SIMILARITIES AND DIFFERENTIABILITY

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1. Introduction. Prof. K. Yano mentioned in a lecture that a Riemann space is euclidean if it possesses a one-parameter group H of non-isometric similarities. With "Minkowskian" replacing "euclidean" the theorem holds also for Finsler spaces, but fails to hold when differentiability hypotheses are altogether omitted by substituting G-spaces[1] for Finsler spaces. This remains so, even under very strong supplementary hypotheses: *without being Minkowskian the space may, in addition to H, possess groups of motions of a rather high dimension and its geodesics may be the euclidean straight lines.* On the other hand, a very mild differentiability suffices for concluding from the existence of H and the axioms of a G-space that the space is Minkowskian. Yet, nothing in the formulation of the original theorem suggests the necessity of smoothness requirements.

The author is not aware of any similarly striking example where differentiability assumptions in their usual form conceal strong purely geometric implications. Therefore a systematic analysis of the situation seems justified,

We begin by discussing similarities in general G-spaces, then convince ourselves by examples[2] that the above mentioned phenomena actually occur. Next, we discuss a simple intrinsic, geometric condition for differentiability. Examples show that this condition is still too weak to deduce the Minkowskian character of the metric from the existence of H, because, in fact, the local metric need not be Minkowskian.

However, strengthening the condition slightly into an analogue of continuous differentiability proves sufficient: *A G-space which admits a similarity with dilation factor $k \neq 1$ (a group H of similarities is not needed) and is continuously differentiable at one of the (always existing) fixed points of the similarity, is Minkowskian in the small when $k > 1$, and in the large when $k < 1$.*

The local metric is Minkowskian at a point of a G-space where the space is continuously differentiable and regular[3]. We use these methods to partially solve the interesting problem of *deciding from the intrinsic*

1) G-spaces are defined in [2, page 37]; although they have no differentiability properties, a large part of differential geometry holds for them, see [2].

2) A cone in E^3 with total angle $\alpha<2\pi$ or $\alpha>2\pi$ at its apex a and with its intrinsic metric provides a simple example for a space which possesses a group H of similarities (and the group of rotations about a). However, for $\alpha<2\pi$ prolongation of a segment for a point b to the apex a beyond a is impossible, and for $\alpha>2\pi$ it is not unique, so that the axioms for a G-space are not satisfied.

3) An other approach to differentiability of G-spaces is found in [1, Chapter II]. The present conditions are simpler and the proofs shorter. Because the author suspected the existence of such an approach, the method of [1, Chapter II] is not discussed in [2]. The remaining results of [1] are also found in [2].

distances whether the space is a Finsler space with given smoothness properties.

2. Global and local similarities. The mapping $\alpha: x \to x\alpha = x'$ of the metric space R (with distance xy) on itself is called a *global similarity* with dilation factor $k > 0$ if

(1) $$x\alpha y\alpha = kxy \text{ for any two points } x, y \text{ in } R.$$

If $k = 1$ then α is a motion; if $k \neq 1$ we speak of a proper similarity. α is one-to-one because $x\alpha y\alpha > 0$ for $xy > 0$, hence α^{-1} is defined and also a similarity. If α_1 and α_2 are similarities of R with factors k_1 and k_2 then $\alpha_1\alpha_2$ is a similarity with dilation factor k_1k_2. The global similarities of a metric space form, therefore, a group G. The motions form an invariant subgroup G_1. Mapping each similarity on its dilation factor yields a homomorphism of G on a multiplicative subgroup of the positive reals with G_1 as kernel. A compact space deos not possess a proper similarity.

(2) *A proper global similarity α of a complete metric space has exactly one fixed point f which is called the center of α.*

For a proof we put generally $x_\nu = x\alpha^\nu$. Then

(3) $$x_\nu y_\nu = k^\nu xy, \quad x_\nu x_{\nu+1} = k^\nu x_0 x_1.$$

Since the fixed points of α and α^{-1} are identical we may assume that $k < 1$. Then (3) implies for $\rho > \nu$ that

$$x_\nu x_\rho < \sum_{i=\nu}^{\rho-1} x_i x_{i+1} = x_0 x_1 \sum_{i=\nu}^{\rho-1} k^i < k^\nu (1-k) x_0 x_1,$$

so that $\{x_\nu\}$ is a Cauchy sequence and hence converges to a point f. Clearly $f\alpha = f$ and (3) implies $y_\nu \to f$. Consequently, f is the only fixed point of α.

(4) *A G-space which admits a proper global similarity is straight*[4].

We have to show that for two given distinct points x, y a point z with (xyz), i.e., distinct from y with $xy + yz = xz$ exists. With the previous notations, $x_\nu \to f$, $y_\nu \to f$ implies $x_\nu, y_\nu \in S(f, \rho(f))$ for large ν; here $S(f, \rho)$ is the open sphere $fx < \rho$ and $\rho(f)$ is defined in [2, p. 33]: hence there is a point z with $(x_\nu y_\nu z)$. Then $z^{-\nu}$ satisfies $(xyz^{-\nu})$.

Instead of global similarities, differential geometry considers local similarities which are special cases of conformal mappings. This is also feasible in the present general setup; we discuss briefly the necessary details:

The mapping $\alpha: x \to x\alpha = x'$ of a complete metric space R without isolated points on the metric space R' is *conformal* if it is interior and

(5) $$\lim_{x \to p, y \to p, x \neq y} \frac{x\alpha y\alpha}{xy} = \delta(p)$$

exists for each $p \in R$ and $0 < \delta(p) < \infty$.

We note some trivial consequences of the definition, using the assumption that α be interior only in (11).

4) For the definition of a straight space, see [2, p. 38]. If the space is smooth enough, then it is, in the language of the calculus of variations, a simply connected space without conjugate points.

(6) *α is continuous.*

For $x_\nu \to p$ and $\delta(p) < \infty$ imply $x_\nu \alpha p \alpha \to 0$.

(7) *$\delta(p)$ is a continuous function of p.*

For if $p_\nu \to p$ we can choose $x_\nu \neq p_\nu$ with $x_\nu p_\nu < \nu^{-1}$ and $|x'_\nu p'_\nu / x_\nu p_\nu - \delta(p_\nu)| < \nu^{-1}$. Then $x_\nu \to p$ hence $x'_\nu p'_\nu / x_\nu p_\nu \to \delta(p)$ and $\delta(p_\nu) \to \delta(p)$.

(8) *For any compact set C in R and a given $\varepsilon > 0$ there is an $\varepsilon' > 0$ such that $x, y \in S(p, \varepsilon')$ and $p \in C$ imply $|x'y'/xy - \delta(p)| < \varepsilon$.*

Otherwise an $\varepsilon > 0$ and triples $x_\nu \neq y_\nu, p_\nu \in C$ with $x_\nu y_\nu \in S(p_\nu, \nu^{-1})$ and $|x'_\nu y'_\nu / x_\nu y_\nu - \delta(p_\nu)| \geqq \varepsilon$ would exist. For a subsequence $\{p_\rho\}$ of $\{p_\nu\}$ we have $p_\rho \to p \in C$ because C is compact. Therefore $x_\rho \to p, y_\rho \to p, \delta(p_\rho) \to \delta(p)$ and $|x'_\rho y'_\rho / x_\rho y_\rho - \delta(p)| \geqq \frac{\varepsilon}{2}$ for large ν, contradicting (5).

A corollary of (8) is

(9) *If $x(t)$, $a \leqq t \leqq b$ is a rectifiable curve in R, then $x'(t) = x(t)\alpha$ is a rectifiable curve in R'.*

(10) *If $y(s), 0 \leqq s \leqq L$ is the representation of a rectifiable curve in terms of arclength and $y'(s) = y(s)\alpha$, then the length L' of $y'(s)$ equals*

$$L' = \int_0^L \delta(y(s))\, ds$$

For (8) yields for any sufficiently fine partition $s_0 = 0 < s_1 < \dots < s_n = L$ the inequality

$$\left| \sum y'(s_i) y'(s_{i+1}) - \sum y(s_i) y(s_{i+1}) \delta(y(s_i)) \right| < \varepsilon \sum y(s_i) y(s_{i+1}) \leqq \varepsilon L.$$

(11) *The mapping α of R on R' is locally topological.*

There is a positive π_p such that the restriction of α to $S(p, \pi_p)$ is one-to-one. Otherwise a sequence of pairs $x_\nu \to p, y_\nu \to p$, $x_\nu \neq y_\nu$ with $x'_\nu = y'_\nu$ would exist, but then $\delta(p) = 0$.

Because of their importance in other connections we notice the following facts which are easy to prove and explicitly contained in [2, (7.5)]. If $\pi(p)$ is the least upper bound of these π_p, then the restriction α_p of α to $S(p, \pi(p))$ is one-to-one and either $\pi(p) \equiv \infty$, i. e., α is one-to-one in the large, or $0 < \pi(p) < \infty$ and $|\pi(p) - \pi(q)| \leqq pq$.

$S' = S(p, \pi(p))\alpha = S(p, \pi(p))\alpha_p$ is open because α is interior, and for the same reason α_p^{-1} is continuous on S'.

The conformal mapping α of R on R' is called a *local similarity* of R on R' if the corresponding function $\delta(p)$ is constant, say $\delta(p) = k$. We conclude from (10) that the lengths of any rectifiable curve $x(t)$ in R and $L(x')$ of its image $x'(t) = x(t)\alpha$ in R satisfy

(12) $$L(x') = kL(x).$$

From now on we assume that both R and R' are G-spaces. Then we have for a segment T from x to y in R

(13) $$kxy = kL(T) = L(T\alpha) \geqq x'y'.$$

We notice the corollary

(14) *A compact space does not admit a local similarity on itself with dilation factor $k<1$.*

For (13) would imply shrinking of the diameter of the space. Local similarities with $k>1$ are, of course, possible; they exist, for example, for a torus with a euclidean metric.

Locally, (13) can be improved:

(15) *For a given point p in R there is a positive ε such that α maps $S(p, k\varepsilon)$ on $S(p\alpha, \varepsilon)$ with $x'y' = kxy$ for any $x, y \in S(p, k\varepsilon)$.*

For $S' = S(p, \pi(p))\alpha$ is open, see the proof of (11), and contains a sphere $S(p', 2\varepsilon), \varepsilon > 0$. For $x', y' \in S(p', \varepsilon)$ a segment $T(x', y')$ lies in $S(p', 2\varepsilon)$, see [2, (6.9)] and is mapped by α_p^{-1} on a curve C in $S(p, \pi(p))$ leading from $x = x'\alpha_p^{-1}$ to $y = y'\alpha_p^{-1}$. Then by (12) and (13) $x'y' \leqq kxy \leqq kL(C) = L(T(x', y')) = x'y'$.

Denote by R_k the remetrization of R obtained by multiplying all distances in R by k. Then α may be interpreted as a mapping of R_k on R' and (14) shows that this mapping is locally isometric. From the discussion of such mappings in Chapter IV of [2] we obtain the following results:

(16) THEOREM. *A local similarity of R on R' with factor k induces a global similarity with factor k of $S(p, k\rho(p)/2)$ on $S(p', \rho(p')/2)$,* (see [2, Theorem (27. 10)]).

(17) *If the fundamental group of R is not isomorphic to a proper subgroup of the fundamental group of R',* (in particular, if R' is simply connected) *then $kxy = x'y'$ for any x, y in R.*

This follows from [2, (27.17)] applied to R_k and R'. Finally we prove:

(18) THEOREM. *If a G-space R admits a local similarity α on itself with dilation factor $k<1$ then R is straight and α is global.*

It suffices to prove that the geodesic through two given distinct points of R is unique. For then R is by [2, Theorem (31.2)] either straight or of the elliptic type. The latter is impossible because of (14). The similarity will be global by (17), since straight spaces are simply connected.

For any $\nu > 0$ the mapping α^ν is a local similarity of R on itself with factor k^ν, hence $x\alpha^\nu y\alpha^\nu \leqq k^\nu xy$, see (13), and we conclude as in the proof of (2) that α has exactly one fixed point f and $x\alpha^\nu \to f$.

If there were two different geodesics through x and $y \neq x$ then two distinct geodesic curves $z_i(\tau), 0 \leqq \tau \leqq L_i, i = 1, 2$, with $z_i(0) = x, z_i(L_i) = y$ would exist. Here L_i is the length of $z_i(\tau)$, and one of the two curves may be chosen as a segment $T(x, y)$. We choose ν so large that for $i = 1, 2$

$$\alpha^\nu L_i < \rho(f)/2 \text{ and } z_i(\tau)\alpha^\nu \in S = S(f, \rho(f)/2), 0 \leqq \tau \leqq L_i.$$

Then $z_1(\tau)\alpha^\nu$ and $z_2(\tau)\alpha^\nu$ are two distinct geodesic curves in S. But $T(x\alpha^\nu, y\alpha^\nu)$ is the only geodesic curve from $x\alpha^\nu$ to $y\alpha^\nu$ in S. Hence at least one of the two curves $z_i(\tau)\alpha^\nu$ must leave S, but then its length would exceed $\rho(f)$. We know that (18) does not hold for $k > 1$, but

(19) THEOREM. *If a G-space R admits a local similarity α on itself with dilation factor $k > 1$, then the universal covering space $\bar{R}$ of R is straight and possesses a global similarity $\bar{\alpha}$ which lies over α and also has factor k.*

As in [2, Chapter IV] we assume that $\bar{R}$ is related to R by a definite locally isometric mapping Ω; we realize the fundamental group of R by the group of motions of $\bar{R}$ which lie over the identity of R. The existence of a mapping $\bar{\alpha}$ of $\bar{R}$ on itself which lies over α, i. e., $\bar{\alpha}\,\Omega = \Omega\alpha$ follows exactly as in the proof of [2, (28. 7)] or also from the Covering Homotopy Theorem. We conclude from the fact that Ω is locally isometric and from (16) that $\bar{\alpha}$ is a local isometry of $\bar{R}$ on itself with the same factor k. Because of (17), $\bar{\alpha}$ is global and $\bar{R}$ is straight.

For a later application we notice:

If $\bar{f}$ is the fixed point of $\bar{\alpha}$ then $\bar{f}\Omega$ is a fixed point of α since $f\alpha = \bar{f}\Omega\alpha = \bar{f}\bar{\alpha}\Omega = \bar{f}\Omega = f$. Conversely, if f is a fixed point of α, then an $\bar{\alpha}$ over α exists whose fixed point lies over f. This is easily seen by choosing f as the distinguished point in the proof of [2, (28. 7)]. The mapping $e^{i\varphi} \to e^{3i\varphi}$ of the unit circle of the complex plane on itself shows that α may have more than one fixed point.

3. Non-Minkowskian spaces with proper similarities.

We now give *two examples* of straight spaces which possess proper similarities without being Minkowskian[5].

The *first* consists of a suitable metrization of Moulton's well known example for a non-Desarguesian curve system. Consider an (x_1, x_2)-plane with the auxiliary euclidean distance $e(x,y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$. The curves of Moulton's system W consist

1) Of the lines $x_2 = \tan\beta\ (x_1 - a) \quad \pi/2 < \beta \leqq \pi$
2) $x_2 = \text{const.}$ on $\beta = \pi/2$
3) The broken lines

$$x_2 = \begin{cases} 2\tan\beta(x_1 - a) \text{ for } x_2 \leqq 0, \\ \tan\beta(x_1 - a) \text{ for } x_2 > 0. \end{cases} \qquad 0 < \beta < \pi/2$$

Two curves with different β intersect. For any two points x, y and a given β we denote by $\delta_\beta(x,y)$ the eucliden distance of the lines with slope $\tan\beta$ (in $x_2 > 0$ for the broken lines) which pass through x and y.

Clearly

$$\delta_\beta(x,y) \leqq e(x,y)$$

and for $k > 0$

$$\delta_\beta(kx, ky) = k\delta_\beta(x,y).$$

5) They also show that the statement (22) in the Appendix of [2] is false, without additional assumptions, for example continuous differentiability, see the present Theorem (29).

Hence

$$\delta(x,y) = \int_0^{\pi} \delta_\beta(x,y)\,d\beta$$

is finite and satisfies

$$\delta(kx,ky) = k\delta(x,y) \quad \text{for } k > 0.$$

Moreover, δ_β *is invariant under the translations parallel to the* x_1*-axis, hence* δ *is too,* so that a global similarity with an arbitrary factor $k > 0$ and an arbitrary point on the x_1-axis as center exists. That the curves in W are the geodesics is seen exactly as in the more general situation studied in [2, p. 61], or as in the second example.

The product of this space with a euclidean or Minkowskian space, as it is defined in [2, Theorem (8.15)] is a higher dimensional space which possesses a proper global similarity with an arbitrary factor $k > 0$ and an arbitrary point of a fixed hyperplane as center.

In the *second example the geodesics will be the euclidean lines, so that the Theorem of Desargues holds; moreover, the rotations about the center of the similarities will exist, as for the cases mentioned in Footnote* 2).

Consider again an (x_1, x_2)-plane, put for fixed positive ρ

$$g_\rho(x,\beta) = \operatorname{sign}(x_1\cos\beta + x_2\sin\beta)\ |x_1\cos\beta + x_2\sin\beta|^\rho$$

and

$$\delta_\rho(x,y) = \int_{-\pi/2}^{\pi/2} |g_\rho(x,\beta) - g_\rho(y,\beta)|\,d\beta.$$

Obviously $0 < \delta_\rho(x,y) = \delta_\rho(y,x) < \infty$ for $x \neq y$. The triangle inequality holds for δ_ρ because of

$$|g_\rho(x,\beta) - g_\rho(z,\beta)| \leqq |g_\rho(x,\beta) - g_\rho(y,\beta)| + |g_\rho(y,\beta) - g_\rho(z,\beta)|. \tag{20}$$

If $y = (1-t)x + tz, 0 < t < 1$, then for arbitrary β

$$g_\rho(x,\beta) \geqq g_\rho(y,\beta) \geqq g_\rho(z,\beta) \text{ or } g_\rho(x,\beta) \leqq g_\rho(y,\beta) \leqq g_\rho(z,\beta),$$

because the line with normal direction β through y lies between (or coincides with) the lines through x and z and because of our agreement on the sign of g_ρ. Therefore we have equality in (20) for every β and $\delta_\rho(x,z) = \delta_\rho(x,y) + \delta_\rho(y,z)$.

If y does not lie on the euclidean segment from x to z, then we can find a value β such that

$$g_\rho(y,\beta) > \max\{g_\rho(x,\beta), g_\rho(z,\beta)\} \text{ or } g_\rho(y,\beta) < \min\{g_\rho(x,\beta), g_\rho(z,\beta)\},$$

hence we have inequality in (20) for this and all neighboring β, so that $\delta_\rho(x,z) < \delta_\rho(x,y) + \delta_\rho(y,z)$.

Thus, the geodesics are the euclidean straight lines, and the metric is clearly invariant under the euclidean rotations about the origin. Since

$$g_\rho(kx,\alpha) = k^\rho g_\rho(x,\alpha) \text{ for } k > 0$$

we have

$$\delta_\rho(kx, ky) = k^\rho \delta_\rho(x, y), \quad k > 0.$$

Hence $x_i = k^{1/\rho} x_i$ is a global similarity with factor k and the origin as center. The metric is Minkowskian only for $k = 1$ (in which case it is euclidean).

4. **Differentiability**. The usual definition of differentiability of a function involves two spaces, one for the independent and one for the dependent variable. This method can, of course, be followed in our case by postulating the existence of suitable coordinates. However, it is much more satisfactory to establish the existence of such coordinates directly from intrinsic properties of the distance. One is led to such a formulation of differentiability by the following observation:

Consider a real valued function $f(x) = f(x_1, \ldots, x_n)$ defined in a neighborhood of z. If $f(x)$ possesses at z a differential, then for $x^\nu \to z, y^\nu \to z, x^\nu \neq y^\nu$ and $0 < t_\nu < M$,

$$\lim_{\nu\to\infty} \frac{f[(1-t_\nu)z + t_\nu x^\nu] - f[(1-t_\nu)z + t_\nu y^\nu]}{t_\nu [f(x^\nu) - f(y^\nu)]} = 1, \tag{21}$$

provided $f(x^\nu) \neq f(y^\nu)$. This formulation has the advantage of using only the dependent variable and suggests a procedure for G-spaces:

In a G-space denote by $a(\beta, q), \beta \geqq 0$, a point satisfying

$qa(\beta, q) = \beta qa$ and $qa(\beta, q) + a(\beta, q)a = qa$ or $qa + aa(\beta, q) = qa(\beta, q)$.

For $\beta \leqq 1$ and $a(\beta, q)$ exists but may not be unique when $0 < \beta < 1$. For $\beta > 1$ it may not exist, but is unique. However, for a given point p and a given $M > 0$ there is always a positive ρ such that $a(\beta, q)$ exists and is unique for $\beta < M$ and $a, q \in S(p, \rho)$, see [1, (8.11)]. Because in our considerations a and q will be points tending to a point p, the point $a(\beta, q)$ will eventually exist and be unique. Therefore we will not mention this question again. We notice that

$$a(\beta, q)(\alpha, q) = a(\alpha\beta, q). \tag{22}$$

In analogy to (21) we say that *the space is differentiable at p if*

$$D_p: \lim_{\nu\to\infty} \frac{a_\nu(\beta_\nu, p) b_\nu(\beta_\nu, p)}{\beta_\nu a_\nu b_\nu} = 1, \text{ for } a_\nu \to p, b_\nu \to p, \ a_\nu \neq b_\nu \text{ and } 0 < \beta_\nu < M.$$

We notice the following immediate consequence of D_p:

(23) *If D_p holds, then $\delta > 0$ ($\delta < \rho(p)$) exists such that $a \neq b$, $ap \leqq \delta$, $bp \leqq \delta$ and $0 < \beta < 1$ imply*

$$1/2 < a(\beta, p) b(\beta, p)/\beta ab < 3/2.$$

The principal consequences of D_p are summarized in

(24) THEOREM. *If D_p holds, then*

$$m(a, b) = \lim_{\beta\to 0+} \beta^{-1} a(\beta, p)\, b(\beta, p) \tag{25}$$

exists and yields a metrization of $S(p, \delta)$ which is topologically equivalent to ab. Moreover $m(p, a) = pa$,

$$\lim m(a_\nu, b_\nu)/a_\nu b_\nu = 1 \textit{ for } a_\nu \to p, \ b_\nu \to p \textit{ and } a_\nu \neq b_\nu$$

and

(26) $$m[a(\beta, p), b(\beta, p)] = \beta m(a, b).$$

In all considerations of this section and many of the next, the second point q in $a(\beta, q)$ will be p. We therefore put for simplicity

$$a(\beta, p) = a(\beta).$$

Obviously $m(a, a) = 0$. Assume $a \neq b$.

To prove (24) we conclude first from (22) and (23) that

$$a(\beta) = a(\alpha)(\beta/\alpha)$$

hence

$$1/2 < a(\alpha)b(\alpha)\beta/a(\beta)b(\beta)\alpha < 3/2,$$

so that

$$0 < \liminf_{\beta \to 0+} \beta^{-1}a(\beta)b(\beta) < \limsup_{\beta \to 0+} \beta^{-1}a(\beta)b(\beta) < \infty.$$

Choose $\beta_\nu \to 0+$ such that $\lim \beta_\nu^{-1}a(\beta_\nu)b(\beta_\nu)$ exists and let $\alpha_\nu \to 0+$. Denote by β'_ν the smallest $\beta_\rho > \alpha_\nu$, so that $\beta'_\nu \to 0+$. Then (22) and D_p yield

$$\lim_\nu a(\alpha_\nu)b(\alpha_\nu)\alpha_\nu^{-1}/a(\beta'_\nu)b(\beta'_\nu)\beta'^{-1}_\nu = 1.$$

Therefore the above limit $m(a, b)$ in (25) exists and is different from 0 and ∞. The symmetry and triangle inequality for ab yield the same properties for $m(a, b)$.

The equivalence of the metrics ab and $m(a, b)$ follows from (23) by letting $\beta \to 0+$.

If $a_\nu \to p, b_\nu \to p$ and $a_\nu \neq b_\nu$ choose $\beta_\nu < \nu^{-1}$ such that

$$|m(a_\nu, b_\nu) - \beta_\nu^{-1} a_\nu(\beta_\nu)b_\nu(\beta_\nu)| < \nu^{-1}a_\nu b_\nu.$$

After dividing this equation by $a_\nu b_\nu$ the condition D_p yields $m(a_\nu, b_\nu)/a_\nu b_\nu \to 1$. The relation (26) follows from (22).

Under the usual assumptions of differential geometry $m(a, b)$ is, of course, part of the *normal Minkowskian metric* of the space at p[6]. In order to obtain all of this space we define $a(\beta)$ for $ap = \delta$ and $0 \leqq \beta \leqq 1$ as before. For $1 < \beta < \infty$ we define a new point $a(\beta)$ not in the given space and metrize the set of all $a(\beta)$, $ap = \delta$, by

$$m(a_1(\beta_1), a_2(\beta_2)) = \alpha m(a_1(\beta_1/\alpha), a_2(\beta_2/\alpha))$$

where α is so large that $\beta_i/\alpha \leqq \delta$. We conclude from (22) and (25) that this definition is independent of the choice of α and consistent for $a_i p \leqq \delta$. Moreover (22), (25) and (26) hold in all of the new space, which we call the normal tangential space T_p at p.

The space T_p is finitely compact: if $a_\nu \in T_p$ and $m(a_\nu, p) < N$, then $m[a_\nu(\delta N^{-1}), p] < \delta$, so that the points $a_\nu(\delta N^{-1})$ have an accumulation point a and $a(\delta N^{-1})$ is an accumulation point of a_ν.

We prove next

(27) *For a given $\rho > 0$ and given points a, b in T_p there exists a point c such*

6) For the following statements see [2, Section 15].

that

$$m(a,c) = \rho m(a,b) \text{ and either } m(a,c) + m(c,b) = m(a,b),$$
$$\text{or } m(a,c) + m(b,c) = m(a,c)$$

(i. e., c is a point $b(\rho, a)$ for the metric m, but we avoid this notation to prevent confusion).

PROOF. If β is sufficiently small then

$$d^\beta = b(\beta)(\rho, a(\beta))$$

exists; we put $c^\beta = d^\beta(\beta^{-1})$. From

$$m(p,d^\beta) = pd^\beta \leqq pa(\beta) + a(\beta)d^\beta = pa(\beta) + \rho a(\beta)b(\beta)$$
$$\leqq (1+\rho)pa(\beta) + \rho pb(\beta) = [(1+\rho)\, m(\beta a) + \rho m(p,b)]\beta$$

follows

$$m(p,c^\beta) \leqq (1+\rho)\ m(p,a) + \rho\ m(p,c).$$

Hence the points $\{c^\beta\}$ are bounded. We show, and emphasize for a later application, that every limit c of a converging sequence c^{β_ν} with $\beta_\nu \to 0+$, (i. e., $m(c^{\beta_\nu}, c) \to 0$) satisfies the assertion. As in the preceding proof $\beta_\nu^{-1}c((\beta_\nu)$ $d^{\beta_\nu} \to 0$, hence

$$a(\beta_\nu)d^{\beta_\nu} - c(\beta_\nu)d^{\beta_\nu} < a(\beta_\nu)c(\beta_\nu) < a(\beta_\nu)d^{\beta_\nu} + c(\beta_\nu)d^{\beta_\nu}$$

yield

$$m(a,c) = \lim_{\nu\to\infty} \beta_\nu^{-1}a(\beta_\nu)c(\beta_\nu) = \lim \beta_\nu^{-1}a(\beta_\nu)d^{\beta_\nu}$$
$$= \rho \lim \beta_\nu^{-1}\, a(\beta_\nu)b(\beta_\nu) = \rho\ m(a,b).$$

Similarly we see that $m(a,c) + m(c,b) = m(a,b)$ for $\rho < 1$ and $m(a,b) + m(b,c) = m(a,c)$ for $\rho > 1$.

The space T_p is therefore convex in Menger's sense and prolongation is possible in the large (for the terminology see [2, pp. 28, 33]). It will be straight if prolongation is unique, i. e., the point c is unique for $\rho > 1$. It is then also unique for $\rho < 1$, [2, pp. 38, 39]. Under the usual assumptions of differential geometry, the space T_p is Minkowskian, hence straight if, and only if, the unitsphere of the Minkowski metric is strictly convex, see [2, Section 17]. In that case the space, or its line element $ds = F(x, dx)$, or also the function $F(x, \xi)$ is called regular at the point p. We therefore introduce the *Regularity Condition at* p:

R_p: *Prolongation is unique in the normal tangential space* T_p.

Regularity presupposes that T_p is defined, hence that D_p holds. Our further investigation could be carried out without assuming regularity, and this would be justified by the fact that there are G-spaces of class C^∞, for instance quasi-hyperbolic planes (see [2, Appendices (46)]), which are nowhere regular. However, matters would become much more complicated.

Because of (26) it suffices to postulate R_p in $px < \delta$ or, which is the same, $m(\rho, x) < \delta$. Therefore R_p reads in terms of the original metric:

R_p: If a, b, c_1, c_2 lie in $S(p,\delta)$, $a \neq b$, and $m(x,y) = \lim_{\beta\to 0+} \beta^{-1}x(\beta,p)y(\beta,p)$, then

$$m(a,b)+m(b,c_i)=m(a,c_i) \text{ and } m(b,c_1)=m(b,c_2)$$

imply $c_1=c_2$.

In the first example of the preceding section the space and T_p are identical for every p on the x_1-axis, in the second they are identical for $p=(0,0)$ and the space is everywhere differentiable and regular. Thus

(28) THEOREM. *Differentiability and regularity of a G-space at a point p do not imply that the normal tangential space T_p is Minkowskian. Differentiability and regularity at p (or everywhere) and the existence of global similarities with center p and arbitrary dilation factors do not imply that the space is Minkowskian.*

5. **Continuous Differentiability**. In the examples the space is differentiable at $p=(0,0)$, but, one feels, not continuously so. This can easily be made precise by observing that, if the function $f(x)$ in (21) has continuous first partial derivatives at z, then the fixed z in (21) may, without impairing the validity of (21), be replaced by a variable point z_ν tending to z. Thus we are led to the following definition:

The space is continuously differentiable at p if

$$CD_p:\lim_{\nu\to\infty}\frac{a_\nu(\beta_\nu,p_\nu)b_\nu(\beta_\nu,p_\nu)}{\beta_\nu a_\nu b_\nu}=1 \textit{ for } a_\nu\to p,\ b_\nu\to p,\ a_\nu\neq b_\nu,\ p_\nu\to p$$
$$\textit{and } 0<\beta_\nu<M.$$

With CD_p replacing D_p both the occurrences in (28) become impossible: First we discuss the generalization of the result mentioned by Yano:

(29) THEOREM. *Let the G-space R possess a local similarity with dilation factor $k\neq 1$, and assume R to be continuously differentiable at a fixed point of the similarity (which is unique for $k<1$). When $k<1$ then R is Minkowskian; when $k>1$ then the universal covering space of R is Minkowskian.*

Theorems (18), (19) and the remark after (19) reduce (29) to the following assertion: A straight space is Minkowskian if it possesses a global similarity α on itself with $k\neq 1$ and is continuously differentiable at the center of α. We may assume $k<1$. The space will be Minkowskian, see [2, p. 237 and p. 261], if for any three distinct points a,b,c and the midpoints b' of a,b (i.e., $ab=2ab'=2b'b$) and c' of a,c the relation $2b'c'=bc$ holds. This is very easy to see.

If $x_\nu=x\alpha_\nu$ then

$$b'_\nu c'_\nu/b_\nu c_\nu=k\,b'_{\nu-1}c'_{\nu-1}/kb_{\nu-1}c_{\nu-1}=b'_{\nu-1}c'_{\nu-1}/b_{\nu-1}c_{\nu-1}.$$

If p is the fixed point of α, then a_ν, b_ν, c_ν tend to $p, b_\nu\neq c_\nu$ and $c'_\nu=c_\nu(2^{-1},a_\nu)$, $b'_\nu=b_\nu(2^{-1},a_\nu)$ because α is a global similarity. Therefore CD_p yields

$$2b'c'/bc=\lim 2b'_\nu c'_\nu/b_\nu c_\nu=1.$$

Next we show:

(30) THEOREM. *If a G-space is continuously differentiable and regular at p*

(i. e., satisfies CD_p and R_p), then the normal tangential metric at p is Minkowskian.

As in the previous proof it suffices to show that

$$m(a,b) = 2m(a,b') = 2m(b',b) \text{ and } m(a,c) = 2m(a,c') = 2m(c',c)$$

imply

$$m(b,c) = 2m(b',c').$$

Because of (26) we may assume that the points a, b, c, b', c' lie in $S(p, \delta)$.

Put $d_\beta = b(\beta)(2^{-1}, a(\beta))$, $e_\beta = c(\beta)(2^{-1}, a(\beta))$. In the proof of (27) we established that any converging sequence $d_{\beta_\nu}(\beta_\nu^{-1})$ with $\beta_\nu \to 0+$ tends to a midpoint of a and b. Because of R_p this midpoint is unique, namely b', hence $\lim_{\beta\to 0+} d_\beta(\beta^{-1}) = b'$ and similarly $\lim_{\beta\to 0+} e_\beta(\beta^{-1}) = c'$. The condition CD_p yields

$$\lim_{\beta\to 0+} 2d_\beta e_\beta / b(\beta)c(\beta) = 1,$$

or

$$m(b,c) = \lim_{\beta\to 0+} \beta^{-1} b(\beta)c(\beta) = 2 \lim \beta^{-1} d_\beta e_\beta.$$

But

$$b'(\beta)c'(\beta) - d_\beta b'(\beta) - e_\beta c'(\beta) \leqq d_\beta e_\beta \leqq b'(\beta)c'(\beta) + d_\beta b'(\beta) + e_\beta c'(\beta)$$

and by (23)

$$d_\beta b'(\beta) \leqq 3\beta^{-1} d_\beta(\beta^{-1})b'.$$

Therefore $\beta^{-1} d_\beta b'(\beta) \to 0$ and similarly $\beta^{-1} e_\beta c'(\beta) \to 0$ so that

$$2 \lim \beta^{-1} d_\beta e_\beta = 2 \lim \beta^{-1} b'(\beta)c'(\beta) = 2m(b',c').$$

This theorem has the corollary:

(31) *A G-space which is continuously differentiable and regular at one point p, is a topological manifold.*

For $m(a,b)$ and ab are topologically equivalent metrizations of $S(p,\delta)$, and $S(p,\delta)$ is, because of $m(a,p) = ap$, an open sphere in the Minkowski space, therefore homeomorphic to E^n for a suitable finite n. It follows from [2, Theorem (10. 1)] that every point of the space has a neighborhood homeomorphic to E^n.

The methods which we have developed give a partial answer to the following very interesting problem:

Consider a Finsler space of class C^n with an $F(x,\xi)$ of class C^{n-1}, $\infty \geqq n \geqq 4$ and the usual conditions on $F(x,\xi)$, see [2, Section 15] or [4]. By passing from an admissible coordinate system to a non-admissible one, we can destroy some or all of the smoothness properties of the metric in terms of coordinates; distance may not even be definable in terms of a function $F(x,\xi)$. But the intrinsic distances have not changed. *How can we recognize from the intrinsic distances that this space can be written as a manifold of class C^n with a metric derived from a function $F(x,\xi)$ of class C?*

Our condition CD_p does not use coordinates and will therefore be satisfied. Normal coordinates of the space at p are, as usual, defined in $S(p,\delta)$ as affine coordinates with origin p belonging to the local Minkowskian geometry, see [3] or [4]. These coordinates are of class C^{n-2}, see [4, pp. 90 ss], except at the origin p where they are in general only of class C', see [3]. Hence if q is a given point we merely need to choose $p \neq q$ such that q lies in a neighborhood of p where normal coordinates are defined. In terms of these coordinates the space is of class C^{n-2}. The function $F(x, \xi)$ is determined by the given intrinsic distance. Thus our methods *solve the problem completely in the case of* C^∞ *and allow us to recover class* C^{n-2}, *if the original space was of class* C^n. No method for reestablishing class C^n is known.

The same problem may be put differently: *given a G-space which is continuously differentiable and everywhere, to find out whether the space is a Finsler space of a certain class.* We procure coordinates at q as before. If q has normal coordinates belonging to different points p_1, p_2, distinct from q, then the transition functions from one system to the other will be class C^n if the space can be obtained as above as a Finsler space of class C^{n+2} with an $F(x, \xi)$ of class C^{n+1}. Thus our methods, together with the classical methods of [4] allow us to establish that the space is of class C^n, when it is optimally of class C^{n+2}. Again, we have a complete answer for $n = \infty$.

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