ON THE INTERPOLATION OF ANALYTIC FAMILIES OF OPERATORS ACTING ON H^p-SPACES

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1. Introduction. Let $\mathfrak{P}$ be the class of all polynomials $P(w) = a_0 + a_1w + \dots + a_kw^k$, where the a_j's $(j = 1, 2, \dots, k)$ are complex numbers and w is a complex variable. If $p > 0$ we can form the space H^p, [11], of all functions $F(w)$, analytic in the interior of the unit circle, satisfying

$$(1.1) \qquad \mu_p(r; F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta \leqq M < \infty,$$

where M is independent of r, $0 \leqq r < 1$. It is well known that $\mu_p(r; F)$ is a non-decreasing function of r, and, if $p \geqq 1$,

$$(1.2) \qquad \|F\|_p = \{\lim_{r\to 1} \mu_p(r; F)\}^{1/p}$$

is a norm. In fact, with this norm, H^p is complete; i. e. it is a Banach space. In case $p < 1$, however, $\|F\|_p$ is not a norm since the triangle inequality is no longer satisfied.[1] H^p, nevertheless, can be made into a complete topological vector space by introducing the metric $d_p(F, G) = \|F - G\|_p^p$. In either of these cases the class $\mathfrak{P}$ is dense in $H^p, p > 0$. We will make repeated use of the fact that, if $0 < p_1 \leqq p_2$, then $\|F\|_{p_1} \leqq \|F\|_{p_2}$.

Let (M, μ) be a measure space, where M is the point set and μ the measure. If $q > 0$, $L^q(M, \mu) = L_q$ will denote the space of all complex-valued measurable functions, f, defined on M such that

$$(1.3) \qquad \|f\|_q = \left\{\int_M |f|^q d\mu\right\}^{1/q} < \infty.$$

We will refer to $\|f\|_q$ as the norm of f. Remarks analogous to the ones made about $\|F\|_p$ apply here: if $q \geqq 1$, L^q becomes a Banach space; while, if $q < 1$, $d_q(f, g) = \|f - g\|_q^q$ is a metric.[2]

We say that a linear transformation, T, mapping $\mathfrak{P}$ into a class of measurable functions defined on M is of type (p, q) in case there exists a constant $A > 0$ such that

$$(1.4) \qquad \|TP\|_q \leqq A\|P\|_p,$$

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1) In order to avoid introducing unnecessary terminology, we shall still refer to $\|F\|_p$ as "the norm of F" when $p<1$.

2) No confusion should arise from the fact that we use the same [notation for the H^p-norm and the L^q-norm.

for all $P \in \mathfrak{P}$. It is well known that if T is of type (p, q), then it has a unique linear extension to all of H^p, preserving (1.4). The least A for which (1.4) is satisfied for all $P \in \mathfrak{P}$ is called the norm of T.

Calderon and Zygmund, [1] and [2], have proved the following theorem:

THEOREM I. *Let T be a linear transformation from $\mathfrak{P}$ into a class of measurable functions on M, of types (p_0, q_0) and (p_1, q_1), with norms A_0 and A_1 respectively. If $0 \leqq t \leqq 1$, set*

$$\frac{1}{p_t} = (1-t)\frac{1}{p_0} + t\frac{1}{p_1} \quad \text{and} \quad \frac{1}{q_t} = (1-t)\frac{1}{q_0} + t\frac{1}{q_1}. \tag{1.5}$$

Then T is of type (p_t, q_t) with norm not exceeding $K A_0^{1-t} A_1^t$, where K depends only on p_0, q_0, p_1 and q_1, but not on t.

The chief purpose of this paper is to extend this theorem to the case where the linear transformation is a function of t. To make this statement more precise we will need a few more concepts.

Let S be the domain consisting of all complex numbers $z = x + iy$ such that $0 < x < 1$. Let $\Gamma(z)$ be a function defined on the closure, $\bar{S}$, of S. We say that Γ is of *admissible growth* in case there exists positive constants B and $b, b < \pi$, such that

$$\Gamma(z) = \Gamma(x + iy) \leqq B\, e^{b|y|} \tag{1.6}$$

for all x and y satisfying $0 \leqq x \leqq 1$ and $-\infty < y < +\infty$.

Suppose $\{T_z\}$, $z \in \bar{S}$, is a family of linear transformations mapping $\mathfrak{P}$ into $L^1(M, \mu)$. We say that $\{T_z\}$ is *admissible and analytic* in case

$$\int_M (T_z P)\, g\, d\mu \tag{1.7}$$

is an analytic function for each $P \in \mathfrak{P}$ and $g \in L^\infty(M, \mu)$,[3] and $\log \|T_z P\|_1$ is of admissible growth for each $P \in \mathfrak{P}$.

We shall prove the following theorem:

THEOREM II. *Let $\{T_z\}$, $z \in \bar{S}$, be an admissible and analytic family of linear transformations. Let p_0, p_1, q_0 and q_1 be positive numbers and assume that for all y, $-\infty < y < +\infty$,*

$$\|T_{iy} P\|_{q_0} \leqq A_0(y) \|P\|_{p_0} \quad \text{and} \quad \|T_{1+iy} P\|_{q_1} \leqq A_1(y) \|P\|_{p_1} \tag{1.8}$$

for all $P \in \mathfrak{P}$, where $\log A_j(y) \leqq C_j e^{d_j|y|}$, *$0 < d_j < \pi$ and $0 < C_j$, for $j = 0, 1$. Then, for each t satisfying $0 \leqq t \leqq 1$, we have*

3) This condition is equivalent to the following seemingly stronger one: for each $z_0 \in \bar{S}$ there exists a circle, K, with center z_0, such that $T_z P = \sum_0^\infty a_k (z - z_0)^k$ for all $z \in K$, where $a_k \in L^1$, $k = 0, 1, 2, \ldots\ldots$, and the series converges absolutely and uniformly in the L^1 norm on each closed subdomain of K (see [3], pg. 57). In general, a function, Γ, defined on a complex domain, whose values are in a Banach space, having such a power series developments is called analytic, Then, the theorem reads: If $\xi[\Gamma(z)]$ is analytic for each ξ in the dual space, Γ is analytic.

(1.9) $$\|T_t P\|_{q_t} \leqq A\|P\|_{p_t}$$

for all $P \in \mathfrak{P}$, *where* p_t *and* q_t *are given by* (1.5) *and* A *depends on* t, p_j, q_j, C_j, d_j $(j = 0, 1)$, *but not on* P[4].

We will devote the next section of this paper to the proof of theorem II. In the third and final part we will apply this theorem obtaining a new proof of the following theorem of G. Sunouchi, [7]. The case $p = 1$ of this theorem is due to Zygmund, [9], who also conjectured, in [10] and [12], the full statement of theorem III:

THEOREM III. *Let* $F \in H^p$, $0 < p \leqq 1$, $\alpha = \frac{1}{p} - 1$ *and* $\sigma_n^\alpha(\theta; F)$ *the Cesàro means of order* α *for the Fourier series development of* $F(e^{i\theta})$. *Then*

(1.10) $$\left\{ \int_0^{2\pi} \sup_{n \geqq 0} \left[\frac{|\sigma_n^\alpha(\theta; F)|^p}{\log(n+2)} \right] d\theta \right\}^{1/p} \leqq A_p \|F\|_p$$

for all $F \in H^p$, *where* A_p *depends on* p *but not on* F.

We make no claim that the proof of theorem III given here is simpler than G. Sunouchi's original proof; it is, however, an illuminating example of the applicability of theorem II. We are indebted to Prof. Zygmund for suggesting the possibility of this application.

2. Proof of Theorem II. We present the proof in a sequence of lemmas.

LEMMA 1. *Let* $\Gamma(z)$ *be an upper-semi-continuous real-valued function defined on* $\bar{S}$, *of admissible growth there, and subharmonic*[5] *in* S. *Then, for each* $z_0 = x_0 + iy \in S$ *we have*

(2.1) $$\Gamma(z_0) \leqq \int_{-\infty}^{\infty} \Gamma(i[y + y_0])\, \omega(1 - x_0, y)\, dy + \int_{-\infty}^{\infty} \Gamma(1 + i[y + y_0])\, \omega(x_0, y)\, dy$$

where $\omega(x, y) = \frac{1}{2} \frac{\sin \pi x}{\cos \pi x + \cosh \pi y}$[6].

Consider the mapping of S into the closed disc $C = \{\zeta\,;\, |\zeta| \leqq 1\}$ given by

(2.2) $$\zeta = \frac{e^{\pi i z} - i}{e^{\pi i z} + i}.$$

4) A similar theorem dealing with transformations acting on L^p spaces was proved by E. M. Stein, [6].

5) Γ is subhoarmonic in an open domain D if

(i) Γ is upper semi-continuous in D;

(ii) Let D' be any subdomain which, together with its boundary B', lies in D. Let ψ be any harmonic function in D', continuous in $D' \cap B'$, such that $\Psi \geqq \Gamma$ on B'. Then $\Psi \geqq \Gamma$ in D'.

Subharmonic functions may assume the value $-\infty$. We refer the reader to [5] for the material on subharmonic functions used in this paper.

6) This lemma, in the setting when Γ is the logarithm of the modulus of an analytic function was first used by I. I. Hirschman, [4], to prove certain convexity theorems for linear transformations on L^p spaces.

The inverse of this mapping, h, is defined for $\zeta \neq 1, -1$ and has the form

$$z = h(\zeta) = \frac{1}{\pi i} \log \left\{ i \frac{1+\zeta}{1-\zeta} \right\},$$

where we are taking that branch of the logarithm for which $\log 1 = 0$. We can then form the function $\Phi(\zeta) = \Gamma(h(\zeta))$. This function will then be subharmonic in the interior of C and upper semicontinuous in C with the exception of 1 and -1. Let $\zeta = \rho e^{i\theta}$, $0 \leqq \rho < 1$, denote an arbitrary point in the interior of C and $z = x + iy$ its corresponding point under the mapping h. Γ is of admissible growth; thus, there exist positive constants B and b, $b < \pi$, such that $\Gamma(z) \leqq Be^{b|y|}$ for all $z \in \bar{S}$. We investigate this last inequality in terms of $\zeta = \rho e^{i\theta}$ and the function Φ. For the moment assume $\rho \geqq 1/4$. It follows immediately from the expression for h that

$$y = -\frac{1}{\pi} \log \frac{|1+\zeta|}{|1-\zeta|}.$$

Thus

$$|y| = \frac{1}{\pi} \left| \log \frac{|1+\zeta|}{|1-\zeta|} \right|$$
$$\leqq \frac{1}{\pi} \left\{ \log 4 + \log \frac{1}{|\cos(\theta/2)|} + \log \frac{1}{|\sin(\theta/2)|} \right\}.$$

Hence, the condition for admissible growth becomes

$$\Psi(\rho e^{i\theta}) \leqq 4B\, |\cos(\theta/2)|^{-b/\pi}\, |\sin(\theta/2)|^{-b/\pi}.$$

Since $b < \pi$, this inequality asserts that $\Psi(\rho e^{i\theta})$ is bounded from above by an integrable function of θ, independently of $\rho \geqq 1/4$. But, if $\rho < 1/4$, then $\Psi(\rho e^{i\theta})$ is certainly bounded from above since Ψ is subharmonic. We thus have

$$\Psi(\rho e^{i\theta}) \leqq g(\theta) \tag{2.3}$$

for all $\rho < 1$, where $g \in L^1(-\pi, \pi)$.

Let $P(r,t) = (1/2)(1-r^2)/(1-2r\cos t + r^2)$ be the Poisson kernel. It follows from the subharmonic character of Ψ that, if $\rho < R < 1$,

$$\Psi(\rho e^{i\theta}) \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi(R e^{i\phi}) P(\rho/R, \theta - \phi)\, d\phi. \tag{2.4}$$

For ρ fixed, $P(\rho/R, \theta-\phi)$ is positive and bounded as long as $\rho < R < 1$. Thus, by (2.3), the integrand in (2.4) is bounded from above by an integrable function $G(\phi)$. Let $f_R(\phi)$ denote this integrand and $f(\phi) = \Psi(e^{i\phi}) P(\rho, \theta - \phi)$ (the last functions being defined for all ϕ except 0 and π). By the upper semicontinuity of Ψ we have

$$\limsup_{R \to 1} f_R(\phi) = f(\phi). \tag{2.5}$$

The functions $G(\phi) - f_R(\phi)$ being non-negative, an application of Fatou's lemma yields

$$\int_{-\pi}^{\pi} \liminf_{R\to 1} \{G(\phi) - f_R(\phi)\}\, d\phi \leqq \liminf_{R\to 1} \int_{-\pi}^{\pi} \{G(\phi) - f_R(\phi)\}\, d\phi.$$

From this it follows that

$$\limsup_{R\to 1} \int_{-\pi}^{\pi} f_R(\phi)\, d\phi \leqq \int_{-\pi}^{\pi} \limsup_{R\to 1} f_R(\phi)\, d\phi.$$

This result, together with (2.4), yields, upon comparing with (2.5),

$$\psi(\rho e^{i\theta}) \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\phi}) \frac{1-\rho^2}{1 - 2\rho\cos(\theta - \phi) + \rho^2}\, d\phi. \tag{2.6}$$

The lemma will now follows from (2.6) by a change of variables. We first restrict our attention to the case $\theta = \pm \pi/2$. Under the mapping (2.2), these values of ζ correspond to the segment $0 \leqq x \leqq 1$ of $\bar{S}$. Letting $z = x$, then, we have

$$\zeta = -i \frac{\cos \pi x}{1 + \sin \pi x}, \quad \rho = |\zeta| = \mp \frac{\cos \pi x}{1 + \sin \pi x}. \tag{2.7}$$

Clearly, the line in S given by $x = 0$ corresponds to the lower semicircle of points $e^{i\phi}$ of C such that $0 \geqq \phi \geqq -\pi$ in such a way that, as y ranges from $-\infty$ to $+\infty$, ϕ ranges from 0 to $-\pi$. Similarly, the line given by $x = 1$ corresponds to the upper semicircle in such a way that, as y ranges from $-\infty$ to $+\infty$, ϕ ranges from 0 to π. For $-\pi < \phi < 0$ we have

$$\sin\phi = -\frac{1}{\cosh \pi y}, \quad \frac{d\phi}{dy} = -\frac{\pi}{\cosh \pi y},$$

while, for $0 < \phi < \pi$ we have " + " instead of " − " in these two equations. Thus, for x fixed, say $0 < x \leqq 1/2$ (this corresponds to $\theta = -\pi/2$; for $1/2 < x < 1$ we have $\theta = \pi/2$ and we take the negative sign in the second part of (2.7). The final result remains unchanged), we have:

$$\int_{-\pi}^{\pi} \Psi(e^{i\phi}) \frac{1-\rho^2}{1 - 2\rho\cos((-\pi/2) - \phi) + \rho^2}\, d\phi = \int_{-\pi}^{0} + \int_{0}^{\pi}$$

$$= \pi \left\{ \int_{-\infty}^{\infty} \Gamma(iy) \frac{\sin \pi x}{\cosh \pi y - \cos \pi x}\, dy + \int_{-\infty}^{\infty} \Gamma(1 + iy) \frac{\sin \pi x}{\cosh \pi y + \cos \pi x}\, dy \right\}.$$

Thus, letting $\omega(x, y) = \frac{1}{2} \frac{\sin \pi x}{\cosh \pi y + \cos \pi x}$ and comparing with (2.6), we have, for $0 < x < 1$,

$$\Gamma(x) \leqq \int_{-\infty}^{\infty} \Gamma(iy)\, \omega(1 - x, y)\, dy + \int_{-\infty}^{\infty} \Gamma(1 + iy)\, \omega(x, y)\, dy. \tag{2.8}$$

If $z_0 = x_0 + iy_0$ is an arbitrary point of S, (2.1) is an immediate consequence of the result (2.8) applied to the function $\Gamma(z + iy_0)$.

We remark that, since $\int_{-\pi}^{\pi} P(r,t)\,dt = \pi$, in the special case of $\Gamma \equiv 1$ we have proved

$$\int_{-\infty}^{\infty} \omega(1-x,y)\,dy + \int_{-\infty}^{\infty} \omega(x,y)\,dy = 1, \qquad 0 < x < 1. \tag{2.9}$$

In the next few lemmas we will assume that $\mu(M) < \infty$. Then, if $0 < q_1 \leqq q_2$, there exists a constant C_{q_1,q_2} such that $\|f\|_{q_1} \leqq C_{q_1,q_2}\|f\|_{q_2}$ for all $f \in L^{q_2(L')}$.

LEMMA 2. *If $V(z)$ is an analytic function mapping a domain in the complex plane into $L^1(M,\mu)$ and $0 < q \leqq 1$, then $D(z) = \int_M |V(z)|^q\,d\mu$ is continuous and $\log D(z)$ is subharmonic.*

Our assumption on V implies that, for each z_0 in the domain, $V(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ where $a_k \in L^1$, $k = 0, 1, 2, \ldots,$ and the series converges absolutely and uniformly in the L^1-norm in a closed disc, K, with center z_0. We have, for any $f \in L^1$,

$$\|f\|_q \leqq C_{q\,1}\,\|f\|_1. \tag{2.10}$$

Thus, the continuity of $D(z)$ is immediate. Hence, to show that $\log D(z)$ is subharmonic, it is sufficient to show that $D(z)$ is the uniform limit of logarithmically subharmonic functions in each such disc K. Using (2.10) we see that $D(z)$ is the uniform limit, on K, of the sequence $\{D_n(z)\}$, where

$$D_n(z) = \int_M \left| \sum_{k=0}^{n} a_k\,(z-z_0)^k \right|^q d\mu.$$

But, for each k, there exists a sequence, $\{s_j^{(k)}\}$, of simple functions such that $\lim_{j\to\infty} \|a_k - s_j^{(k)}\|_1 = 0$. Thus, since for any two numbers a, b

$$\big|\,|a|^q - |b|^q\big| \leqq |a-b|^q, \qquad (0 < q \leqq 1)$$

we have

$$\left| D_n(z) - \int_M \left| \sum_{k=0}^{n} s_j^{(k)}\,(z-z_0)^k \right|^q d\mu \right|$$

$$= \left| \int_M \left| \sum_{k=0}^{n} a_k(z-z_0)^k \right|^q d\mu - \int_M \left| \sum_{k=0}^{n} s_j^{(k)}\,(z-z_0)^k \right|^q d\mu \right|$$

$$\leqq \int_M \left| \sum_{k=0}^{n} (a_k - s_j^{(k)})\,(z-z_0)^k \right|^q d\mu$$

$$\leqq \int_M \sum_{k=0}^{n} |a_k - s_j^{(k)}|^q |z - z_0|^{kq} \, d\mu$$

$$\leqq \sum_{k=0}^{n} |z - z_0|^{kq} \|a_k - s_j^{(k)}\|_1^q \to 0 \qquad \text{as} \quad j \to \infty,$$

uniformly in $z, z \in K$.

Thus, it suffices to show that $\log \int_M \left| \sum_{k=0}^{n} s_j^{(k)} (z - z_0)^k \right|^q d\mu$ is subharmonic. Clearly, we can write

$$\sum_{k=0}^{n} s_j^{(k)} (z - z_0)^k = \sum_{i=1}^{m} p_i(z) \chi_i,$$

where each p_i is a polynomial and χ_i is the characteristic function of a measurable set E_i, with $E_i \cap E_j = 0$ if $i \neq j$. We obtain

$$\log \int_M \left| \sum_{k=0}^{n} s_j^{(k)} (z - z_0)^k \right|^q d\mu = \log \sum_{i=1}^{m} |p_i(z)|^q \mu(E_i).$$

But $\sum_{i=1}^{m} |p_i(z)|^q \mu(E_i)$, being the sum of logarithmically subharmonic functions is itself logarithmically subharmonic. Thus lemma 2 is proved.

LEMMA 3. *Let $p = \max\{p_0, p_1\}$. For $z \in \bar{S}$ we then have*

$$\|T_z P\|_q \leqq A(z) \|P\|_p, \tag{2.11}$$

where $q \leqq \min\{1, q_0, q_1\}$, $A(z)$ is independent of $P \in \mathfrak{P}$ and $\log A(z)$ is of admissible growth on S.

It suffices to establish (2.11) when $\|P\|_p = 1$. By lemma 2, $\log \|T_z P\|_q$ is subharmonic in S and upper semi-continuous on $\bar{S}$. Since $\|T_z P\|_q \leqq C_{q,1} \|T_z P\|_1$, $\log \|T_z P\|_q$ is of admissible growth on $\bar{S}$. We have

$$\|T_{iy} P\|_{q_0} \leqq C_{q,q} \|T_{iy_0} P\|_{q_0} \leqq C_{q,q_0} A_0(y) \|P\|_{p_0}$$
$$\leqq C_{q,q_0} A_0(y) \|P\|_p = C_{q,q_1} A_0(y).$$

Similarly, $\|T_{1+iy} P\|_q \leqq C_{q,q_1} A_1(y)$.
Letting $C = \max\{C_{q,q_0}, C_{q,q_1}\}$, $d = \max\{d_0, d_1\}$ and applying lemma 1, we have

$$\log \|T_{z_0} P\|_q \leqq C e^{d|y_0|} \left\{ C_0 \int_{-\infty}^{\infty} e^{d_0|y|} \omega(1 - x_0, y) \, dy + C_1 \int_{-\infty}^{\infty} e^{d_1|y|} \omega(x_0, y) \, dy \right\}.$$

But it is easily checked that the expression in curly brackets is bounded independently of $x_0, 0 < x_0 < 1$ (this is an easy consequence of, say, (2.9)). Thus the lemma is proved.

This lemma assures us that for each $z \in \bar{S}$ we can uniquely extend T_z to all of H^p, preserving (2.11). We denote this extension by U_z.

We fix a number $t, 0 < t < 1$ and let $a = \frac{1}{p_t}$, $b = \frac{1}{q_t}$, $a(z) = (1-z)\frac{1}{p_0} + z\frac{1}{p_1}$, and $b(z) = (1-z)\frac{1}{q_0} + z\frac{1}{q_1}$, where $z = x + iy$ is any complex number. Thus, $a(t) = a$ and $b(t) = b$. Let f be any simple function defined on $(0, 2\pi)$ and put

$$F(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{is}+w}{e^{is}-w} f(s)\, ds, \qquad |w| < 1. \tag{2.12}$$

It is then well-known that F, and thus any positive integral power of F, belongs to all H^p spaces, $p > 0$. We can set $f(t) = \Re\{F(e^{is})\} + i\Im\{F(0)\}$; in other words, f is a real-valued function plus a pure imaginary constant. Let

$$F_z(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{is}+w}{e^{is}-w} |f(s)|^{a(z)/a} e^{i\arg f(s)}\, ds. \tag{2.13}$$

Let n be a positive integer satisfying np_0, nq_0, np_1, $nq_1 > 1$. We make the simplifying assumption that $n = 2$, the proof being easily extensible to the general case. We also assume that

$$\int_0^{2\pi} |f(t)|^{2/a}\, dt = 1. \tag{2.14}$$

Let g be a non-negative simple function on M satisfying

$$\int_M |g|^{1/(1-(b/2))}\, d\mu = 1, \tag{2.15}$$

and set $G_z = g^{(1-(b(z)/2))/(1-(b/2))}$. We have

$$\|U_t(F^2)\|_{q_t}^{1/2} = \|\, |U_t(F^2)|^{1/2}\|_{2q_t} = \sup \int_M |U_t(F^2)|^{1/2} g\, d\mu^{7)}$$

the supremum being taken over all g satisfying (2.15). We will show that theorem II can be deduced from the fact that a constant B exists, independent of F given by the formula (2.12), such that $\|U_t(F^2)\|_{q_t} \leq B\|F^2\|_{p_t}$. B, of course, will depend on the other parameters mentioned in theorem II. The method employed here extends that found in [8].

We thus study the function

$$\Phi(z) = \int_M |U_z(F_z^2)|^{1/2} g^{(1-(b(z)/2))/(1-b/2)}\, d\mu,$$

7) Since all positive integral powers of F_z are in H^p, $U_z(F_z^2)$ is defined.

which, for $z = t$, reduces to $\int_M |U_z(F^2)|^{1/2} g\, d\mu$.

LEMMA 4. *$\log \Phi(z)$ is subharmonic on S and upper semi-continuous on $\bar{S}$.*

It suffices to show that $\Phi(z)$ is the uniform limit, on each bounded subset of S, of a sequence $\{\Phi_n(z)\}$ such that Φ_n is continuous on $\bar{S}$ and logarithmically subharmonic.

Set $g = \sum_l d_l \chi'_l$, where $d_l > 0$ and $\{\chi'_l\}$ is a finite collection of characteristic functions of mutually disjoint measurable sets $E'_l \subset M$, and $f = \sum_k c_k \varepsilon_k \chi_k$, where $c_k > 0$, $|\varepsilon_k| = 1$, and $\{\chi_k\}$ is a finite collection of characteristic functions of mutually disjoint measurable sets $E_k \subset (0, 2\pi)$. Then

$$\Phi(z) = \sum_l \int_{E'_l} d_l^{(1-b(z)/2)/(1-b/2)} |U_z(F_z^2)|^{1/2} d\mu$$

$$= \sum_l \int_{E'_l} |U_z d_l^{2(1-b(z)/2)/(1-b/2)} (F_z^2)|^{1/2} d\mu.$$

Let

$$\psi_z = \psi_{l,z} = d_l^{2(1-b(z)/2)/(1-b/2)} (F_z^2)$$

and

$$\Psi(z) = \Psi_l(z) = \int_{E'_l} |U_z \psi_z|^{1/2} d\mu.$$

It suffices to show that each Ψ is the uniform limit, on each bounded subset of S, of a sequence $\{\Psi_n\}$ such that Ψ_n is continuous on $\bar{S}$ and logarithmically subharmonic. We have

$$\psi_z = d^{2(1-b(z)/2)/(1-b/2)} \sum_{j,k} (c_j c_k)^{\alpha(z)/a} \varepsilon_j \varepsilon_k \Delta_j \Delta_k$$

where

$$\Delta_j(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{is} + w}{e^{is} - w} \chi_j(s)\, ds.$$

Let $\{P_{jk}^{(n)}\}$, $n = 1, 2, 3, \ldots,$ be a sequence of polynomials such that

$$\|P_{jk}^{(n)} - \Delta_j \Delta_k\|_p \to 0 \qquad \text{as } n \to \infty,$$

where $p = \max\{p_0, p_1\}$ (see lemma 3). Put

$$\Psi_n(z) = \int_{E'_l} \left| d_l^{2(1-b(z)/2)/(1-b/2)} \sum_{j,k} (c_j c_k)^{\alpha(z)/a} \varepsilon_j \varepsilon_k T_z P_{jk}^{(n)} \right|^{1/2} d\mu.$$

The function inside the absolute value sign satisfies the hypothesis in Lemma 2. Thus $\Psi_n(z)$ is continuous and logarithmically subharmonic on $\bar{S}$. Let Ω be a bounded subset of $\bar{S}$. Then for $z \in \Omega$, since $T_z P_{jk}^{(n)} = U_z P_{jk}^{(n)}$, and by Lemma 3 with $q = 1/2$, we have:

$$|\Psi_n(z) - \Psi(z)|$$

$$\leqq \int_{E'_l} \left| d^{2(1-b(z)/2)(1-b/2)} \sum_{j,k} (c_j c_k)^{\alpha(z)/a} \varepsilon_j \varepsilon_k U_z (P_{jk}^{(n)} - \Delta_j \Delta_k) \right|^{1/2} d\mu$$

$$\leqq |d^{l(1-b(z)/2)/(1-b/2)}| \left\{ \sum_{j,k} |(c_j c_k)^{\alpha(z)/a}|^{1/2} [A(z)]^{1/2} \|P_{jk}^{(n)} - \Delta_j \Delta_k\|_p^{1/2} \right\}$$

and the last term converges to 0, uniformly in Ω, as $n \to \infty$ since $A(z)$ is bounded for z in a bounded domain (since $\log A(z)$ is of admissible growth). This proves the lemma.

Since,

$$\Phi(z) \leqq \sum_l d_l^{(1-b(x)/2)/(1-b/2)} \left\{ \sum_{j,k} (c_j (c_k)^{\alpha(x)/2a} \int_{E'_l} |U_z \Delta_j \Delta_k|^{1/2} d\mu \right\}$$

$$\leqq \sum_l d_l^{(1-b(x)/2)/(1-b/2)} \left\{ \sum_{j,k} (c_j c_k)^{\alpha(x)/2a} \|\Delta_j \Delta_k\|_p^{1/2} [A(z)]^{1/2} \right\}$$

and $\log A(z)$ is of admissible growth (lemma 3), $\log \Phi(z)$ is admissible growth.

We also have, using Hölder's inequality and (2.15)

$$\Phi(iy) \leqq \left\{ \int_M |U_{iy}(F_{iy}^2)|^{q_0} d\mu \right\}^{1/2q_0} \left\{ \int_M g^{1/(1-b/2)} d\mu \right\}^{(2q_0-1)/2q_0}$$

$$= \left\{ \int_M |U_{iy}(F_{iy})|^{q_0} d\mu \right\}^{1/2q_0}.$$

But (1.8) asserts that T_{iy} is of type (p_0, q_0) with norm not exceeding $A_0(y)$. Thus T_{iy} has a unique extension to H^{p_0} preserving this norm. It is easily checked that this extension must argee with U_{iy} on F_{iy}^2. Thus,

$$\left\{ \int_M |U_{iy}(F_{iy}^2)|^{q_0} d\mu \right\}^{1/2q_0} \leqq [A_0(y)]^{1/2} \|F_{iy}^2\|_{p_0}^{1/2}$$

$$= [A_0(y)]^{1/2} \|F_{iy}\|_{2p_0}.$$

But, by the M. Riesz inequality,[8] and by (2.14)

$$\|F_{iy}\|_{2p_0} \leqq A_{2p_0} \| |f|^{a(iy)/a} \|_{2p_0}$$

8) The M. Riesz inequality states that, if $f \in L^q(0, 2\pi)$, $p>1$, and $F(w)$ is given by formula (2.12), then there exists a constant, A_p, such that $\|F\|_p \leqq A_p \|f\|_p$.

$$= A_{2p_0}\left\{ \int |f|^{1/p_0 \alpha} |f|^{iy(p_0-p_1)/\alpha p_1 p_0} |^{2p_0} d\mu \right\}^{1/2p_0} = A_{2p_0}.$$

Thus,

$$\Phi(iy) \leqq A_{2p_0}[A_0(y)]^{1/2}. \tag{2.16}$$

Similarly

$$\Phi(1+iy) \leqq A_{2p_1}[A_1(y)]^{1/2}. \tag{2.17}$$

We can thus apply lemma 1 to the function $\Gamma(z) = \log \Phi(z)$ and obtain $\Phi(t) \leqq C$, where C depends only on t and the behaviour of Φ on the lines $x = 0$ and $x = 1$. This can be summed up by the inequality

$$\|U_t(F^2)\|_{q_t} \leqq C^2 = B, \tag{2.18}$$

where B depends only on t, p_0, p_1, d_0, d_1, C_0 and C_1. In particular, it is independent of the function F satisfying (2.12) and (2.14) (where f is a simple function).

Let f be an arbitsary simple function and F the function in (2.12). Then $\|f\|_{2p_t} \leqq 2\|F\|_{2p_t}$. Thus, by (2.18), since $F/\|f\|_{2p_t}$ is a function of type (2.12) satisfying (2.14),

$$\frac{1}{2\|F\|^2_{2p_t}} \|U_t(F^2)\|_{q_t} \leqq \frac{1}{\|f\|^2_{2p_t}} \|U_t(F^2)\|_{q_t} = \left\|U_t\left(\frac{F^2}{\|f\|^2_{2p_t}}\right)\right\|_{q_t} \leqq B.$$

But $\|F\|^2_{2pt} = \|F^2\|_{p_t}$, thus

$$\|U_t(F^2)\|_{q_t} \leqq 2B\|F^2\|_{p_t}, \tag{2.19}$$

for all F of type (2.12) with f simple.

Now let G be any function in $H^p, p = \max\{p_0, p_1\}$, that is the square of a function $F \in H^{2p}$. Let f be the function in $L^{2p}(0, 2\pi)$ ($f(s) = \Re\{F(e^{is})\} + \Im\{F(0)\}$) such that

$$F(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{is}+w}{e^{is}-w} f(s)\,ds.$$

Let $\{f_n\}$ be a sequence of simple functions such that $\|f_n - f\|_{2p} \to 0$ as $n \to \infty$, and set

$$F_n(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{is}+w}{e^{is}-w} f_n(s)\,ds.$$

Then, by lemma 3 (the extension U_t of T_t satisfies (2.11) for all functions in H^p) and by the M. Riesz inequality, we have

$$\|U_t(F^2 - F_n^2)\|_q \leqq A(t)\|F^2 - F_n^2\|_p \leqq A(t)\|F - F_n\|_{2p} \cdot \|F + F_n\|_{2p}$$
$$\leqq A(t)A_{2p}\|f - f_n\|_{2p} \cdot A_{2p}\|f + f_n\|_{2p}.$$

But the last factor is bounded, since $\|f - f_n\|_{2p} \to 0$ and, by this very last fact, the entire right-hand side of the inequality tends to 0 as n tends to ∞. Thus there exists a subsequence, $\{U_t(F^2 - F^2_{n_k})\}$, such that $U_t(F^2 - F^2_{n_k}) \to 0$ almost everywhere, thus, $U_t F^2_{n_k} \to U_t F^2 = U_t G$ almost everywhere. By Fatou's lemma, therefore,

$$\int |U_t G|_{q_t} d\mu \leq \liminf_{k\to\infty} \int |U_t F_{n_k}^2|^{q_t} d\mu$$

$$\leq \liminf_{k\to\infty} \{2B\|F_{n_k}^2\|_{p_t}\}^{q_t} = \{2B\|G\|\}_{q_t}.$$

The last equality being a consequence of the fact that $\|F_{n_k}^2\|_{p_t} \to \|G\|_{p_t}$, which is true, since $\|F_{nk}^2 - G\|_p \to 0$ and $p_t \leq p$.

We have thus shown that

(2.20) $$\|U_t(F^2)\|_{q_t} \leq 2B\|F^2\|_{p_t}$$

for any $F \in H^{2p}$.

But any function $G \in H^{p_t}$ can be written in the form

(2.21) $$G = G_1 + G_2,$$

where G_i $(i = 1, 2)$ has no zeros, is a member of H^{p_t}, and

(2.21)′ $$\|G_1\|_{p_t} = \|G\|_{p_t}, \|G_2\|_{p_t} \leq 2\|G\|_{p_t}.$$

This is easily seen by the Blaschke product decomposition, $G = BG_1$, where B is the Blaschke product and G_1 has no zeros. Thus $G = (B-1)G_1 + G_1$ is the desired decomposition, with $G_2 = (B-1)G_1$. The norm inequalities follow from the fact that $\|G_1\|_{p_t} = \|G\|_{p_t}$ and $|B - 1| \leq 2$ in the unit circle. Since G_1 and G_2 have no zeros, they can be written as the squares of their square roots, and, clearly, these square roots are in H^{2p_t}. Thus, by (2.20), for any $G \in H^p (\subset H^{p_t})$

$$\|U_t G\|_{q_t} = \|U_t(G_1 + G_2)\|_{q_t} \leq \|U_t G_1\|_{q_t} + \|U_t G_2\|_{q_t}$$
$$\leq 2B(\|G_1\|_{p_t} + \|G_2\|_{p_t}) \leq 6B\|G\|_{p_t}.$$

(Here, we used Minkowski's inequality; thus, we have tacitly assumed that $q_t \geq 1$. If $q_t < 1$ we take the q_t powers of the norms and use the triangle inequality).

Putting $A = 6B$, we have

(2.22) $$\|U_t G\|_{q_t} \leq A\|G\|_{p_t}$$

where A depends on t, p_j, d_j, C_j $(j = 0, 1)$[9] but not on $G \in H^p$. Since $\mathfrak{P} \in H^p$ we have proved theorem II. We have, however, assumed that $\mu(M) < \infty$. But this assumption can be easily omitted. For let $P \in \mathfrak{P}$, then $T_t P$ is not zero on, at most, a σ-finite measurable subset, N, of M (for $|T_t P|_{q_t}$ is integrable). Let χ_n be the characteristic function of a measurable subset $E_n \subset N$, of finite measure, where $E_1 \subset E_2 \subset E_3 \ldots$ and $\bigcup_n E_n = N$. We then have, by (2.22), and since $T_t P = U_t P$,

$$\|(T_t P)\chi_n\|_{q_t} \leq A\|P\|_{p_t},$$

where A is independent of n. An application of Fatou's lemma now gives

9) If $q_t < 1$, $A = (2^{q_t} + 1)^{1/q_t} 2B$; thus, in this case, A depends on q_0 and q_1 as well.

$$\|T_t P\|_{q_t} \leqq A \|P\|_{p}{}^{t},$$

for all $P \in \mathfrak{P}$, and the theorem is prored.

3. An Application of Theorem II. Let $\lambda = \alpha + i\beta$ be a complex number, where $\alpha = \Re(\lambda) > -1$. Furthermore, let

$$\sum_{\nu=0}^{\infty} u_\nu \tag{3.1}$$

be a given numerical series. We wish to define the Cesàro means of complex order λ of the series (3. 10).[10]

Let

$$s_n^\lambda = \sum_{\nu=0}^{n} A_{n-\nu}^\lambda u_\nu$$

where A_ν^λ is the ν-the coefficient of the binomial expansion of $(1-x)^{-1-\lambda}$. Thus

$$\frac{1}{(1-x)^{1+\lambda}} = \sum_{\nu=0}^{\infty} A_\nu^\lambda x^\nu, \qquad -1 < x < 1.$$

and

$$A_n^\lambda = \frac{(\lambda+1)(\lambda+2)\ldots.(\lambda+n)}{n!} = (1+\lambda)\left(1+\frac{\lambda}{2}\right)\ldots.\left(1+\frac{\lambda}{n}\right). \tag{3.2}$$

The Cesàro means σ_n^λ are then defined by letting

$$\sigma_n^\lambda = s_n^\lambda / A_n^\lambda. \tag{3.3}$$

Thus, at least formally, if $g(r) = \sum_{\nu=0}^{\infty} u_\nu r^\nu$, $\qquad 0 \leqq r < 1$,

$$\frac{g(r)}{(1-r)^{1+\lambda}} = \sum_{\nu=0}^{\infty} s_\nu^\lambda r^\nu \tag{3.4}$$

Moreover, since

$$\frac{g(r)}{(1-r)^{1+\lambda+\delta}} = \frac{g(r)}{(1-r)^{1+\lambda}} \frac{1}{(1-r)^{\delta}},$$

we obtain

$$s_n^{\lambda+\delta} = \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} s_\nu^\lambda, \tag{3.5}$$

where δ may be a complex number.

We next recall the following well-known fact, [11],

LEMMA 5. *If $\alpha > -1$, then $A_n^\alpha \simeq n^\alpha / \Gamma(\alpha+1)$*[11]*, and, in particular,*

10) In the following paragraphs we recall the standard notions of Cesàro summability, [11], only to make clear their extensibility to complex orders. The idea of complex order of summability in the context of multiple Fourier series was used by the first author in a not yet published paper.

11) "$\simeq$" means that $A_n^\alpha \Gamma(\alpha+1)/n^\alpha \to 1$ as $n \to \infty$.

there exists a constant b_α such that

$$(n+1)^\alpha/b_\alpha \leqq A_n^\alpha \leqq b_\alpha(n+1)^\alpha, \qquad n \geqq 0. \tag{3.6}$$

The next lemma extends the above asymptotic estimate to complex parameters.

LEMMA 6. *Let $\alpha > -1$, $-\infty < \beta < +\infty$, then*

$$1 \leqq |A_n^{\alpha+i\beta}/A_n^\alpha| \leqq C_\alpha e^{2\beta^2} \tag{3.7}$$

for all integral n, where C_α depends only on α.[12]

By (3.3)

$$A_n^{\alpha+i\beta} = \prod_{k=1}^n \left(1+\frac{\alpha+i\beta}{k}\right) \text{ and } A_n^\alpha = \prod_{k=1}^n \left(1+\frac{\alpha}{k}\right).$$

Therefore,

$$\frac{A_n^{\alpha+i\beta}}{A_n^\alpha} = \prod_{k=1}^n \left(1+\frac{i\beta}{k+\alpha}\right)$$

and, therefore,

$$\left|\frac{A_n^{\alpha+i\beta}}{A_n^\alpha}\right|^2 = \prod_{k=1}^n \left(1+\frac{\beta^2}{(k+\alpha)^2}\right). \tag{3.8}$$

The left-hand side of inequality (3.7) is now an immediately consequence of (3.8). For the right-hand side we proceed as follows.

$$\prod_{k=1}^n \left(1+\frac{\beta^2}{(k+\alpha)^2}\right) = \left(1+\frac{\beta^2}{(1+\alpha)^2}\right)\prod_{k=2}^n \left(1+\frac{\beta^2}{(k+\alpha)^2}\right)$$
$$\leqq \left(1+\frac{\beta^2}{(1+\alpha)^2}\right)\prod_{k=2}^n \left(1+\frac{\beta^2}{(k-1)^2}\right)$$

since $\alpha > -1$. We now use the fact that $1+x \leqq e^x$. Thus

$$\prod_{k=2}^n \left(1+\frac{\beta^2}{(k-1)^2}\right) \leqq \exp\left\{\beta^2 \sum_{k=2}^n \frac{1}{(k-1)^2}\right\} \leqq \exp(\beta^2\pi^2/6)$$

$\leqq e^{3\beta^2}$, since $\pi^2/6 \leqq 3$.

Finally, we notice that

$$1+[\beta^2/(1+\alpha^2)] \leqq C_\alpha^2(1+\beta^2) \leqq C_\alpha^2 e^{\beta^2},$$

for some appropriate C_α.

Combining these estimates with (3.8) gives,

$$|A_n^{\alpha+i\beta}/A_n^\alpha|^2 \leqq C_\alpha^2 e^{4\beta^2}.$$

This concludes the proof of the lemma.

Let now $f(\theta)$ be an integrable function over $(0, 2\pi)$ and let it have the Fourier development

12) The estimate (3.7) is not the best possible, but certainly suffices for our purposes.

$$\sum_{-\infty}^{\infty} c_\nu e^{i\nu\theta}.$$

Let $\tau_n^{\alpha+i\beta}(\theta) = \tau_n^{\alpha+i\beta}(\theta; f)$ denote the Cesàro means of order $\alpha + i\beta$ (as defined above) of the numerical series

$$c_0 + \sum_{\nu=1}^{\infty} (c_\nu e^{i\nu\theta} + c_{-\nu} e^{-i\nu\theta}).$$

Similarly, if $F(w) = F(\rho e^{i\theta})$, $\rho < 1$, is of class H^p, $\sigma_n^{\alpha+i\beta}(\theta) = \sigma_n^{\alpha+i\beta}(\theta; F)$ will denote the Cesàro means of order $\alpha + i\beta$ of the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta},$$

where the Taylor expansion of $F(w)$ is

$$F(w) = \sum_{\nu=0}^{\infty} c_\nu w^\nu.$$

We shall also define

$$\tau_*^{\alpha+i\beta}(\theta) = \tau_*^{\alpha+i\beta}(\theta; f) = \sup_{n\geqq 0} |\tau_n^{\alpha+i\beta}(\theta; f)|,$$

and similarly,

$$\sigma_*^{\alpha+i\beta}(\theta) = \sigma_*^{\alpha+i\beta}(\theta; F) = \sup_{n\geqq 0} |\sigma_n^{\alpha+i\beta}(\theta; F)|.$$

LEMMA 7. *Let* $\alpha > 0$, *Then*

(1) $$\tau_*^{\alpha+i\beta}(\theta; f) \leqq A_\alpha e^{2\beta^2} f^*(\theta)^{13)}$$

and

(2) $$\|\tau_*^{\alpha+i\beta}(\theta; f)\|_p \leqq A_{p,\alpha} e^{2\beta^2} \|f\|_p, \qquad 1 < p < \infty,$$

where $f^*(\theta)$ denotes the maximal function of Hardy and Littlewood,

$$f^*(\theta) = \sup_{\pi > |h| > 0} \frac{1}{h} \int_0^h |f(\theta + s)| ds^{14)}$$

We take for granted the following two well-known facts (see [11]):

If $\alpha > 0$, then

(3.9) $$\tau_*^{\alpha/2}(\theta; f) \leqq A_\alpha f^*(\theta),$$

and

(3.10) $$\|f^*(\theta)\|_p \leqq A_p \|f(\theta)\|_p, \qquad 1 < p.$$

Now let us make use of (3.5) where we take $\lambda = \alpha/2$, $\delta = (\alpha/2) + i\beta$ and $s_\nu^{\alpha/2}$, the Cesàro sums of the Fourier series of $f(\theta)$. We thus have

13) From now on, A_α will denote a general constant depending only on α which may be different in various occasions.

14) In what follows, we shall only need the lemma in the case $\alpha = 1/2$ and $p = 2$. Since the proof, however, is no simpler in this special case, we include the general case.

$$s_n^{\alpha+i\beta}(\theta) = \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha/2)-1+i\beta} s^{\alpha/2}(\theta). \tag{3.11}$$

By lemma 5, however,

$$|\tau_n^{\alpha/2}(\theta)| = |s_n^{\alpha/2}(\theta)/A_n^{\alpha/2}| \geqq A_\alpha(n+1)^{-\alpha/2}|s_n^{\alpha/2}(\theta)|,$$

and, therefore, by (3.9)

$$|s_n^{\alpha/2}(\theta)| \leqq A_\alpha(n+1)^{\alpha/2}f^*(\theta). \tag{3.12}$$

Substituting this estimate in (3.11), we get

$$|s_n^{\alpha+i\beta}(\theta)| \leqq A_\alpha \Big(\sum_{\nu=0}^{n} |A_{n-\nu}^{(\alpha/2)-1+i\beta}|\,(\nu+1)^{\alpha/2}\Big)f^*(\theta). \tag{3.13}$$

Lemma 5 and lemma 6, together, yield

$$|A_{n-\nu}^{(\alpha/2)-1+i\beta}| \leqq A_\alpha\, e^{2\beta^2}(n+1-\nu)^{(\alpha/2)-1}, \quad \nu \leqq n. \tag{3.14}$$

Using (3.14) in (3.13) we obtain

$$\begin{aligned}|s_n^{\alpha+i\beta}(\theta)| &\leqq A_\alpha\, e^{2\beta^2} f^*(\theta)\Big(\sum_{\nu=0}^{n}(n+1-\nu)^{(\alpha/2)-1}(\nu+1)^{\alpha/2}\Big)\\ &\leqq A_\alpha(n+1)^\alpha\, e^{2\beta^2} f^*(\theta).\end{aligned}$$

However,

$$|\tau_n^{\alpha+i\beta}(\theta)| = |s_n^{\alpha+i\beta}(\theta)/A_n^{\alpha+i\beta}| \leqq |S_n^{\alpha+i\beta}(\theta)/A_n^\alpha|,$$

by lemma 6, thus

$$|\tau_n^{\alpha+i\beta}(\theta)| \leqq A_\alpha\, |s_n^{\alpha+i\beta}(\theta)|(n+1)^{-\alpha},$$

by lemma 5. Therefore,

$$|\tau_n^{\alpha+i\beta}(\theta)| \leqq A_\alpha\, e^{2\beta^2} f^*(\theta),$$

and part (1) of the lemma is proved. Part (2) follows from part (1) by making use of (3.10). This concludes the proof of the lemma.

It will be our next aim to prove the following theorem

THEOREM A. *Let $F \in H^1$, then*

$$\int_0^{2\pi} \sup_{n\geqq 0} \frac{|\sigma_n^{i\beta}(\theta\,;F)|}{\log(n+2)}\,d\theta \leqq A\, e^{2\beta^2} \int_0^{2\pi} |F(e^{i\theta})|\,d\theta. \tag{3.15}$$

REMARK: We note that, when $\beta = 0$, the above theorem reduces to the case $p = 1$ of theorem III.

For the proof of this theorem we need the following lemma:

LEMMA 8. *Let $F \in H^2$ and define*

$$\Omega^\beta(\theta\,;F) = \Big(\sum_{n=0}^{\infty} \frac{|\sigma_n^{(1/2)+i\beta}(\theta\,;F) - \sigma_n^{(-1/2)+i\beta}(\theta\,;F)|^2}{(n+1)\log(n+2)}\Big)^{1/2}.$$

Then

$$(3.16)\qquad \left(\int_0^{2\pi} [\Omega^\beta(\theta\,;F)]^2\,d\theta\right)^{1/2} \leqq A\,e^{2\beta^2}\left\{\int_0^{2\pi} |F(e^{i\theta})|^2\,d\theta\right\}^{1/2}.$$

Let $F(e^{i\theta}) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta}$. Then, as may easily be verified

$$\sigma_n^{(1/2)+i\beta}(\theta\,;F) - \sigma_n^{(-1/2)+i\beta}(\theta\,;F)$$
$$= -\left\{\left(\frac{1}{2}+i\beta\right)A_n^{(1/2)+i\beta}\right\}^{-1}\sum_{\nu=0}^{n} A_{n-\nu}^{(-1/2)+i\beta}\,\nu\,c_\nu\,e^{i\nu\theta}.$$

Using Parseval's identity and lemmas 5 and 6 we obtain

$$\int_0^{2\pi} |\sigma_n^{(1/2)+i\beta}(\theta) - \sigma_n^{(-1/2)+i\beta}(\theta)|^2\,d\theta \leqq A(n+1)^{-1}\sum_{\nu=0}^{n} |A_{n-\nu}^{(-1/2)+i\beta}|^2\nu^2|c_\nu|^2$$
$$\leqq Ae^{4\beta^2}(n+1)^{-1}\sum_{\nu=0}^{n}(n+1-\nu)^{-1}|c_\nu|^2\nu^2.$$

Therefore,

$$\sum_{n=0}^{\infty}\int_0^{2\pi}\frac{|\sigma_n^{(1/2)+i\beta}(\theta;F) - \sigma_n^{(-1/2)+i\beta}(\theta\,;F)|^2}{(n+1)\log(n+2)}\,d\theta$$
$$\leqq Ae^{4\beta^2}\sum_{n=0}^{\infty}(n+1)^{-2}[\log(n+2)]^{-1}\sum_{\nu=0}^{n}(n+1-\nu)^{-1}|c_\nu|^2\nu^2$$
$$\leqq Ae^{4\beta}\sum_{\nu=0}^{\infty}|c_\nu|^2,$$

where the last inequality follows by inverting the order of summation and noting that

$$\sum_{\nu=0}^{\infty}\nu^2|c_\nu|^2\sum_{n=\nu}^{\infty}(n+1)^{-2}[\log(n+2)]^{-1}(n+1-\nu)^{-1}$$
$$= \sum_{\nu=0}^{\infty}\nu^2|c_\nu|^2\left\{\sum_{n=\nu}^{2\nu}+\sum_{n=2\nu+1}^{\infty}\right\} \leqq \sum_{\nu=0}^{\infty}\nu^2|c_\nu|^2\left\{\frac{A}{\nu^2}\right\}$$

Therefore,

$$\int_0^{2\pi}[\Omega^\beta(\theta\,;F)]^2\,d\theta \leqq Ae^{4\beta^2}\int_0^{2\pi}|F(e^{i\theta})|^2\,d\theta,$$

which is the desired result.

We now proceed to the proof of Theorem A. By the decomposition (2.12) and (2.21)′, we see that it is sufficient to restrict our attention to the case when F has no zeros in $|w|<1$. We then write $F(w) = G^2(w)$. We next use the well-known relation

$$(3.17) \qquad s_n^{\lambda}(\theta\,;F) = \sum_{\nu=0}^{n} s_\nu^{(\lambda-1)/2}(\theta\,;G)\, s_{n-\nu}^{(\lambda-1)/2}(\theta\,;G)$$

(which follows easily from (3.4), for example).

We set $\lambda = i\beta$ and then obtain

$$|\sigma_n^{i\beta}(\theta\,;F)| = |s_n^{i\beta}(\theta\,;F)/A_n^{i\beta}| \leqq |s_n^{i\beta}(\theta\,;F)|$$

by lemma 6, But, from (3.17) and Schwarz's inequality we get

$$|s_n^{i\beta}(\theta\,;F)| \leqq \sum_{\nu=0}^{n} |s_\nu^{(-1/2)+i\beta/2}(\theta\,;G)|^2.$$

Therefore,

$$|\sigma_n^{i\beta}(\theta\,;F)| \leqq \sum_{\nu=0}^{n} |s_\nu^{(-1/2)+i\beta/2}(\theta\,;G)|^2.$$

By lemmas 5 and 6, however,

$$\sum_{\nu=0}^{n} |s_\nu^{(-1/2)+i\beta/2}(\theta\,;G)|^2 = \sum_{\nu=0}^{n} |A_\nu^{(-1/2)+i\beta/2}|^2 |\sigma_\nu^{(-1/2)+i\beta/2}(\theta\,;G)|^2$$
$$\leqq Ae^{\beta^2} \sum_{\nu=0}^{n} (\nu+1)^{-1} |\sigma_\nu^{(-1/2)+i\beta/2}(\theta\,;G)|^2$$

Thus, for $n \geqq 0$,

$$\left| \frac{\sigma_n^{i\beta}(\theta\,;F)}{\log(n+2)} \right|$$
$$\leqq \frac{Ae^{\beta^2}}{\log(n+2)} \sum_{\nu=0}^{n} \frac{|\sigma^{(-1/2)+i\beta/2\nu}(\theta\,;G)|^2}{\nu+1}$$
$$\leqq \frac{2Ae^{\beta^2}}{\log(n+2)} \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(-1/2)+i\beta/2}(\theta\,;G) - \sigma_\nu^{(1/2)+i\beta/2}(\theta\,;G)|^2}{\nu+1}$$
$$+ \frac{2Ae^{\beta^2}}{\log(n+2)} \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(1/2)+i\beta/2}\,\theta\,;G)|^2}{\nu+1}$$
$$\leqq Ae^{\beta^2} \sum_{\nu=0}^{n} \frac{|\sigma_\nu^{(-1/2)+i\beta/2}(\theta\,;G) - \sigma_\nu^{(1/2)+i\beta/2}(\theta\,;G)|^2}{(\nu+1)\log(\nu+2)}$$
$$+ Ae^{\beta^2} \sup_\nu |\sigma_\nu^{(1/2)+i\beta/2}(\theta\,;G)|^2.$$

Finally,

$$|\sigma_n^{i\beta}(\theta\,;F)/\log(n+2)| \leqq Ae^{\beta^2}\{\Omega^{\beta/2}(\theta\,;G)\}^2 + Ae^{\beta^2}\{\sigma_*^{(1/2)+i\beta/2}(\theta\,;G)\}^2.$$

Therefore, by lemma 7, when $p = 2$, and lemma 8,

$$\int_0^{2\pi} \sup_{n\geqq 0} \left| \frac{\sigma_n^{i\beta}(\theta\,;F)}{\log(n+2)} \right| d\theta \leqq Ae^{2\beta^2} \int_0^{2\pi} |G(e^{i\theta})|^2\, d\theta$$

$$\leqq Ae^{2\beta^2}\int_0^{2\pi} |F(e^{i\theta})|\, d\theta,$$

since $F(e^{i\theta}) = [G(e^{i\theta})]^2$. This concludes the proof of theorem A.

We next prove a modification of theorem III in the case $p = 2^{-k}$, $k = 0, 1, 2, \ldots\ldots$

THEOREM B. *Let* $p = 2^{-k}$, $k = 0, 1, 2, \ldots,$ *and let* $\alpha = (1/p) - 1 = 2^k - 1$. *Suppose* $F \in H^p$. *Then*

$$\left(\int_0^{2\pi} \sup_{n\geqq 0} \frac{|\sigma_n^{\alpha+i\beta}(\theta\,;F)|^p}{\log(n+2)}\, d\theta\right)^{1/p} \leqq A_k\, e^{2\beta^2}\, \|F\|_p.$$

This theorem reduces to theorem A when $p = 1$ (i. e. $k = 0$). We therefore assume the theorem known for the case $k-1$ and then deduce it for the case k. As in the proof of theorem A, we may assume that $F(w)$ has no zeros when $|w| < 1$. Again we write $F = G^2$, and use the relation (3.17). We take $\lambda = \alpha + i\beta$, where $\alpha = (1/p) - 1 = 2^k - 1$. We thus obtain

$$|s_n^{\alpha+i\beta}(\theta\,;F)| \leqq \sum_{\nu=0}^{n} |s_\nu^{(\alpha-1)/2+i\beta/2}(\theta\,;G)|^2$$

$$\leqq A_\alpha e^{\beta^2} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} |\sigma_\nu^{(\alpha-1)/2+i\beta/2}(\theta, G)|^2,$$

and, therefore,

$$|\sigma_n^{\alpha+i\beta}(\theta\,;F)| \leqq \frac{A_\alpha e^{\beta^2}}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} |\sigma_\nu^{(\alpha-1)/2+i\beta/2}(\theta\,;G)|^2.$$

Hence,

$$\frac{|\sigma_n^{\alpha+i\beta}(\theta\,;F)|}{[\log(n+2)]^{1/p}} \leqq \frac{A_\alpha e^{\beta^2}}{(n+1)^\alpha} \sum_{\nu=0}^{n} \frac{(\nu+1)^{\alpha-1} |\sigma_\nu^{(\alpha-1)/2+i\beta/2}(\theta\,;G)|^2}{[\log(\nu+2)]^{1/p}}$$

$$\leqq A_\alpha\, e^{\beta^2} \sup_{\nu\geqq 0} \{|\sigma_\nu^{(\alpha-1)/2+i\beta/2}(\theta\,;G)|^2/[\log(\nu+2)]^{1/p}\}.$$

From this it follows immediately that

$$\int_0^{2\pi} \sup_{n\geqq 0} \frac{|\sigma_n^{\alpha+i\beta}(\theta\,;F)|^p}{\log(n+2)}\, d\theta \leqq A_\alpha^p\, e^{p\beta^2} \int_0^{2\pi} \sup_{\nu\geqq 0} \frac{|\sigma_\nu^{(\alpha-1)/2+i\beta/2}(\theta\,;G)|^{2p}}{\log(\nu+2)}\, d\theta.$$

We now apply the case $k-1$ to the right-hand side. Since $p = 2^{-k}, 2p = 2^{-k+1}$ and, since $\alpha = (1/p) - 1$, $(\alpha-1)/2 = (1/2p) - 1$. Therefore, the case $k-1$ yields

$$\int_0^{2\pi} \sup_{n\geqq 0} \frac{|\sigma_n^{\alpha+i\beta}(\theta\,;F)|^p}{\log(n+2)}\, d\theta \leqq A_\alpha^p\, e^{p\beta^2} A_{k-1}^{2p}\, e^{p\beta^2} \int_0^{2\pi} |G(e^{i\theta})|^{2p}\, d\theta$$

$$= A_k^p\, e^{2p\beta^2} \int_0^{2\pi} |F(e^{i\theta})|^p\, d\theta.$$

This concludes the proof of theorem B.

PROOF OF THEOREM III. (Forthe statement of the theorem, see section 1).

We assume that $2^{-k-1} < p < 2^{-k}$, since the case $p = 2^{-k}$ is already covered in theorem B. Let $p_0 = 2^{-k-1}$ and $p_1 = 2^{-k}$. We shall prove the theorem by interpolating (i.e. using theorem II) between the indices p_0 and p_1.

Now fix $n(\theta)$ as an integer-valued step-function on $[0, 2\pi]$, so that $2 \leqq n(\theta)$. We first show that

$$(3.18) \qquad \left(\int_0^{2\pi} \frac{|\sigma^{\alpha}_{n(\theta)}(\theta; P)|^p}{\log n(\theta)} d\theta\right)^{1/p} \leqq A_p \left(\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right)^{1/p},$$

for every polynomial $P(w)$, where A_p does not depend on $n(\theta)$ and P(Theorem III will then be an easy consequence of this).

Consider now the family of operators on $\mathfrak{P}$ (= class of polynomials defined in §1) defined by

$$(3.19) \qquad T_z(P) = \sigma^{\alpha(z)}_{n(\theta)}(P)$$

where

$$\alpha(z) = 2^{k+1} - 1 - z\,2^k$$

We notice that $\alpha(0) = 2^{k+1} - 1 = (1/p_0) - 1$ and $\alpha(1) = 2^k - 1 = (1/p_1) - 1$. Keeping $P \in \mathfrak{P}$ fixed we claim that the following three facts hold

(1) $T_z(P)$ is analytic from z in the strip $S = \{z; 0 \leqq \Re(z) \leqq 1\}$, into $L^1(M, \mu)$, where $M = (0, 2\pi)$ and $d\mu = d\theta/\log n(\theta)$.

$$(2) \qquad \log \int_0^{2\pi} \frac{|T_z(P)|}{\log n(\theta)} d\theta \text{ is of admissible growth in } S.$$

$$(3) \qquad \left(\int_0^{2\pi} \frac{|T_{iy}(P)|^{p_0}}{\log n(\theta)} d\theta\right)^{1/p_0} \leqq A_0(y)\|P\|_{p_0}$$

and

$$\left(\int_0^{2\pi} \frac{|T_{1+iy}(P)|^{p_1}}{\log n(\theta)} d\theta\right)^{1/p_1} \leqq A_1(y)\|P\|_{p_1}$$

where $\log A_j(y) \leqq ay^2 + b$, $j = 0, 1$, and a and b are independent of P and $n(\theta)$.

We now show (1). Suppose that

$$P(w) = \sum_{k=0}^{n} c_k w^k.$$

By the linearity of T_z, it is sufficient to restrict ourselves to the case $P(w) = w^k$. Now

$$T_z(w^k) = \sigma^{\alpha(z)}_{n(\theta)}(e^{ik\theta}) = \begin{cases} A^{\alpha(z)}_{n(\theta)-k}\, e^{ik\theta}/A^{\alpha(z)}_{n(\theta)} & \text{if } k \leqq n(\theta) \\ 0 & \text{otherwise} \end{cases}$$

Since $n(\theta)$ is constant on each of a finite number of intervals, then either

$$T_z(w^k) = A^{\alpha(z)}_{n'-k} e^{ik\theta} / A^{\alpha(z)}_{n'}$$

or

$$T_z(w^k) = 0$$

on each such interval. However,

$$\frac{A^{\alpha(z)}_{n'-k}}{A^{\alpha(z)}_{n'}} = \frac{(1+\alpha(z))\ldots(1+\alpha(z)/(n'-k))}{(1+\alpha(z))\ldots(1+\alpha(z)/n')}$$

is a rational function in z, since $\alpha(z) = 2^{k+1} - z2^k - 1$. Moreover, since

$$2^k - 1 \leqq \Re(\alpha(z)) \leqq 2^{k+1} - 1 \qquad \text{when } 0 \leqq \Re z \leqq 1,$$

this rational function has no poles in $\bar{S}$. Clearly, therefore, we may write

$$(3.20) \qquad T_z(w^k) = \sum_{j=1}^{m} [Q_j(z)\chi_j(\theta)]\, e^{ik\theta},$$

where m is the number of different values the function $n(\theta)$ assumes, χ_j is a characteristic function of an interval and Q_j is a rational function which is analytic in $\bar{S}$.

We now show (2). It is sufficient to see that

$$\log \int_0^{2\pi} \frac{|T_z(w^k)|}{\log n(\theta)}\, d\theta$$

is of admissible growth in S. This last follows from (3.20) and the fact that each $\log|Q_j(z)|$ is of admissible growth in $\bar{S}$, since Q_j is a rational function with no poles there.

Finally, we prove (3). The two inequalities follow directly from theorem B. For example,

$$T_{iy}(P) = \sigma^{\alpha(iy)}_{n(\theta)}(\theta\,;P)$$

$\alpha(iy) = 2^{k+1} - 1 - iy2^k$. Thus, the case $k+1$ of theorem B implies

$$\left(\int_0^{2\pi} \frac{|T_z(P)|^{p_0}}{\log n(\theta)}\, d\theta\right)^{1/p_0} \leqq A_{k+1}\, e^{y \quad k+1} \|P\|_{p_0}$$

(whee $p_0 = 2^{-k-1}$), and where A_{k+1} is independent of $n(\theta)$ and P. Similarly for p_1. Thus we have (3) with $a = 2^{2k+1}$ and $b = \max\{\log A_{k+1}, \log A_k\}$.

By (1), (2) and (3), we may now apply the interpolation theorem (theorem II) to the operator family $\{T_z\}$. For this purpose we choose t, so that $0 < t < 1$, and set $p = p_t$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} = (1-t)2^{k+1} + t2^k.$$

The conclusion is

$$\left(\int_0^{2\pi} \frac{|T_t(P)|^p}{\log n(\theta)}\, d\theta\right)^{1/p} = \left(\int_0^{2\pi} \frac{|\sigma^{\alpha(t)}_{n(\theta)}(\theta\,;P)|^p}{\log \quad ;}\, d\theta\right)^{1/p}$$

$$\leqq A_t \|P\|_p.$$

A_t depends only on t and the bounds obtained for $A_0(y)$ and $A_1(y)$ in (3) above. More simply,

$$\left(\int_0^{2\pi} \frac{|\sigma_{n(\theta)}^{\alpha(t)}(\theta;P)|^p}{\log n(\theta)}\, d\theta\right)^{1/p} \leq A_t\left(\int_0^{2\pi} |P(e^{i\theta})|^p\, d\theta\right)^{1/p}, \tag{3.21}$$

with A_t independent of $n(\theta)$ and P.

However, $\alpha(t) = 2^{k+1} - t2^k - 1$, thus $\alpha(t) = (1/p) - 1$. To conclude the proof of theorem III we first observe that (3.21) may be extended to all functions in H^p. Then we see that $n(\theta)$ may be chosen in such a way that

$$\left(\int_0^{2\pi} \frac{|\sigma_{n(\theta)}^{\alpha(t)}(\theta;F)|^p}{\log n(\theta)}\, d\theta\right)^{1/p}$$

is as close to

$$\left(\int_0^{2\pi} \sup_{n\geq 0} \frac{|\sigma_n^{\alpha(t)}(\theta;F)|^p}{\log(n+2)}\, d\theta\right)^{1/p}$$

as we wish.

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