ON FUNCTIONS REGULAR IN A HALF-PLANE

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1. Let $\varphi(z)$ be an analytic function, regular for $y > 0$, and let

$$\int_{-\infty}^{\infty} |\varphi(x+iy)|^p \, dx \leqq K^{*)} \qquad (p > 0)$$

for all value $y > 0$. Then we say that $\varphi(z)$ belongs to the class $\mathfrak{H}_p$ (= The Hille-Tamarkin Class). E. Hille and J. D. Tamarkin [2], for $p \geqq 1$ and T. Kawata [3], for $1 > p > 0$, proved the following theorems.

THEOREM A. *(1) A function $\varphi(z) \in \mathfrak{H}_p$ tends to a limit function $\varphi(x)$ in the mean of order p, and*

$$\int_{-\infty}^{\infty} |\varphi(x+iy)|^p \, dx \uparrow \int_{-\infty}^{\infty} |\varphi(x)|^p \, dx \qquad \text{as } y \downarrow 0.$$

(2) Any $\varphi(z) \in \mathfrak{H}_p$ for almost all x tends to its limit function $\varphi(x)$ along any non-tangential path.

THEOREM B. *A function $\varphi(z) \in \mathfrak{H}_p$ can be represented as a product $\varphi(z) = B(z)\psi(z)$ where $B(z)$ is the Blaschke product and $\psi(z) \in \mathfrak{H}_p$ which does not vanish in $y > 0$.*

THEOREM C. *If the limit function $\varphi(x) \in L_p$, $1 \leqq p \leqq \infty$ has a Fourier transform $\Phi(x)$ in L_q $(1 \leqq q \leqq \infty)$, then the Poisson integral associated with $\varphi(x)$ can be written in the form*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(t) \frac{y\,dt}{(t-x)^2 + y^2} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{ixt} e^{-yt} \Phi(t) dt.$$

These theorems are counterparts of theorems on functions belonging to class H_p $(p > 0)$ in a unit circle. Recently D. Waterman [6] proved $\mathfrak{H}_p$ $(p > 1)$ analogue of the Littlewood-Paley and Zygmund theorems. In the present note, the author shows some generalized theorems following on his former paper [5].

We put by the definition

$$g_\alpha^*(x) \equiv g_\alpha^*(x;\varphi) = \left\{ \frac{1}{\pi} \int_0^{\infty} y^{2\alpha} dy \int_{-\infty}^{\infty} \frac{|\varphi'(t+iy)|^2}{|t-z|^{2\alpha}} dt \right\}^{1/2}$$

*) Throughout this paper, $A, B\ldots\ldots$ are constants and may be different from one occurence to another.

$$= \left\{ \frac{1}{\pi} \int_0^\infty y^{2\alpha}\, dy \int_{-\infty}^\infty \frac{|\varphi'(x+t+iy)|^2}{(y^2+t^2)^\alpha}\, dt \right\}^{1/2}.$$

If $\alpha = 1$, this reduces to Waterman's $g^*(x)$, which is a counterpart of $g^*(\theta)$ of Littlewood-Paley. Then we have

THEOREM 1. *If $\varphi(z) \in \mathfrak{H}_p$, then*

$$\int_{-\infty}^\infty \{g_\alpha^*(x)\}^p\, dx \leqq A_{p,\alpha} \int_{-\infty}^\infty |\varphi(x)|^p\, dx,$$

where $\alpha > 1/p$ for $0 < p \leqq 2$ and $\alpha > 1/2$ for $p > 2$.

For the proof, we need some lemmas.

LEMMA 1. *If $\alpha > 1/2$, then*

$$\int_{-\infty}^\infty \frac{dt}{(y^2+t^2)^\alpha} = O(y^{1-2\alpha}) \qquad \textit{for } y > 0.$$

PROOF.

$$\int_{-\infty}^\infty \frac{dt}{(y^2+t^2)^\alpha} = 2\int_0^\infty \frac{dt}{(y^2+t^2)^\alpha} = 2\left\{ \int_0^y + \int_y^\infty \right\} \frac{dt}{(y^2+t^2)^\alpha}$$

$$= O\left(\int_0^y \frac{dt}{y^{2\alpha}}\right) + O\left(\int_y^\infty \frac{dt}{t^{2\alpha}}\right) = O(y^{1-2\alpha}).$$

PROOF OF THEOREM 1. The case $p = 2$. Since $\alpha > 1/2$, we have

$$\int_{-\infty}^\infty \{g_\alpha^*(x)\}^2\, dx = \frac{1}{\pi}\int_{-\infty}^\infty dx \int_0^\infty y^{2\alpha}\, dy \int_{-\infty}^\infty \frac{|\varphi'(x+t+iy)|^2}{(y^2+t^2)^\alpha}\, dt$$

$$= \frac{1}{\pi}\int_0^\infty y^{2\alpha}\, dy \int_{-\infty}^\infty \frac{dt}{(y^2+t^2)^\alpha} \int_{-\infty}^\infty |\varphi'(x+t+iy)|^2\, dx.$$

By Theorem C and Parseval's relation, this yields

$$= \frac{1}{\pi}\int_0^\infty y^{2\alpha}\, dy \int_{-\infty}^\infty \frac{dt}{(y^2+t^2)^\alpha} \int_0^\infty x^2 e^{-yx}\, \Phi^2(x)\, dx$$

$$= \frac{1}{\pi}\int_0^\infty x^2\Phi^2(x)\, dx \int_0^\infty y^{2\alpha} e^{-yx}\, dy \int_{-\infty}^\infty \frac{dt}{(y^2+t^2)^\alpha}$$

$$\leqq A\int_0^\infty x^2\Phi^2(x)\, dx \int_0^\infty y^{2\alpha} e^{-2yx} y^{1-2\alpha}\, dy \qquad \text{(by Lemma 1)}$$

$$\leqq B\int_0^\infty x^2\Phi^2(x)\, dx \int_0^\infty y\, e^{-2yx}\, dy$$

$$\leqq C\int_0^\infty x^2\,\Phi^2(x)\,x^{-2}\,dx = C\int_0^\infty \Phi^2(x)\,dx = D\int_{-\infty}^\infty \varphi^2(x)\,dx.$$

Thus we get theorem for the case $p = 2$. For the sake of proving the case $0 < p < 2$, we need a more lemma.

LEMMA 2. *If* $\varphi(z) \in \mathfrak{H}_2$, *and* $1 < k < 2$, *then*

$$|\varphi(x+t+iy)| \leqq A_k\,\varphi_k^*(x)\left\{1 + \frac{|t|}{y}\right\}^{1/k}$$

where

$$\varphi_k^*(x) = \sup_{0<|h|<\infty}\left|\frac{1}{h}\int_0^h |\varphi(x+u)|^k\,du\right|^{1/k}$$

and

$$\int_{-\infty}^\infty |\varphi_k^*(x)|^2\,dx \leqq B_k\int_{-\infty}^\infty |\varphi(x)|^2\,dx.$$

For the proof, see Waterman's paper [6].

The case $0 < p < 2$. In the view of Theorem B, we can suppose that $\varphi(z)$ is zero point free. Put

$$\psi(z) = \{\varphi(z)\}^{p/2}$$

then

$$\psi(z) \in \mathfrak{H}_2.$$

Since

$$\varphi'(z) = \frac{2}{p}\{\psi(z)\}^{\frac{2}{p}-1}\psi'(z),$$

we have

$$\{g_\alpha^*(x;\varphi)\}^2 = \frac{4}{\pi p^2}\int_0^\infty y^{2\alpha}\,dy\int_{-\infty}^\infty \frac{|\psi(y+t+iy)|^{2\left(\frac{2}{p}-1\right)}|\psi'(x+t+iy)|^2}{(y^2+t^2)^\alpha}\,dt$$

$$\leqq A_{p,k}\{\psi_k^*(x)\}^{\frac{2}{p}(2-p)}\int_0^\infty y^{2\alpha}\,dy\int_{-\infty}^\infty\left\{1+\frac{|t|}{y}\right\}^{\frac{2}{k}\left(\frac{2}{p}-1\right)}\frac{|\psi'(x+t+iy)|^2}{(y^2+t^2)^\alpha}\,dt$$

$$\leqq A_{p,k}\{\psi_k^*(x)\}^{\frac{2(2-p)}{p}}\int_0^\infty y^{2\alpha-\frac{2}{k}\left(\frac{2}{p}-1\right)}\,dy\int_{-\infty}^\infty\frac{|\psi'(x+t+iy)|^2}{(y^2+t^2)^{\alpha-\frac{1}{k}\left(\frac{2}{p}-1\right)}}\,dt.$$

If we put $\alpha = (1+\varepsilon)/p$ $(\varepsilon > 0)$, $\beta = \alpha - \dfrac{1}{k}\left(\dfrac{2}{p}-1\right)$, and take $k\,(<2)$ near enough to 2, then

$$2\beta - 1 = \left(\frac{2}{p}-1\right)\left(1-\frac{2}{k}\right) + \frac{2\varepsilon}{p} > 0,$$

whence

$$\{g_\alpha^*(x;\,\varphi)\}^2 \leqq A_{p,k}\{\psi_k^*(x)\}^{\frac{2}{p}(2-p)}\{g_\beta^*(x;\,\psi)\}^2.$$

Applying Hölder's inequality, the case $p = 2$ and Lemma 2, successively

$$\int_{-\infty}^{\infty}\{g_\alpha^*(x;\,\varphi)\}^p\,dx \leqq A_{p,k}\int_{-\infty}^{\infty}\{\psi_k^*(x)\}^{2-p}\{g_\beta^*(x;\,\psi)\}^p\,dx$$

$$\leqq A_{p,k}\left[\int_{-\infty}^{\infty}\{\psi_k^*(x)\}^2\,dx\right]^{(2-p)/2}\left[\int_{-\infty}^{\infty}\{g_\beta^*(x;\,\beta)\}^2\,dx\right]^{p/2}$$

$$\leqq A_{p,k}\int_{-\infty}^{\infty}|\psi(x)|^2\,dx \leqq A_{p,k}\int_{-\infty}^{\infty}|\varphi(x)|^p\,dx.$$

Thus we get theorem for the case $0 < p < 2$. For the case $p > 2$, the proof is done by the standard argument of A. Zygmund, [cf. 6]. So we omit the proof.

2. In the present section, we show some applications of the last theorem. If we suppose the limit function $\varphi(x) \in L_p$ $(1 \leqq p \leqq \infty)$, and has a Fourier transform $\Phi(x)$ in L_q $(1 \leqq q \leqq \infty)$, then

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{ixt}\,e^{-yt}\,\Phi(t)\,dt, \qquad y > 0.$$

Put

$$\sigma^{\alpha}(\omega, x) = \frac{1}{\Gamma(\alpha)\,\omega^{\alpha}}\int_0^{\omega}(\omega - t)^{\alpha}\,\Phi(t)\,e^{ixt}\,dt, \quad \alpha > -1$$

and

$$\tau^{\alpha}(\omega, x) = \frac{1}{\Gamma(\alpha-1)\omega^{\alpha}}\int_0^{\omega}(\omega - t)^{\alpha-1}\,t\Phi(t)\,e^{ixt}\,dt, \quad \alpha > 0$$

then we get easily

$$\tau^{\alpha}(\omega, x) = \alpha\{\sigma^{\alpha-1}(\omega, x) - \sigma^{\alpha}(\omega, x)\} = \omega\frac{d}{d\omega}\sigma^{\alpha}(\omega, x).$$

THEOREM 2. *If $\varphi(z) \in \mathfrak{H}_p$ $(1 < p < \infty)$, then*

$$\int_{-\infty}^{\infty}dx\left\{\int_0^{\infty}\frac{|\tau^{\alpha}(\omega, x)|^2}{\omega}\,d\omega\right\}^{p/2} \leqq A_{p,\alpha}\int_{-\infty}^{\infty}|\varphi(x)|^p\,dx$$

where $\alpha > 1/p$ for $1 < p < 2$, and for $\alpha > 1/2$ for $2 < p < \infty$.

PROOF. Since

$$\varphi'(x + t + iy) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} i\,e^{itu}\,e^{-yu}\,u\Phi(u)\,e^{ixu}\,du \quad (y > 0)$$

and

$$\frac{1}{(y-it)^{\alpha}} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} u^{\alpha-1} e^{-uy} e^{itu}\, du \quad (\alpha > 0,\ y > 0),$$

the convolution theorem yields

$$\int_{-\infty}^{\infty} \frac{\varphi'(x+t+iy)}{(y-it)^{\alpha}} e^{-it\omega}\, dt = \frac{\Gamma(\alpha)}{\sqrt{2\pi}}\int_0^{\omega} (\omega-u)^{\alpha-1} e^{-(\omega-u)y} e^{ixu-yu} u\Phi(u)\, du$$

$$= \frac{\Gamma(\alpha)}{\sqrt{2\pi}} e^{-\omega y}\int_0^{\omega} (\omega-u)^{\alpha-1} u\,\Phi(u) e^{ixu}\, du = \frac{\Gamma(\alpha)}{\sqrt{2\pi}} e^{-\omega y} \omega^{\alpha} \tau^{\alpha}(\omega, x).$$

Applying Parseval's identity, we have

$$\int_{-\infty}^{\infty} \frac{|\varphi'(x+t+iy)|^2}{(t^2+y^2)^{\alpha}}\, dt = \{\Gamma(\alpha)\}^2 \int_0^{\infty} \omega^{2\alpha}\{\tau^{\alpha}(\omega, x)\}^2 e^{-2\omega y}\, d\omega$$

and

$$\int_0^{\infty} y^{2\alpha}\, dy \int_{-\infty}^{\infty} \frac{|\varphi'(x+t+iy)|^2}{(t^2+y^2)^{\alpha}}\, dt$$

$$= A\int_0^{\infty} \omega^{2\alpha} |\tau^{\alpha}(\omega, x)|^2\, d\omega \int_0^{\infty} e^{-2\omega y} y^{2\alpha+1-1}\, dy$$

$$= A\int_0^{\infty} \omega^{2\alpha} |\tau^{\alpha}(\omega, x)|^2 \frac{\Gamma(2\alpha+1)}{(2\omega)^{2\alpha+1}}\, d\omega$$

$$\geqq B\int_0^{\infty} \frac{|\tau^{\alpha}(\omega, x)|^2}{\omega}\, d\omega, \qquad (B \neq 0).$$

Thus we get Theorem 2 by Theorem 1.

From this theorem we can easily deduce a strong summability theorem and an absolute summability theorem, cf. [5].

3. Before proceeding to Theorem 3, we need some preliminary remarks.

Let $X(\alpha, t)$, $t \in (0, 1)$ be the Wiener process over $(0, 1)$ and $\xi(\alpha, t)$, $t \in (-\infty, \infty)$ be the same process over an infinite range, [cf. Paley, Wiener and Zygmund [4]], then for any $\beta(t) \in L_2(-\infty, \infty)$,

$$(3.1)\quad \left\{\int_0^1 d\alpha \left|\int_{-\infty}^{\infty} \beta(t)\, d\xi(\alpha, t)\right|^{2m}\right\}^{\frac{1}{2m}} = \left\{\frac{1\cdot3\cdot5\ldots.(2m-1)}{2m}\right\}^{\frac{1}{2m}} \left\{\int_{-\infty}^{\infty} |\beta(t)|^2\, dt\right\}^{\frac{1}{2}},$$

$$m = 1, 2, \ldots..$$

On the other hand, if we write $\tilde{f}(x)$ for the conjugate function of $f(x)$, then

$$(3.2)\qquad \int_{-\infty}^{\infty} |\tilde{f}(x)|^p\, dx \leqq A_p \int_{-\infty}^{\infty} |f(x)|^p\, dx, \qquad p > 1.$$

Further let us suppose $f(x, t) \equiv f(x, \cdot)$ be L_2-valued and B_{2m}-integrable in the Bochner sense over $-\infty < x < \infty$ and

$$\int_{-\infty}^{\infty} \tilde{f}(x, t)\, d\xi(\alpha, t)$$

be conjugate*) to

$$\int_{-\infty}^{\infty} f(x, t)\, d\xi(\alpha, t).$$

Then (3.2) gives

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \tilde{f}(x, t)\, d\xi(\alpha, t) \right|^{2m} dx \leqq A_{2m} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x, t)\, d\xi(\alpha, t) \right|^{2m} dx.$$

Integrating with respect to α,

$$\int_0^1 d\alpha \int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} \tilde{f}(x, t)\, d\xi(\alpha, t) \right|^{2m} \leqq A_{2m} \int_0^1 d\alpha \int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} f(x, t)\, d\xi(\alpha, t) \right|^{2m}$$

and changing the order of integration and applying (3.1), we have

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |\tilde{f}(x, t)|^2\, dt \right\}^{\frac{2m}{2}} dt \leqq A_{2m} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2\, dt \right\}^{\frac{2m}{2}} dx,$$

$$m = 1, 2, \ldots\ldots$$

By the device of Boas and Bochner [1], applying generalized M. Riesz's convexity theorem and the conjugacy method, we establish

THEOREM 3. *If* $f(x, \cdot) \in B_p\{L_2\}$ $(p > 1)$ and $\tilde{f}(x, \cdot)$ *is its conjugate function, then*

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |\tilde{f}(x, t)|^2\, dt \right\}^{p/2} dx \leqq B_p \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2\, dt \right\}^{p/2} dx.$$

Moreover

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |s_u(x, t)|^2 dt \right\}^{p/2} dx \leqq C_p \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2\, dt \right\}^{p/2} dx,$$

where

$$s_u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} F(u, t)\, e^{iux}\, du$$

and $F(u, t)$ *is the transform of* $f(x, t)$, *that is*

*) It is sufficient to consider simple functions $f(x, \cdot)$ only for our Theorem 3. Then we may define $\tilde{f}(x, \cdot)$ as the function whose transform is $-iF(x, \cdot)\operatorname{sgn} x$, where $F(x, \cdot)$ is the transform of $f(x, \cdot)$.

$$F(u,t)=\underset{a\to\infty}{\overset{(p)}{\text{l.i.m.}}}\ \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} f(x,t)e^{-ixu}\,dx.$$

From this theorem we can prove

THEOREM 4.*) *Let $f(x)\in L_p$ $(1<p<\infty)$, $F(t)$ be its transform and put*

$$\Delta_n(x)=\frac{1}{\sqrt{2\pi}}\int_{2^n}^{2^{n+1}} F(t)\,e^{ixt}\,dt$$

then

$$\int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty}|\Delta_n(x)|^2\right)^{p/2}dx\leqq C_p\int_{-\infty}^{\infty}|f(x)|^p\,dx.$$

PROOF. Since

$$\tau^1(\omega,x)=\frac{1}{\omega}\int_0^{\omega} t\,F(t)\,e^{ixt}\,dt=\frac{i}{\omega}\,s'(\omega,x)$$

where $s(\omega,z)$ is $\sigma^0(\omega,x)$, we have

$$\int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}|\tau^1(2^n,x)|^2\right\}^{p/2}dx$$

$$\leqq A_p\int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{1}{(2^n)^2}\left|\int_0^{2^n} tF(t)\,e^{ixt}\,dt\right|^2\right\}^{p/2}dx$$

$$\leqq B_p\int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\left|\int_0^{2^n} tF(t)\,e^{ixt}\,dt\right|^2\int_{2^n}^{2^{n+1}}\frac{d\omega}{\omega^3}\right\}^{p/2}dx$$

$$\leqq C_p\int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}\frac{1}{\omega^3}\left|\int_0^{\omega} tF(t)\,e^{ixt}\,dt\right|^2 d\omega\right\}^{p/2}dx \text{ (by Theorem 3)}$$

$$\leqq C_p\int_{-\infty}^{\infty}\left\{\int_0^{\infty}\frac{|\tau^1(\omega,x)|^2}{\omega}\,d\omega\right\}^{p/2}dx.$$

Now

$$|\sigma(2^{n+1},x)-\sigma(2^n,x)|^2\leqq A\left\{\int_{2^n}^{2^{n+1}}\left|\frac{d}{d\omega}\sigma(\omega,x)\right|d\omega\right\}^2$$

$$\leqq B\left\{\int_{2^n}^{2^{n+1}}\omega\left|\frac{d}{d\omega}\sigma(\omega,x)\right|^2d\omega\right\}^{1/2}\left\{\int_{2^n}^{2^{n+1}}\frac{d\omega}{\omega}\right\}^{1/2}$$

*) This Theorem was stated without proof by D. L. Guy, Weighted p-norms and Fourier transforms (Preliminary report), Bull. Amer. Math. Soc, 62(1956) p. 159, but my paper is independent of his result.

$$\leqq B\left\{\int_{2^n}^{2^{n+1}} \omega \left|\frac{d}{d\omega}\sigma(\omega, x)\right|^2 d\omega\right\}^{1/2} \leqq C\int_{2^n}^{2^{n+1}} \frac{|\tau^1(\omega, x)|^2}{\omega} d\omega,$$

and

$$|\Delta_n(x)|^2 = |s(2^{n+1}, x) - s(2^n, x)|^2$$
$$\leqq |s(2^{n+1}, x) - \sigma(2^{n+1}, x)|^2 + |s(2^n, x) - \sigma(2^n, x)|^2 + |\sigma(2^{n+1}, x) - \sigma(2^n, x)|^2.$$

Thus we establish

$$\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}|\Delta_n(x)|^2\right\}^{p/2} dx \leqq A_p\int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}|\tau^1(2^n, x)|^2\right\}^{p/2} dx$$
$$+ B_p\int_{-\infty}^{\infty}\left\{\int_0^{\infty}\frac{|\tau^1(\omega, x)|^2}{\omega} d\omega\right\}^{p/2} dx$$
$$\leqq C_p\int_{-\infty}^{\infty}|f(x)|^p dx.$$

This is the required result.

References

[1] R. P. Boas, Jr. and S. Bochner, On a theorem of M. Riesz for Fourier series, Journ. London Math. Soc., 14(1939), 62–72.

[2] E. Hille and J. D. Tamarkin, On the absolute integrability of Fourier transform, Fund. Math., 25(1935), 329–352.

[3] T. Kawata, On analytic functions regular in the half-plane (I), Japanese Jonrn. Math., 13(1936), 421–430.

[4] R. E. A. C. Paley, N. Wiener and A. Zygmund, Notes on random functions, Math. Zeitschr., 37(1933), 647–668.

[5] G. Sunouchi, Theorems on power series of the class H^p, Tôhoku Math. Journ., 8(1950), 125–146.

[6] D. Waterman, On functions analytic in a half-plane, Trans. Amer. Math. Soc., 81(1956), 167–194.

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