}}')$ except $\widetilde{v}$. Hence by virtue of (1.3) $\widetilde{\mathfrak{U}}_2$ intersects $\widetilde{\mathfrak{U}}_1$ at a point. On the other hand, the rays $\widetilde{\mathfrak{U}}_1$ and $\widetilde{\mathfrak{U}}_2$ lie over asymptotes to $\mathfrak{l}$. Hence $\mathfrak{U}_2\Phi$ is disjoint from $\widetilde{\mathfrak{U}}_1\Phi$. But this is a contradiction. Similarly $\widetilde{K}$ does not intersect $\widetilde{M}'$ at a point distinct from $\widetilde{p}'$. Hence $\widetilde{p}'$ lies on $\widetilde{K}$, which proves $K(\mathfrak{l}) - K = \phi$.

If the tube U_0 is non-expanding, the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$ are corays each other. Let $\widetilde{x}$ be any point on $\widetilde{\mathfrak{R}}$ and $\widetilde{\mathfrak{x}}$ the ray issuing from the point $\widetilde{x}(\in \widetilde{B})$ which lies over a coray $\mathfrak{x}$ from x to $\mathfrak{l}$. Then $\widetilde{\mathfrak{x}}$ is a coray to $\widetilde{\mathfrak{l}}$. For if $\widetilde{\mathfrak{x}}$ is a coray to a ray $\widetilde{\mathfrak{l}}''$ which lies over $\mathfrak{l}$, by virtue of (1.3) and (1.4) the rays $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}''$ are corays each other. Hence there exists only one coray from x to $\mathfrak{l}$ which proves $K(\mathfrak{l}) = \phi$. Thus we complete the proof.

Let $\mathfrak{R}$ be a Finsler space of class $C^r (r \geqq 4)$ (or a Riemann space of class C^r $(r \geqq 3)$). In the above proof it is easily seen that $\widetilde{M} \cup \widetilde{M}'$ lies over the limit circle $L(\widetilde{s},\widetilde{\mathfrak{l}})$. If the tube U_0 is expanding, $L(\widetilde{s},\widetilde{\mathfrak{l}})$ does not coincide with $L(\widetilde{s'},\widetilde{\mathfrak{l}'})$. From this it follows that, if a limit circle L with respect to $\mathfrak{l}$ contains an asymptotic conjugate point a, then L does not contain another asympotic conjugate point and is a simple closed curve of class C^1 except a. If the tube U_0 is non-expanding, every limit circle with respect to $\mathfrak{l}$ is a simple closed curve of class C^1.

(5.3) THEOREM. *If for a ray $\mathfrak{l}$ the tube U_0, which contains $\mathfrak{l}$ or a subray of $\mathfrak{l}$, is non-expanding, there exists a subtube of $\widetilde{U}_0$ disjoint from the set $K(\mathfrak{l})$. But if the tube U_0 is expanding there does not exist a subtube of U_0 disjoint from the set $K(\mathfrak{l})$.*

PROOF. At first we shall consider the case where the tube U_0 is non-expanding.

Suppose that the ray $\mathfrak{l}$, the tube U_0 and a part $\widetilde{U}_0$ of $\widetilde{\mathfrak{R}}$ has been chosen so as described in §1. Let $\widetilde{s}$ be a point on $\widetilde{\mathfrak{l}}$ such that

$$\alpha = \inf_{x\in C_0}\alpha(x,\mathfrak{l}) > \alpha(\widetilde{s},\widetilde{\mathfrak{l}})$$

and $\widetilde{s'}$ the point on $\widetilde{\mathfrak{l}'}$ which lies over the point $s\,(=\widetilde{s}\Phi)$. Then the limit circle $L(\widetilde{s},\widetilde{\mathfrak{l}})$ passes through the point $\widetilde{s'}$ and under Φ the arc M of $L(\widetilde{s},\widetilde{\mathfrak{l}})$ from $\widetilde{s}$ to $\widetilde{s'}$ is mapped onto the limit circle $L(\widetilde{s},\widetilde{\mathfrak{l}})$. The exterior of $L(\widetilde{s},\widetilde{\mathfrak{l}})$ does not contain the segment $T(\widetilde{s},\widetilde{s'})$. If we put $T(s,s')\Phi = C_0'$, the subtube U_0' of U_0 bounded by C_0' is disjoint from the set $K(\mathfrak{l})$. For let x be any point of U_0'. Then we have

$$\alpha > \alpha(\widetilde{x},\widetilde{\mathfrak{l}})\ (\widetilde{x}\in \widetilde{U}_0').$$

As we see from the proof of (3.7), the coray from every point $\widetilde{x}$ of $\widetilde{U}_0'$ to $\widetilde{\mathfrak{l}}$ lies over a coray from x to $\mathfrak{l}$ and there exists only one coray from x to $\mathfrak{l}$. From this it follows that $K(\mathfrak{l})\cap U_0' = \phi$.

Next we consider the case where the tube U_0 is expanding.

Let $\widetilde{s}$ and $\widetilde{s'}$ be any point on $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}'}$ respectively such that $\widetilde{s}\Phi = \widetilde{s'}\Phi = s$ and $\widetilde{C}_0'$ any geodesic polygon which connects $\widetilde{s}$ to $\widetilde{s'}$. It is sufficient to prove that there exists on $\widetilde{C}_0'$ the point which lies over a point of the set $K(\mathfrak{l})$. If $s\in K(\mathfrak{l})$, there is nothing to say. If $s\notin K(\mathfrak{l})$, then the coray from every point x of $S(\widetilde{s},\beta_s)\cap\widetilde{C}_0'$ to $\widetilde{\mathfrak{l}}$ lies over a coray from x to $\mathfrak{l}$. But the coray from $\widetilde{x}$ to another ray, which lies over $\mathfrak{l}$ but is not a coray to $\widetilde{\mathfrak{l}}$, does not lie over acoray from x to $\mathfrak{l}$. Similarly the coray from every point $\widetilde{x'}$ of $S(\widetilde{s'},\beta_s)\cap\widetilde{C}_0'$ to $\widetilde{\mathfrak{l}'}$ lies over a coray from $\widetilde{x'}$ to $\widetilde{\mathfrak{l}}$. But the coray from x' to another ray, which lies over $\mathfrak{l}$ but is not a coray to $\widetilde{\mathfrak{l}'}$, does not lie over a coray from x' to $\mathfrak{l}$. From this it follows that there exists on $\widetilde{C}_0$ a point $\widetilde{p}$ such that the corays from $\widetilde{p}$ to at least two rays lying over $\mathfrak{l}$ lie over corays from p to $\mathfrak{l}$. Under Φ such a point $\widetilde{p}$ is mapped onto a point of the set $K(\mathfrak{l})$.

Thus the theorem is proved.

(5.4) THEOREM. *If $\mathfrak{l}$ is homeomorphic to a sphere punctured at three points at least, i.e., $k \geqq 2$, then for any ray $\mathfrak{l}$ the set $K(\mathfrak{l})$ is not vacuous.*

PROOF. Suppose that the ray $\mathfrak{l}$ is a subray of a straight line whose opposite contains a subray belonging to another tube U_1 (if this is not so, in the same way as in the proof of Theorem (5.2) we use another suitable ray instead of $\mathfrak{l}$). To prove the theorem we suppose $K(\mathfrak{l}) = \phi$. Then it is sufficient to show that we arrive at a contradiction.

Let r be a point $\mathfrak{l}$ and C_0' the geodesic one-gon with r as its vertex which is homotopic to C_0 (or the closed geodesic through r homotopic to C_0 if it exists). Then by choosing suitably the boundary C_1 of the tube U_1, if necessary, C_0' is supposed to be disjoint from C_1. Let x' be any point of C_1. Then the asymptote through x' intersects C_0' at only one point x. By virtue of the assumption all asymptotes are straight lines. Hence the correspondence: $x' \rightarrow x$ is coninuous. Since C_0' is a simple closed curve, the image of C_1 coincides with C_0'. From this it follows that C_1 is continuously deformable to C_0. This contradicts $k \geqq 2$. Thus the theorem is proved.

At the end we prove the following

(5.5) THEOREM. *For a ray $\mathfrak{l}$ the set $K(\mathfrak{l})$ consists of the finite number of unbounded and continuous curves. If $K(\mathfrak{l})$ has branch points, the number of these points is finite.*

PROOF. As before we assume that $\mathfrak{l}$ is a subray of a straight line whose opposite contains a subray belonging to another tube and the boundary C_0 of U_0 is a geodesic one-gon whose vertex coincides with the initial point r of $\mathfrak{l}$ (or the closed geodesic through r if it exists). For a part $\widetilde{U}_0$ of $\widetilde{\mathfrak{R}}$ which lies over U_0 we use the same notation as before.

We prove at first that, if the tube U_0 contains branch points of the set $K(\mathfrak{l})$, the number of these points is finite. If the tube U_0 is non-expanding, the tube U_0 contains a subtube U_0' disjoint from the set $K(\mathfrak{l})$. Since $U_0 - U_0'$ is bounded, we see from the proof of (5.1) that the number of the branch points contained in $U_0 - U_0'$ is finite, which implies that the tube U_0 contains the finite number of branch points.

Now we consider the case where the tube U_0 is expanding.

Since the set $\widetilde{K}(\mathfrak{l})$ ($= K(\mathfrak{l})\Phi^{-1}$) is closed, the set $\widetilde{K}(\mathfrak{l}) \cap T(\widetilde{r}, \widetilde{r}')$ is compact. Hence there exists the finite number of points $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_m$ of $\widetilde{K}(\mathfrak{l}) \cap T(\widetilde{r}, \widetilde{r}')$ such that $\bigcup_{i=1}^{m} \overline{W(\widetilde{x}_i)}$ covers $K(\mathfrak{l}) \cap T(\widetilde{r}, \widetilde{r}')$. By Theorem (4.1) each $\widetilde{K}(\mathfrak{l}) \cap \overline{W(\widetilde{x}_i)}$ consists of arcs issuing from $\widetilde{x}_i$ which are finite in number. We denote by $\widetilde{K}_i$ each $\overline{W(\widetilde{x}_i)} \cap \widetilde{K}(\mathfrak{l})$. Suppose further that on $T(\widetilde{r}, \widetilde{r}')$ each $\widetilde{x}_i$ precedes $\widetilde{x}_{i+1}$. Then the curve $\widetilde{K}_1$ contains a subarc $\widetilde{K}_1'$ such that the coray from every point of $\widetilde{K}_1'$ to $\widetilde{\mathfrak{l}}$ lies oves over an asymptote to $\mathfrak{l}$.

Let $\widetilde{\mathfrak{F}}_1'$ be the system of the corays from the points of $\widetilde{K}_1'$ to $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{U}}$ the coray from an end point $\widetilde{y}$ of $\widetilde{K}_1'$ to $\widetilde{\mathfrak{l}}$. Let $\widetilde{z}$ be a point on $\widetilde{\mathfrak{U}}$ which follows $\widetilde{y}$. Then for a positive number β less than β_z $\widetilde{S(z,\beta)} \cap \widetilde{L(z,\mathfrak{l})}$ is an arc disjoint from the set $\widetilde{K}(\mathfrak{l})$. We denote by L the arc $\widetilde{S(z,\beta)} \cap \widetilde{L(z,\mathfrak{l})}$. Let x be any point of L and $\mathfrak{V}$ the asymptote through x. Then, under Φ, $\mathfrak{V}$ is the image of a ray (or a straight line) $\widetilde{\mathfrak{V}}$ through $\widetilde{x}$ $(\in \widetilde{L})$ which is a coray (or an asympotote) to $\widetilde{\mathfrak{l}}$. Such $\mathfrak{V}$'s form a system $\widetilde{\mathfrak{F}}_1''$. Obviously $\widetilde{\mathfrak{F}}_1''$ conta anins element not belonging to $\widetilde{\mathfrak{V}}_1'$. The system $\widetilde{\mathfrak{F}}_1' \cup \widetilde{\mathfrak{F}}_1''$ is considered as an xtension of the system $\widetilde{\mathfrak{F}}_1'$. In such a way we have the largest system $\widetilde{\mathfrak{F}}_1$ such that every element of $\widetilde{\mathfrak{F}}_1$ is a coray or an asymptote to $\widetilde{\mathfrak{l}}$ and lies over an asymptote to $\mathfrak{l}$.

Let a curve $K_i (i \neq 1)$ contains a subarc $\widetilde{K}_i'$ such that the coray from every point of $\widetilde{K}_i'$ to $\widetilde{\mathfrak{l}}$ lies over an asymptote to $\mathfrak{l}$. Then in the same way as in the above we have the largest system $\widetilde{\mathfrak{F}}_i$ such that every element of $\widetilde{\mathfrak{F}}_i$ is a coray or an asymptote to $\widetilde{\mathfrak{l}}$ and lies over an asymptote to $\mathfrak{l}$. We show that $\widetilde{\mathfrak{F}}_i$ coincides with the system $\widetilde{\mathfrak{F}}_1$.

Suppose that $\widetilde{\mathfrak{F}}_i$ does not coincide with $\widetilde{\mathfrak{F}}_1$. Let $\widetilde{\mathfrak{F}}_i'$ be the system of corays from the point of $\widetilde{K}_i'$ to $\widetilde{\mathfrak{l}}$. Then there exists at least one unbounded and continuous curves of $\widetilde{K}(\mathfrak{l})$ which lies between the systems $\widetilde{\mathfrak{F}}_1'$ and $\widetilde{\mathfrak{F}}_i'$. Let $\widetilde{s}$ be a point on $\widetilde{\mathfrak{l}}$ such that

$$\inf_{v \in C_0} (x, \mathfrak{l}) > \alpha(\widetilde{s}, \widetilde{\mathfrak{l}}) \; (= \alpha(s, \mathfrak{l})) \tag{5.6}$$

and $\widetilde{v}$ the the point at which the arc $\widetilde{L(s,\mathfrak{l})} \cap \widetilde{U}_0$ intersects a curve of $\widetilde{K}(\mathfrak{l})$ at first $\widetilde{s}$. Then we have

$$\alpha(\widetilde{v}, \widetilde{\mathfrak{l}}) = \alpha(\widetilde{s}, \widetilde{\mathfrak{l}})$$

and there exists a ray $\widetilde{\mathfrak{C}}$ issuing from $\widetilde{v}$ which lies over an asymptote to $\widetilde{\mathfrak{l}}$ but is not a coray to $\widetilde{\mathfrak{l}}$. The ray $\widetilde{\mathfrak{C}}$ is not a coray to $\widetilde{\mathfrak{l}}'$. For if $\widetilde{\mathfrak{C}}$ is a coray to $\widetilde{\mathfrak{l}}'$. $\widetilde{\mathfrak{C}}$ intersects the rays of $\widetilde{\mathfrak{F}}_1'$. But this is impossible. Hence $\widetilde{\mathfrak{C}}$ is a coray to a ray $\widetilde{\mathfrak{l}}''$ which lies over $\mathfrak{l}$ and is distinct from $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{l}}'$. The ray $\widetilde{\mathfrak{l}}''$ is not contained in $\widetilde{U}_0$. Hence $\widetilde{\mathfrak{C}}$ intersects $\widetilde{C}_0$ at a point $\widetilde{u}$. Then we have

$$\begin{aligned} \inf_{x \in C_0} \alpha(x, \mathfrak{l}) &\leqq \alpha(u, \mathfrak{l}) = \alpha(\widetilde{u}, \widetilde{\mathfrak{l}}'') \\ &< \alpha(\widetilde{v}, \widetilde{\mathfrak{l}}'') = \alpha(\widetilde{v}, \widetilde{\mathfrak{l}}) = \alpha(s, \mathfrak{l}) \end{aligned}$$

which contradicts (5.6). From this we see that $\widetilde{\mathfrak{F}}_i$ coincides with $\widetilde{\mathfrak{F}}_1$.

It is also easy to prove that if two curves $\widetilde{K}_j$ and $\widetilde{K}_l$ $(j \neq l)$ contain subarc $\widetilde{K}_j'$ and $\widetilde{K}_l'$ respectively such that the corays from points of $\widetilde{K}_j'$ and $\widetilde{K}_l'$ to a ray $\widetilde{\mathfrak{l}}'''(\widetilde{\mathfrak{l}}'''\Phi = \mathfrak{l})$ lie over asymptotes to $\mathfrak{l}$, then the largest system in the above sense which contains the system of the corays from the points of

$\widetilde{K}_j'$ to $\widetilde{\mathfrak{l}}'''$ contains that of corays from the points of $\widetilde{K}_t'$ to $\widetilde{\mathfrak{l}}'''$ and vice versa.

From the above we see that $\widetilde{U}_0$ is covered by the finite number of such system. We denote by $\widetilde{\mathfrak{F}}_1, \widetilde{\mathfrak{F}}_2, \ldots, \widetilde{\mathfrak{F}}_t$ these systems. Then every point of the set $\widetilde{K}(\mathfrak{l}) \cap \widetilde{U}_0$ is identical with the common initial point of the rays contained in some of these systems.

Let $\widetilde{p}$ be a branch point of the set $\widetilde{K}(\mathfrak{l}) \cap \widetilde{U}_0$. Then the set $\widetilde{K}(\mathfrak{l}) \cap \overline{W(\widetilde{p})}$ consists of arcs issuing from p which are finite in number. We denote by $\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_l$ these arcs. Suppose that every point of each $\widetilde{M}_i$ is identical with the common initial point of rays of two systems $\widetilde{\mathfrak{F}}_{j_i}$ and $\widetilde{\mathfrak{F}}_{j_{i+1}}$ where $\widetilde{\mathfrak{F}}_{j_{l+1}} = \widetilde{\mathfrak{F}}_{j_1}$, and suppose further that there exists on the prolongation $\widetilde{P}$ of an arc $\widetilde{M}_1$ a branch point $\widetilde{p}'$ contained in $\widetilde{U}_0$. Then we show that the systems $\widetilde{\mathfrak{F}}_{j_3}, \widetilde{\mathfrak{F}}_{j_4}$, and $\widetilde{\mathfrak{F}}_{j_l}$ do not contain any of the rays with $\widetilde{p}'$ as their common initial points.

To do this, let $\widetilde{K}(\mathfrak{l}) \cap \overline{W(\widetilde{p}')}$ be composed of arcs $\widetilde{M}_1', \widetilde{M}_2', \ldots, \widetilde{M}_h'$ issuing from $\widetilde{p}'$ and suppose that every point of an arc $\widetilde{M}_1'$ is identical with the common initial point of rays of a system $\widetilde{\mathfrak{F}}_{j_i}$ $(i \geqq 3)$ and another system. Then we show that this is a contradiction. Let L be the set of the initial points of the rays of $\widetilde{\mathfrak{F}}_{j_i}$. Then for the arcs $\widetilde{M}_i$ and $\widetilde{M}_1'$ we consider the following two cases:

(1) $\widetilde{L}$ contains two unbounded and continuous curves $\widetilde{L}_i$ and $\widetilde{L}_1'$ such that $\widetilde{L}_i \supset \widetilde{M}_i, \widetilde{L}_1' \supset \widetilde{M}_1'$, and $\widetilde{L}_i \cap \widetilde{L}_1' = \phi$.

(2) $\widetilde{L}$ contains an unbounded and continuous curve $\widetilde{L}'$ which contains the arcs $\widetilde{M}_i$ and $\widetilde{M}_1'$.

Suppose that the case (1) holds. There exists a straight line $\widetilde{\mathfrak{H}}$ of $\widetilde{\mathfrak{F}}_{j_i}$ which separates $\widetilde{L}_i$ and $\widetilde{L}_1'$. $\widetilde{\mathfrak{H}}$ lies over an asymptote to $\mathfrak{l}$ which is a straight line. Since $\widetilde{P}$ contains an arc $\widetilde{C}$ which connects $\widetilde{p}$ and $\widetilde{p}'$, the straight line $\widetilde{\mathfrak{H}}$ intersects $\widetilde{C}$ at a point $\widetilde{q}$. The point $\widetilde{q}$ belongs to the set $\widetilde{K}(\mathfrak{l})$. But this is a contradiction.

Suppose that the case (2) holds. There exists an subarc $\widetilde{C}'$ of $\widetilde{L}'$ which connects $\widetilde{p}$ and $\widetilde{p}'$ and is distinct from $\widetilde{C}$. The arcs $\widetilde{C}$ and $\widetilde{C}'$ bound a domain $\widetilde{W}$. Let $\widetilde{w}$ be a point of the boundary of $\widetilde{W}$ which lies on $\widetilde{C}$ but not on $\widetilde{C}'$ (or on $\widetilde{C}'$ but not on $\widetilde{C}$). Then there exists two rays issuing from $\widetilde{w}$ which lie over asymptotes from w to $\mathfrak{l}$. One of these asymptotes contains a subsegment belonging to W and again intersects the boundary of W at a point $\widetilde{w}'$ distinct from w. The point $\widetilde{w}'$ belongs to the set $\widetilde{K}(\mathfrak{l})$. But this is a contradiction.

From the above it is easy to deduce that the number of the branch points of $\widetilde{P} \cap \widetilde{U}_0$ is finite and hence the number of the branch points of

$\widetilde{K}(\mathfrak{l}) \cap \widetilde{U}_0$ is finite.

Since $\mathfrak{R} - \bigcup_{i=0}^{k} U_i$ is a bounded open set, if $\mathfrak{R} - \bigcup_{i=0}^{k} U_i$ contains branch points of $K(\mathfrak{l})$, the number of these points is finite. This is clear from the proof of Theorem (4.1).

To show that if another tube $U_i (i \neq 0)$ contains branch points of $K(\mathfrak{l})$, the number of these points is finite, suppose that the opposite of the straight line $\mathfrak{G}$ contains a subray $\mathfrak{x}$ belonging to U_1. Further let C_i' be the geodesic one-gon with the initial point x of $\mathfrak{x}$ as its vertex which is homotopic to C_i (or the closed geodesic through x homotopic to C_i if it exists) and U_i' the tube bounded by C_i'. Then there exists on $\widetilde{\mathfrak{R}}$ a part $\widetilde{U}_i'$ bounded by two rays $\widetilde{\mathfrak{x}}$ and $\widetilde{\mathfrak{x}}'$ lying over $\mathfrak{x}$ and a segment $\widetilde{C}_i'$ lying over C_i' such that $\widetilde{U}_i'\Phi = U_i'$. In the same way as in the above it is easy to show that, if the tube U_i' contains branch points of $K(\mathfrak{l})$, then the number of these points is finite. Hence the tube U_i also has this property.

From what we have proved above the proof is complete.

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