A GENERAL APPROXIMATION METHOD OF EVALUATING MULTIPLE INTEGRALS

LEE-TSCH C. HSU

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The object of this paper is to investigate a general method concerning the approximate evaluation of multiple integrals with periodic continuous functions as integrands. As it will be shown, our general method has some practical advantage when applied especially to the cases of double integrals and triple integrals which are taken over circular regions and spherical domains respectively.

Throughout the paper we always denote, for a continuous function $f(x_1, \dots, x_n)$ defined on a certain n-dimensional domain D,

$$M_f = \max_D |f(x_1, \dots, x_n)|,$$

$$\omega_f(\delta_1, \dots, \delta_n) = \max_D |f(x_1, \dots, x_n) - f(x_1', \dots, x_n')|, \quad (|x_i - x_i'| \leqq \delta_i)$$

where the maxima are all taken with respect to $(x_1, \dots, x_n) \in D$, $(x_1', \dots, x_n') \in D$, the letter one being restricted to $|x_i - x_i'| \leqq \delta_i$ $(1 \leqq i \leqq n)$. Moreover, for a real number $x > 0$, we shall always denote its decimal part by $<x>$, i.e. $<x> = x - [x]$, $[x]$ being the integral part of x.

1. A fundamental lemma and its consequences. Hereafter D_n is always used to denote an n-dimensional hypercubic domain of $(x_1, \dots, x_n)$ in euclidean n-space, namely, D_n; $0 \leqq x_1 \leqq 1, \dots, 0 \leqq x_n \leqq 1$. We are now going to establish a useful lemma which actually forms a basis of our method.

LEMMA. *Let $f(x_1, \dots, x_k, y_1, \dots, y_k)$ be any continuous function defined on D_{2k}. Then for all positive integers $N_i \geqq 2$ $(i = 1, \dots, k)$ we have*

$$(1) \qquad \left| \int_{D_{2k}} f(x, y)\, dx\, dy - \int_{D_k} f(x, <Nx>)\, dx \right| \leqq 2\omega_f\left(\frac{1}{N_1}, \dots, \frac{1}{N_k}, 0, \dots 0, \right),$$

where x, dx etc. are abbreviations for $(x_1, \dots, x_k)$, $dx_1 \dots dx_k$ etc. respectively; and $<Nx>$ stands for $<N_1x_1>, \dots, <N_kx_k>$.

As is easily seen, the meaning of this lemma is that it replaces the $2k$-fold integral by the k-fold integral with an error estimation expressed by the modulus of continuity $\omega_f(N_1^{-1}, \dots, N_k^{-1}, 0, \dots, 0)$. In order to save space in its proof, we have to adopt more abbreviations here, e.g.

$$\int_{(\nu-1)x}^{\nu\Delta x} \text{ stands for } \int_{(\nu_1-1)\Delta x_1}^{\nu_1\Delta x_1} \cdots \int_{(\nu_k-1)\Delta x_k}^{\nu_k\Delta x_k}; \quad \sum_\nu \text{ stands for } \sum_{\nu_1=1}^{N_1} \cdots \sum_{\nu_k=1}^{N_k};$$

$$f\left(\frac{\nu-1}{N}, y\right) \text{ stands for } f\left(\frac{\nu_1-1}{N_1}, \ldots, \frac{\nu_k-1}{N_k}, y_1, \ldots, y_k\right).$$

Let us now define

$$I(N) = \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} f(x, <Nx>)\, dx, \quad \left(\Delta x_1 = \frac{1}{N_1}, \ldots, \Delta x_k = \frac{1}{N_k}\right),$$

$$A(N) = \sum_\nu \int_{\nu-1}^{\nu} f\left(\frac{\nu-1}{N}, <y>\right) \frac{1}{N_1 \ldots N_k}\, dy,$$

$$B(N) = \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} dx \int_0^1 f(x,y)\, dy,$$

$$C(N) = \int_{(\nu-1)\Delta x}^{\nu\Delta x} dx \int_0^1 f\left(\frac{\nu-1}{N}, y\right) dy,$$

$$D(N) = \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} f\left(\frac{\nu-1}{N}, <Nx>\right) dx.$$

It is clear that $C(N) = A(N) = D(N)$ and

$$I(N) = \int_{D_k} f(x, <Nx>)\, dx, \quad B(N) = \int_{D_{2k}} f(x,y)\, dx\, dy.$$

Putting $\rho_1(N) = I(N) - A(N)$ and $\rho_2(N) = A(N) - B(N)$, we may write $I(N) = \rho_1(N) + \rho_2(N) + B(N)$; so it follows that

$$\left| \int_{D_k} f(x, <Nx>)\, dx - \int_{D_{2k}} f(x,y)\, dx\, dy \right| \leqq |\rho_1(N)| + |\rho_2(N)|.$$

Now we have to estimate $|\rho_1(N)|$ and $|\rho_2(N)|$. In fact, we have

$$|\rho_1(N)| = |I(N) - A(N)| = |I(N) - D(N)|$$

$$\leqq \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} \left| f(x, <Nx>) - f\left(\frac{\nu-1}{N}, <Nx>\right) \right| dx$$

$$\leqq \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} \omega_f\left(\frac{1}{N_1}, \ldots, \frac{1}{N_k}, 0, \ldots, 0\right) dx$$

$$= \omega_f\left(\frac{1}{N_1}, \ldots, \frac{1}{N_k}, 0, \ldots, 0\right)$$

$$|\rho_2(N)| = |A(N) - B(N)| = |C(N) - B(N)|$$

$$\leqq \sum_\nu \int_{(\nu-1)\Delta x}^{\nu\Delta x} dx \int_0^1 \left| f\left(\frac{\nu-1}{N}, y\right) - f(x,y) \right| dy$$

$$\leqq \sum_{\nu} \int_{(\nu-1)\Delta x}^{\nu\Delta x} dx \int_0^1 \omega_f\left(\frac{1}{N_1}, \ldots, \frac{1}{N_k}, 0, \ldots, 0\right) dy$$

$$= \omega_f\left(\frac{1}{N_1}, \ldots, \frac{1}{N_k}, 0, \ldots, 0\right).$$

This completes the proof of our lemma.

For $k = 1$ and $k = 2$ the lemma yields some simple but interesting consequences. We now mention a few of them as follows:

COROLLARY 1. *For any continuous function $f(x,y)$ of two variables defined on $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, we have*

$$\lim_{N\to\infty} \int_0^1 f(x, \langle Nx\rangle)\, dx = \int_0^1 \int_0^1 f(x,y)\, dx\, dy, \tag{2}$$

where $N \to \infty$ through the sequence of positive integers.

COROLLARY 2. *Let $f(x,y)$ be a continuous function defined on $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and satisfying the following Lipschitz condition*

$$|f(x,y) - f(x',y)| \leqq A|x - x'|, \quad (0 \leqq x \leqq x' \leqq 1)$$

where A is an absolute constant independent of y in $0 \leqq y \leqq 1$. Then for all positive integers $N \geqq 2$ we have

$$\left| \int_0^1 \int_0^1 f(x,y)\, dx\, dy - \int_0^1 f(x, \langle Nx\rangle)\, dx \right| \leqq \frac{2A}{N}. \tag{3}$$

COROLLARY 3 (*An improvement of Maréchal-Wilkins' theorem*). *Let $g(r, \theta)$ be a continuous function defined on the circular region $S(0 \leqq r \leqq R, 0 \leqq \theta \leqq 2\pi)$, (r, θ) denoting polar coordinates. Denote*

$$\omega_g(\delta, \delta') = \max_S |g(r,\theta) - g(r',\theta')|, \quad (|r - r'| \leqq \delta, \quad |\theta - \theta'| \leqq \delta').$$

Then for all positive integers $N \geqq 2$ we have

$$\left| \iint_S g(r,\theta)\, dS - 2\pi \int_0^R g\left(r, \frac{2N\pi r}{R}\right) r\, dr \right| \tag{4}$$

$$\leqq 4\pi R^2 \left\{ (R+1)\,\omega_g\left(\frac{1}{N}, 0\right) + M_g \frac{1}{N} \right\},$$

where $M_g = \max|g(r, \theta)|$, $g(r,\theta) = g(r, \theta + 2\pi)$, $(0 \leqq \theta < +\infty)$.

PROOF. Making the substitutions

$$r_1 = \frac{1}{R} r, \quad \theta_1 = \frac{1}{2\pi} \theta, \quad g(Rr_1, 2\pi\theta_1) r_1 = f(r_1, \theta_1),$$

and noticing that $f(r_1, \theta_1)$ is of period 1 in θ_1, we get

$$\iint_S g(r,\theta)\, dS = 2\pi R^2 \int_0^1 \int_0^1 f(r_1, \theta_1)\, dr_1\, d\theta_1, \tag{5}$$

$$(6)\qquad 2\pi\int_0^R g\left(r,\ 2N\pi\frac{r}{R}\right)r\,dr = 2\pi R^2\int_0^1 f(r_1,\ Nr_1)\,dr_1 = 2\pi R^2\int_0^1 f(r_1, <Nr_1>)dr_1.$$

Denoting the plane region $0\leqq x\leqq 1,\ 0\leqq y\leqq 2\pi$ by Δ, we clearly have

$$\begin{aligned}\omega_f\left(\frac{1}{N}, 0\right) &= \max_\Delta |g(Rx,y)x - g(Rx',y)x'|, \qquad \left(|x-x'|\leqq\frac{1}{N}\right),\\ &\leqq \max_\Delta |g(Rx,y)-g(Rx',\ y)|x + \max_\Delta |g(Rx',y)|\cdot|x-x'|\\ &\leqq \omega_g\left(\frac{R}{N},\ 0\right) + M_g\cdot|x-x'|\\ &\leqq (R+1)\omega_g\left(\frac{1}{N}, 0\right) + M_g\cdot\frac{1}{N}.\end{aligned}$$

Thus the inequality (4) is implied by (1) in view of (5) and (6).

It is known that J. E. Wilkins has given a proof by means of Fejér's theorem on Fourier series, justifying the limiting process of A. Maréchal (See [1] and [2]; cf. also Grosswald [3]):

$$\frac{1}{2\pi}\iint_S g(r,\theta)dS = \lim_{\rho\to 0+}\int_0^R g(r, r/\rho)(r^2+\rho^2)^{\frac{1}{2}}dr = \lim_{\rho\to 0+}\int_0^R g(r,r/\rho)r\,dr.$$

Apparently our Corollary 3 is more precise than this original result; yet its proof here presented seems more elementary than the original one.

It may be noteworthy that a slight modification of the proof of our lemma (taking for instance $k=1$ and $\Delta x = \Delta x_1 = \frac{1}{N}(\beta-\alpha)$ etc.) will easily show that the formula (2) can be slightly generalized to the following form

$$(7)\qquad \lim_{N\to\infty}\int_\alpha^\beta f(x, <Nx>)\,dx = \int_\alpha^\beta dx\int_0^1 f(x,y)dy,$$

where $f(x,y)$ is assumed to be bounded and continuous in $\alpha < x < \beta,\ 0\leqq y \leqq 1$.

This remark enables us to state the

COROLLARY 4. *For any continuous function $f(x,y,z)$ defined on $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, $0\leqq z\leqq 1$, we have*

$$(8)\qquad \lim_{\mu\to\infty}\lim_{\lambda\to\infty}\int_0^1 f(x, <\lambda x>, <\mu x>)\,dx = \int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz,$$

where $\lambda\to\infty$, $\mu\to\infty$ are assumed through the sequence of positive integers.

PROOF. First, an application of our lemma with $k=2$, $N_1=N_2=N$ gives at once the result

$$\int_0^1\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz\,du = \lim_{\mu\to\infty}\int_0^1\int_0^1 f(x,y, <\mu x>)\,dx\,dy.$$

Now $\phi_\mu(x, y) = f(x,y, <\mu x>)$ is obviously a bounded function, continuous in each rectangular region $\frac{\nu-1}{\mu} < x < \frac{\nu}{\mu}$, $0 \leqq y \leqq 1$, $(\nu = 1, 2, \ldots, \mu)$. Thus, by applying (7) we obtain

$$\int_0^1 \int_0^1 f(x,y, <\mu x>) dx\, dy = \sum_{\nu=1}^{\mu} \int_{(\nu-1)/\mu}^{\nu/\mu} dx \int_0^1 f(x,y, <\mu x>) dy$$

$$= \sum_{\nu=1}^{\mu} \lim_{\lambda\to\infty} \int_{(\nu-1)/\mu}^{\nu/\mu} f(x, <\lambda x, >, <\mu x>)\, dx = \lim_{\lambda\to\infty} \int_0^1 f(x, <\lambda x>, <\mu x>) dx.$$

This is what we need to show.

As a final remark, we mention that the order estimation $O(N^{-1})$ for the left-hand side of (3) cannot be improved. In fact, taking $f(x,y) = xy$ for instance, we easily find

$$\left| \int_0^1 \int_0^1 xy\, dx\, dy - \int_0^1 x < Nx > dx \right| = \frac{1}{12N}, \quad (N = 2, 3, \ldots)$$

the order being precisely of N^{-1} as $N \to \infty$.

2. A reduction principle. Since any $(2k-1)$-fold integral taken over D_{2k-1} can always be expressed as a $2k$-fold integral over the domain D_{2k}, we see that a successive application of the fundamental lemma (with the case $N_1 = N_2 = \ldots = N_k$) enables us to state the following principle:

Let $F(x_1, \ldots, x_n)$ $(n \geqq 3)$ be a continuous function defined on D_n $(0 \leqq x_i \leqq 1)$ and let it be periodic in $x_3, \ldots, x_n$ with the periods all equal to 1. Let s be an integer such that $2^{s-1} < n \leqq 2^s$. Then the multiple integral $\int_{D_n} F(x)\, dx$ can always be reduced to a certain definite integral $\int_0^1 \psi(x_1)\, dx_1$ with an error estimation, $\rho(N_1, \ldots, N_s)$ say, such that $\rho(N_1, \ldots, N_s) \to 0$ as $N_s \to \infty, \ldots, N_1 \to \infty$ successively, where $\psi(x_1)$ is a piece-wisely continuous function of x_1 involving the s integral parameters $N_1, \ldots, N_s$. *More precisely, we may express*

$$\int_{D_n} F\, dx_1 \ldots dx_n = \int_0^1 \psi(x_1) dx_1 + \rho(N_1, \ldots, N_s), \tag{9}$$

where the function $\psi(x_1)$ is of the form $F(x, y_1, \ldots, y_{n-1})$, $(y_1, \ldots, y_{n-1})$ being a certain permutation of $(n-1)$ symbols out of the $2^s - 1$ symbols below:

$$<N_{\nu_1}<N_{\nu_2}< \ldots <N_{\nu_t} x_1> \ldots >>>, \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_t \leqq s;\ 1 \leqq t \leqq s).$$

In fact, it is not difficult to verify that the periodicity assumption upon $x_3, \ldots, x_n$ sufficiently ensures the piece-wise continuity of $F(x_1, y_1, \ldots, y_{n-1})$ in x_1. A simple instance with $n = 4$ will suffice to illustrate this point, as its reasoning is true in general. If $f(x, y, z, u)$ is a continuous function defined on D_4 with the property that $f(x, y, 0, u) = f(x, y, 1, u)$, $f(x, y, z, 0) = f(x, y, z, 1)$, then $\phi(x, y) = f(x, y, <\lambda x>, <\mu x>)$ is easily verified to be a continuous function of (x, y) and $\psi(x) = \phi(x, <\nu x>)$ is piece-wisely continuous in x, where λ, μ, ν are arbitrary integers $\geqq 2$. Also, we easily observe that the further assumption $f(x, 0, z, u) = f(x, 1, z, u)$ will imply the continuity of $\psi(x)$.

As an illustration of the principle, we now give the following result, corresponding to the case $n = 3$:

THEOREM 1. *Let $f(x, y, z)$ be any continuous function defined on D_3 with $f(x, y, 0) = f(x, y, 1)$. Then we may express*

$$\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dx\, dy\, dz = \int_0^1 f(x, <\mu x>, <\lambda x>)\, dx + \rho(\lambda, \mu), \tag{10}$$

where $|\rho(\lambda, \mu)| \leqq 2\{\omega_\phi(\mu^{-1}, 0) + \omega_f(\lambda^{-1}, 0, 0)\}$, and $\phi = \phi_\lambda = f(x, y, <\lambda x>)$ is a continuous function of (x, y), μ, λ being any positive integers $\geqq 2$.

This is obviously a refinement of Corollary 4 in § 1; and its proof is easily completed by repeated application of the fundamental lemma. Moreover, it is clear that $\lim_{\lambda\to\infty} \lim_{\mu\to\infty} \rho(\lambda, \mu) = 0$; since

$$\lim_{\mu\to\infty} \omega_\phi\left(\frac{1}{\mu}, 0\right) = 0, \quad \lim_{\lambda\to\infty} \omega_f\left(\frac{1}{\lambda}, 0, 0\right) = 0.$$

Similar results for the cases $n = 4$ and $n = 5$ can also be obtained by means of our principle. Finally, we recall that a definite integral of a continuous function can always be evaluated approximately by various numerical methods; so that our reduction principle actually asserts a general way of approximately evaluating multiple integrals of periodic continuous functions extended over hypercubic domains.

3. **Some approximation formulas.** For the sake of practical application, we are particularly interested in the approximate evaluation of double integrals and triple integrals. In this section we shall make use of Corollarly 3 (§ 1) and Theorem 1 (§ 2) to obtain some approximation formulas for integrals of the forms:

$$I = \iint_S g(r, \theta)\, dS, \qquad J = \iiint_\sigma F(r, \varphi, \theta)\, d\sigma,$$

where S and σ denote respectively the circular region $S(0 \leqq r \leqq R,\ 0 \leqq \theta \leqq 2\pi)$ and the spherical domain $\sigma(0 \leqq r \leqq R,\ 0 \leqq \varphi \leqq \pi,\ 0 \leqq \theta \leqq 2\pi)$, (r, θ) and (r, φ, θ) being known as polar coordinates and spherical coordinates respectively.

Denote by $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ the zeros of the Legendre polynomial $P_n(x) = \frac{1}{n!2^n} D^n(x^2 - 1)^n$, $(n = 1, 2, \ldots)$ and let

$$A_k^{(n)} = \frac{1}{[1-(\xi_k^{(n)})^2][P_n'(\xi_k^{(n)})]^2}, \quad x_k^{(n)} = \frac{1}{2}\left(1+\xi_k^{(n)}\right), \qquad (1 \leqq k \leqq n).$$

Then by appealing to Gauss' formula for numerical integrations (See for example, [4; pp. 605–611] and [5; pp. 85–86]), we may immediately deduce a theorem from Corollary 3, viz.

THEOREM 2. *For every continuous function* $g(r,\theta)$ *defined on* $S(0 \leqq r \leqq R,$ $0 \leqq \theta \leqq 2\pi)$ *we have*

$$\iint_S g(r,\theta)\,dS = \lim_{n\to\infty} 2\pi R^2 \sum_{k=1}^{n} A_k^{(n)} g(Rx_k^{(n)},\ 2\pi N x_k^{(n)}) x_k^{(n)} + \rho(N) \tag{11}$$

with

$$|\rho(N)| \leqq 4\pi R^2 \left\{(R+1)\omega_g\left(\frac{1}{N}, 0\right) + M_g \cdot \frac{1}{N}\right\}, \tag{12}$$

where $M_g = \max|g(r,\theta)|$, $g(r,\theta) = g(r, \theta + 2\pi)$, $(0 \leqq \theta < +\infty)$, $N = 2, 3, 4, \ldots\ldots$

In fact it is well-known that (cf. [4; p. 605])

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k^{(n)} \cdot g(Rx_k^{(n)},\ 2\pi N x_k^{(n)}) \cdot x_k^{(n)} = \int_0^1 g(Rx,\ 2\pi N x)\, x\, dx.$$

Therefore (11) is inferred from (4).

For large N and n, we may write, in view of (11) and (12),

$$\iint_S g(r,\theta)\,dS \approx 2\pi R^2 \sum_{k=1}^{n} A_k^{(n)} g(Rx_k^{(n)},\ 2\pi N x_k^{(n)}) \cdot x_k^{(n)}. \tag{11'}$$

This formula applies to every continuous function (with period 2π in θ) defined on S, provided that both R and M_g are relatively very small when compared with N. In practical calculation, we usually take N to be very large in order to make $\rho(N)$ very small. As to numerical values of $A_k^{(n)}, x_k^{(n)}$ $(1 \leqq n \leqq 8)$, one may refer to [5: pp. 85–86] or elsewhere.

EXAMPLE. Assuming $R = 1$ and taking $N = 10,000$ and $n = 8$, we may gain at once the following numerical integration formula from (11)$'$:

$$\begin{aligned}\frac{1}{2\pi}\iint_S g(r,\theta)dS \approx\ & 0.0010 \times g(0.0199, 3.4601) + 0.0113 \times g(0.1017, 4.1947) + \\ & + 0.0372 \times g(0.2372, 2.1230) + 0.0740 \times g(0.4083, 5.1949) + \\ & + 0.1073 \times g(0.5917, 1.0882) + 0.1197 \times g(0.7628, 4.1601) + \\ & + 0.0999 \times g(0.8983, 2.0885) + 0.0496 \times g(0.9801, 2.8230).\end{aligned}$$

We have purposely worked out a simple numerical example in which $z = g(r,\theta)$ just represents a hemispherical surface with radius 1 and centre at the origin of the (r,θ)-plane ((r,θ,z) being understood as the cylindrical coordinates), showing that the error produced by the above formula is less than 0.002. It is believed that the accuracy of such a kind of formulas can be considerably increased by taking $N=1,000,000$ and $n=10$ or 12.

In order to obtain an approximatoin formula for the triple integral J_r we need first to establish the

THEOREM 3. *Let $f(x,y,z)$ be any continuous function defined on D_3 with $f(x,y,0)=f(x,y,1)$. Then for all integers $N\geqq 2$ we have*

$$\left|\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz-\int_0^1 f(x,\ <N^2x>,\ <Nx>)\,dx\right| \tag{13}$$
$$\leqq 8\omega_f\left(\frac{1}{N},\ 0,\ \frac{1}{N}\right).$$

PROOF. By Theorem 1 we obtain

$$\left|\int_0^1\int_0^1\int_0^1 f\,dx\,dydz-\int_0^1 f(x,\ <N^2x>,\ <Nx>)\,dx\right| \tag{14}$$
$$\leqq 2\omega_f\left(\frac{1}{N},\ 0,0\right)+2\omega_\phi\left(\frac{1}{N^2},\ 0,\ \right),$$

where $\phi=\phi(x,y,\ <Nx>)$. Let us estimate $|\omega_\phi(N^{-2},0)|$. Define the intervals $I_k=[(k-1)/N,\ k/N]$, $(k=1,2,\ldots.N)$. For a pair of points x,x' belonging to the interior of some I_k with $|x-x'|\leqq N^{-2}$, we clearly have $|<Nx'>-<Nx>|\leqq N^{-1}$; so that

$$|f(x',y,\ <Nx'>)-f(x,y,<Nx>)|\leqq\omega_f\left(\frac{1}{N^2},0,\ \frac{1}{N}\right).$$

For other possible cases we may assume $x\in I_k,\ x'\in I_{k+1},\ |x-x'|\leqq N^{-2}$, and we may write $x=\dfrac{k}{N}-\varepsilon_1, x'=\dfrac{k}{N}+\varepsilon_2$, $(\varepsilon_1\geqq 0,\ \varepsilon_2\geqq 0,\varepsilon_1+\varepsilon_2\leqq N^{-2})$. Then we find

$$\begin{aligned}
&|f(x',y,<Nx'>)-f(x,y,<Nx>)|\\
&\leqq|f(x',y,<Nx'>)-f(x,y,<Nx'>)|+|f(x,y,<Nx'>)-f(x,y,\ <Nx>)|\\
&\leqq\omega_f\left(\frac{1}{N^2},0,\ 0\right)+|f(x,y,N\varepsilon_2)-f(x,y,1-N\varepsilon_1)|\\
&\leqq\omega_f\left(\frac{1}{N^2},\ 0,0\right)+|f(x,y,N\varepsilon_2)-f(x,y,0)|+|f(x,y,1)-f(x,y,1-N\varepsilon_1)|\\
&\leqq\omega_f\left(\frac{1}{N^2},\ 0,\ 0\right)+2\omega_f\left(0,\ 0,\ \frac{1}{N}\right)\leqq 3\omega_f\left(\frac{1}{N},\ 0,\ \frac{1}{N}\right).
\end{aligned}$$

Thus we can conclude that $\omega_\phi(N^{-2},0)\leqq 3\omega_f(N^{-1},0,N^{-1})$. The inequality (13) is therefore inferred from (14).

For a continuous function $F(r,\varphi,\theta)$ defined on the spherical domain $\sigma(0\leqq r\leqq k,\ 0\leqq\varphi\leqq\pi,\ 0\leqq\theta\leqq 2\pi)$, we can always extend its definition to the region $\sigma^*(0\leqq r\leqq R,\ 0\leqq\varphi<+\infty,\ 0\leqq\theta<+\infty)$ by assuming

$$F(r,\varphi,\theta)=F(r,2\pi-\varphi,\theta),$$
$$F(r,\varphi,\theta)=F(r,\varphi+2k\pi,\theta)=F(r,\varphi,\theta+2k\pi),\qquad(k=1,2,3,\ldots.).$$

That is to say, F can always be made continuous and periodic in φ and θ with periods both equal to 2π. Thus we may express

$$J = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\theta \int_0^R F(r, \varphi, \theta) r^2 \sin\varphi \, dr$$
$$= 2\pi^2 R^3 \int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz,$$

where $f(x,y,z) = F(Rx, 2\pi z, 2\pi y)x^2\sin(2\pi z)$, $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1,\ 0 \leqq z \leqq 1)$.

Now in order to apply Theorem 3, it requires to estimate $\omega_f(N^{-1}, 0, N^{-1})$. Clearly we have (with x, x', z, z' being subjected to $|x - x'| \leqq N^{-1}$, $|z - z'| \leqq N^{-1}$):

$$\omega_f\Big(\frac{1}{N}, 0, \frac{1}{N}\Big) = \max|F(Rx, 2\pi z, 2\pi y)x^2 \sin(2\pi z) - F(Rx', 2\pi z', 2\pi y)x'^2 \sin(2\pi z')|$$
$$\leqq \omega_F\Big(\frac{R}{N}, \frac{2\pi}{N}, 0\Big)|x^2\sin(2\pi z)| + M_F\cdot|x^2\sin(2\pi z) - x'^2\sin(2\pi z')|$$
$$\leqq \omega_F\Big(\frac{R}{N}, \frac{2\pi}{N}, 0\Big) + M_F\cdot|x^2 - x'^2|\,|\sin(2\pi z)| + M_F\cdot|x'|^2\cdot$$
$$|\sin(2\pi z) - \sin(2\pi z')|$$
$$\leqq \omega_F\Big(\frac{R}{N}, \frac{2\pi}{N}, 0\Big) + 2M_F\cdot\frac{1}{N} + 2M_F\cdot|\sin\pi(z - z')\cdot\cos\pi(z + z')|$$
$$\leqq \omega_F\Big(\frac{R}{N}, \frac{2\pi}{N}, 0\Big) + 2M_F\cdot\frac{1}{N} + 2M_F\cdot\Big|\sin\frac{\pi}{N}\Big|$$
$$\leqq (R+1)(2\pi+1)\,\omega_F\Big(\frac{1}{N}, \frac{1}{N}, 0\Big) + (2+2\pi)M_F\cdot\frac{1}{N}, \quad (N \geqq 2).$$

In an analogous manner, if we define $g(x, y, z) = F(Rx, 2\pi y, 2\pi z)x^2\sin(2\pi y)$, then it can be found that

$$\omega_g\Big(\frac{1}{N}, 0, \frac{1}{N}\Big) \leqq (R+1)(2\pi+1)\,\omega_F\Big(\frac{1}{N}, 0, \frac{1}{N}\Big) + 2M_F\cdot\frac{1}{N}$$

Thus we see that Theorem 3 entails the following

THEOREM 4. *Let $F(r,\varphi,\theta)$ be any continuous function defined on the spherical domain $\sigma(0 \leqq r \leqq R,\ 0 \leqq R \leqq \pi, 0 \leqq \theta \leqq 2\pi)$ with $F(r,\varphi,\theta) = F(r, 2\pi - \varphi, \theta)$ and with period 2π in both φ and θ. Then for all integers $N \geqq 2$ we have*

$$\left|\frac{1}{2\pi^2R^3}\iiint_\sigma F(r,\varphi,\theta)\,d\sigma - \int_0^1 F(Rx, 2\pi Nx, 2\pi N^2x)x^2\sin(2\pi Nx)\,dx\right|$$
$$\leqq 8\left\{(R+1)(2\pi+1)\,\omega_F\Big(\frac{1}{N}, \frac{1}{N}, 0\Big) + (2\pi+2)M_F\cdot\frac{1}{N}\right\}. \tag{15}$$

THEOREM 5. *Let $F(r,\varphi,\theta)$ be defined as in Theorem 4. Then*

$$\left|\frac{1}{2\pi^2R^3}\iiint_\sigma F(r,\varphi,\theta)\,d\sigma-\int_0^1 F(Rx,2\pi N^2x,2\pi Nx)\,x^2\sin(2\pi N^2x)\,dx\right|$$

(16)

$$\leqq 8\left\{(R+1)(2\pi+1)\,\omega_F\left(\frac{1}{N},\,0,\,\frac{1}{N}\right)+2M_F\cdot\frac{1}{N}\right\}.$$

Moreover, as an immediate consequence of Theorem 4, we obtain

COROLLARY 5. *For every continuous function $F(r,\varphi,\theta)$ as in Theorem 4, we have*

$$\frac{1}{2\pi^2R^3}\iiint_\sigma F(r,\varphi,\theta)\,d\sigma=\lim_{N\to\infty}\int_0^1 F(Rx,2\pi Nx,2\pi N^2x)x^2\sin(2\pi Nx)\,dx, \tag{17}$$

$$\frac{1}{2\pi^2R^3}\iiint_\sigma F\,d\sigma=\lim_{n\to\infty}\sum_{k=1}^n A_k^{(n)}F(Rx_k^{(n)},2\pi Nx_k^{(n)},\ 2\pi N^2x_k^{(n)})\cdot (x_k^{(n)})^2\sin(2\pi Nx_k^{(n)})+\rho_N, \tag{18}$$

where $|\rho_N|\leqq 8\{(R+1)(2\pi+1)\omega_F(N^{-1},N^{-1},0)+(2\pi+2)M_F\cdot N^{-1}\}$, $(N\geqq 2)$.

The formula (17), or the more precise relation (15), may obviously be regarded as an improvement over a result of E. Grosswald [3] concerning an extension of the Maréchal-Wilkins theorem to the euclidean 3-space. Moreover, (18) actually provides a type of numerical integration formulas for $J=\iiint_\sigma F(r,\varphi,\theta)d\sigma$. In particular, if both R and M_F are relatively very small as compared with N, and if the function F is not very irregular, then we may write, in view of (15),

$$J\approx 2\pi^2R^3\int_0^1 F(Rx,2\pi Nx,\ 2\pi N^2x)\,x^2\sin(2\pi Nx)\,dx.$$

Note added. It is not difficult to justify that the Corollary 4 of § 1 remains valid when '$\lambda\to\infty$' and '$\mu\to\infty$' are assumed through any sequences of real numbers; and consequently we may apply the Corollary to establish the following theorem:

If $\Phi(x,y,z)$ is continuous and bounded on the domain $D(-\infty<x<\infty,\ 0\leqq y\leqq\omega_1,\ 0\leqq z\leqq\omega_2)$, and if it is periodic with periods ω_1,ω_2 in y,z respectively, then for every function $f(x)$ summable over $(-\infty,\infty)$ we have

$$\lim_{\mu\to\infty}\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\Phi(x,\lambda x,\mu x)\,dx$$

$$=\frac{1}{\omega_1\cdot\omega_2}\int_{-\infty}^{\infty} f(x)\,dx\int_0^{\omega_1}dy\int_0^{\omega_2}\Phi(x,y,z)\,dz.$$

This is not only a generalization of Maréchal-Wilkins' theorem, but also a new extension of the well-known Riemann-Lebesgue lemma. In fact, the lemma follows easily by taking

$$\omega_2 = 2\pi, \quad \Phi(x, y, z) = \cos z \text{ or } \sin z.$$

For proof, it suffices to notice that an easy substitution of variables can show that the Corollary 4 implies

$$\lim_{\mu\to\infty}\lim_{\lambda\to\infty}\int_a^b \Phi(x, \lambda x, \mu x)\,dx = \frac{1}{\omega_1\cdot\omega_2}\int_a^b dx\int_0^{\omega_1} dy\int_0^{\omega_2}\Phi(x, y, z)\,dz,$$

where a and b are any two finite numbers. The passage from this relation to the more general one as stated in the theorem goes similarly as in the ordinary case of the Riemann-Lebesgue lemma, since in fact

$$\int_{-\infty}^{\infty} |f(x)|\,dx < +\infty, \quad |\Phi(x, y, z)| < M = \text{constant}.$$

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NORTH-EAST PEOPLE'S UNIVERSITY, CHANGCHUN, CHINA