SOME THEOREMS ON FRACTONAL INTEGRATON

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1. Let $u(\theta) \in L^r(0, 2\pi)$, $(1 < r < \infty)$ and have mean value zero and put

$$u(\theta) \sim \sum_{n=-\infty}^{\infty}{}' a_n e^{in\theta}$$

where $'$ denotes that the term for which $n = 0$ is omitted. The fractional integral $u_\alpha(\theta)$ of order α is defined by

$$u_\alpha(\theta) \sim \sum_{n=-\infty}^{\infty}{}' a_n (in)^{-\alpha} e^{in\theta},$$

where

$$(in)^{-\alpha} = |n|^{-\alpha} \exp\{(\alpha \pi i \operatorname{sgn} n)/2\}.$$

Hirschman [1] proved many interesting results for fractional integration which are related to the work of Littlewood-Paley [4], Marcinkiewicz [5] and Zygmund [9]. But he has not considered the function $g^*(\theta)$ of Littlewood-Paley. In § 2 we shall prove an integral inequality concerning with the function $g^*(\theta)$. This inequality is used by Koizumi [3] to prove other theorems for fractional integration. In the last section we will generalize Theorem 4.2 of Hirschman.

2. If we put[1]

$$u_\alpha(\rho, \theta) \sim \sum_{n=1}^{\infty} c_n (in)^{-\alpha} \rho^n e^{in\theta}, \qquad u_\alpha(1, \theta) = u_\alpha(\theta)$$

and

$$g^*(\alpha, \beta; \theta) = \left\{ \int_0^{2\pi} (1-\rho)^{2(\beta-\alpha)} d\rho \int_0^{2\pi} \frac{|u_{\alpha-1}(\rho, \theta+t)|^2}{|1-\rho e^{it}|^{2\beta}} dt \right\}^{1/2},$$

then $g^*(0, \beta; \theta)$ reduces to the function $g^*_\beta(\theta)$ which is given by the present author [7] and $g^*(0, 1; \theta)$ reduces to the function $g^*(\theta)$ of Littlewood-Paley. We shall introduce another auxiliary function $h^*(\alpha, \beta; \theta)$ by the definition

$$h^*(\alpha, \beta; \theta) = \left\{ \sum_{n=1}^{\infty} \frac{|\tau_n^\beta(\alpha, \theta)|^2}{n^{1-2\alpha}} \right\}^{1/2}$$

where

1) We consider the complex L^r class. This class is isomorphic to the real L^r class by the M. Riesz theorem.

$$\tau_n^\beta(\alpha,\theta) = \frac{1}{A_n^\beta}\sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1}(i\nu)^{-(\alpha-1)} c_\nu e^{i\nu\theta},$$

$$A_n^\beta = \binom{n+\beta}{n}.$$

Then the following lemma is proved.

LEMMA 2.1. *If $\beta > \alpha - 1/2$, then*[2)]

$$A_{\alpha,\beta} \leqq g^*(\alpha,\beta;\theta)/h^*(\alpha,\beta;\theta) \leqq B_{\alpha,\beta}.$$

PROOF. Since

$$u_{\alpha-1}(z,\theta) \sim \sum_{n=1}^{\infty} c_n (in)^{-(\alpha-1)} z^n e^{in\theta},$$

we have

$$\frac{u_{\alpha-1}(z,\theta)}{(1-z)^\beta} = \sum_{n=1}^{\infty} A_n^\beta \tau_n^\beta(\alpha,\theta) z^n.$$

Using the Parseval identity,

$$\int_0^{2\pi} \frac{|u_{\alpha-1}(\rho,\theta+t)|^2}{|1-\rho e^{it}|^{2\beta}}\,dt = \sum_{n=1}^{\infty}(A_n^\beta)^2|\tau_n^\beta(\alpha,\theta)|^2\rho^{2n},$$

and

$$\{g^*(\alpha,\beta;\theta)\}^2$$

$$= \int_0^1 (1-\rho)^{2(\beta-\alpha)}\,d\rho \int_0^{2\pi} \frac{|u_{\alpha-1}(\rho,\theta+t)|^2}{|1-\rho e^{it}|^{2\beta}}\,dt$$

$$= \sum_{n=1}^{\infty}(A_n^\beta)^2|\tau_n^\beta(\alpha,\theta)|^2 \int_0^1 (1-\rho)^{2(\beta-\alpha)}\rho^{2n}\,d\rho$$

$$= O(1)\sum_{n=1}^{\infty}|\tau_n^\beta(\alpha,\theta)|^2 \frac{n^{2\beta}}{n^{2(\beta-\alpha)+1}} \qquad (\text{for } \beta > \alpha - 1/2)$$

$$= O(1)\sum_{n=1}^{\infty}\frac{|\tau_n^\beta(\alpha,\theta)|^2}{n^{1-2\alpha}}$$

where $O(1)$ depends on α and β only.

THEOREM 2.1. *If $\beta > \alpha > -\infty$, and $\beta > 1/r$ when $1 < r \leqq 2$ and $\beta > 1/2$ when $r \geqq 2$, then*

2) $A_{\alpha,\beta}$, $B_{\alpha,\beta}, \ldots$ are constants depending on α, β, and are differ in each occurence.

$$\int_0^{2\pi} \{g^*(\alpha, \beta; \theta)\}^r d\theta \leqq A \int_0^{2\pi} |u(\theta)|^r d\theta.$$

PROOF. (1) *The case* $\beta = 1$. From Lemma 2.1, we have

$$g^*(\theta) \equiv g^*(0, 1; \theta)$$
$$= O(1)\left\{\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n} \nu c_\nu e^{i\nu\theta}|^2}{n}\right\}^{1/2} = O(1)\left\{\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n} t_\nu(\theta)|^2}{n^3}\right\}^{1/2},$$

where $t_\nu(\theta) = \nu c_\nu e^{i\nu\theta}$. On the other hand,

$$g^*(\alpha, 1; \theta) = O(1)\left\{\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n} \nu c_\nu e^{i\nu\theta} \nu^{-\alpha}|^2}{n^{3-2\alpha}}\right\}^{1/2}$$
$$= O(1)\left\{\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n} \nu^{-\alpha} t_\nu(\theta)|^2}{n^{3-2\alpha}}\right\}^{1/2}.$$

If we put

$$T_n(\theta) = \sum_{\nu=1}^{n} t_\nu(\theta),$$

then Abel's lemma gives

$$\sum_{\nu=1}^{n} \nu^{-\alpha} t_\nu(\theta) = \sum_{\nu=1}^{n-1} T_\nu(\theta)\, \Delta \nu^{-\alpha} + T_n(\theta)\, n^{-\alpha},$$

and

$$\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n} t_\nu(\theta)\, \nu^{-\alpha}|^2}{n^{3-2\alpha}} \leqq 2\sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n-1} T_n(\theta)\, \nu^{-\alpha-1}|^2}{n^{3-2\alpha}} + 2\sum_{n=1}^{\infty} \frac{|T_n(\theta) n^{-\alpha}|^2}{n^{3-2\alpha}}$$
$$= 2I_1 + 2I_2$$

say. Then

$$I_2 = \sum_{n=1}^{\infty} \frac{|T_n(\theta)|^2}{n^3} \leqq A\{g^*(\theta)\}^2.$$

By the well known inequality (see Hardy, Littlewood and Polya, *Inequalities* p. 255, Problem 346), we find, for $\alpha < 1$,

$$I_1 = \sum_{n=1}^{\infty} \frac{|\sum_{\nu=1}^{n-1} T_\nu(\theta)\, \nu^{-\alpha-1}|^2}{n^{3-2\alpha}} \leqq \sum_{n=1}^{\infty} \frac{|n| T_n(\theta)| n^{-\alpha-1}|^2}{n^{3-2\alpha}}$$
$$\leqq \sum_{n=1}^{\infty} \frac{|T_n(\theta)|^2}{n^3} \leqq A\{g^*(\theta)\}^2.$$

Thus the theorem is now a consequence of the integral inequality of Little-wood-Paley.

(2) *The case* $0<\beta<1$. Let us put

$$t_n(\theta)=n\,c_n\,e^{in\theta},\quad (t_0(\theta)=0),$$

$$t_n^\beta(\theta)=\frac{1}{A_n^\beta}\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}\,t_\nu(\theta),$$

$$\tau_n(\theta)=n^{-(\alpha-1)}\,c_n\,e^{in\theta},\quad (\tau_0(\theta)=0),$$

$$\tau_n^\beta(\theta)=\frac{1}{A_n^\beta}\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}\,\tau_\nu(\theta),$$

then we have

$$\begin{aligned}A_n^\beta\tau_n^\beta(\theta)&=\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}\,n^{-\alpha}\,n\,c_n\,e^{in\theta}\\&=\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}\,n^{-\alpha}\sum_{\mu=1}^{\nu}A_{\nu-\mu}^{-\beta-1}\,A_\mu^\beta\,t_\mu^\beta(\theta)\\&=\sum_{\mu=1}^{n}A_\mu^\beta\,t_\mu^\beta(\theta)\sum_{\nu=0}^{n-\mu}A_{n-\mu-\nu}^{\beta-1}\,A_\nu^{-\beta-1}\,(\mu+\nu)^{-\alpha}\\&=\sum_{\mu=1}^{n}A_\mu^\beta\,t_\mu^\beta(\theta)\sum_{\nu=0}^{N}A_{N-\nu}^{\beta-1}\,A_\nu^{-\beta-1}\,(n-N+\nu)^{-\alpha},\end{aligned}$$

where $N=n-\mu$. The last term equals

$$A_n^\beta\,t_n^\beta(\theta)\,n^{-\alpha}+\sum_{\mu=1}^{n-1}A_\mu^\beta\,t_\mu^\beta(\theta)\sum_{\nu=0}^{N}A_{N-\nu}^{\beta-1}\,A_\nu^{-\beta-1}\,(n-N+\nu)^{-\alpha}=I,$$

say.

On the other hand, if we put

$$B_{N,\nu}=\sum_{k=0}^{\nu}A_{N-k}^{\beta-1}\,A_k^{-\beta-1}$$

then we have

$$B_{N,N}=\sum_{k=0}^{N}A_{N-k}^{\beta-1}\,A_k^{-\beta-1}=\begin{cases}1, & N=0.\\ 0, & N\geqq 1.\end{cases}$$

Let us assume $B_{N,-1}=0\,(N\geqq 1)$, then

$$\begin{aligned}&\sum_{\nu=0}^{N}A_{N-\nu}^{\beta-1}\,A_\nu^{-\beta-1}\,(n-N+\nu)^{-\alpha}\\&\qquad=\sum_{\nu=0}^{N}(B_{N,\nu}-B_{N,\nu-1})\,(n-N+\nu)^{-\alpha}\\&\qquad=\sum_{\nu=0}^{N}B_{N,\nu}\,\Delta'n-N+\nu)^{-\alpha}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{\nu=0}^{N}|B_{N,\nu}| &= \sum_{\nu=0}^{N}\Big|\sum_{k=0}^{\nu} A_{N-k}^{\beta-1} A_k^{-\beta-1}\Big| \\ &= \sum_{\nu=0}^{N}\Big| -\sum_{k=\nu+1}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1}\Big| \\ &= -\sum_{\nu=0}^{N}\sum_{k=\nu+1}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1} \\ &= -\sum_{k=0}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1}(k+1) = \beta,\end{aligned}$$

we obtain, for $N \geqq 1$,

$$\begin{aligned}&\Big|\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_\nu^{-\beta-1}(n-N+\nu)^{-\alpha}\Big| \\ &\qquad \leqq \sum_{\nu=0}^{N}|B_{N,\nu}|\,|\Delta(n-N+\nu)^{-\alpha}| \\ &\qquad = \sum_{\nu=0}^{N}|B_{N,\nu}|\,(\mu+\nu)^{-\alpha-1} \leqq \begin{cases}\mu^{-\alpha-1}, & (\alpha>-1)\\ n^{-\alpha-1}, & (-\infty<\alpha<-1).\end{cases}\end{aligned}$$

Hence we find that

$$|I| \leqq A_n^\beta t_n^\beta(\theta)\, n^{-\alpha} + \sum_{\mu=1}^{n-1} A_\mu\, t_\mu^\beta(\theta)\,\mu^{-\alpha-1}, \qquad \text{when } \alpha > -1$$

or

$$|I| \leqq A_n^\beta t_n^\beta\, n^{-\alpha} + \sum_{\mu=1}^{n-1} A_\mu\, t_\mu^\beta(\theta)\, n^{-\alpha-1} \qquad \text{when } -\infty < \alpha < -1,$$

and

$$\tau_n^\beta(\theta) \leqq t_n^\beta(\theta)\, n^{-\alpha} + \frac{1}{A_n^\beta}\sum_{\mu=1}^{n-1} A_\mu^\beta(\theta)\, t_\mu^\beta(\theta)\,\mu^{-\alpha-1} \qquad \text{when } \alpha > -1$$

or

$$\tau_n^\beta(\theta) \leqq t_n^\beta(\theta)\, n^{-\alpha} + \frac{n^{-\alpha-1}}{A_n^\beta}\sum_{\mu=1}^{n-1} A_\mu^\beta(\theta)\, t_\mu^\beta(\theta) \qquad \text{when } -\infty < \alpha < -1.$$

For the case $\alpha > -1$, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\tau_n^\beta(\theta)|^2}{n^{-2\alpha+1}} &\leqq 2\sum_{n=1}^{\infty}\frac{|t_n^\beta(\theta)|^2}{n} \\ &\quad + 2\sum_{n=1}^{\infty}\frac{1}{n^{2(\beta-\alpha)+1}}\Big|\sum_{\mu=1}^{n-1} A_\mu^\beta\, t_\mu^\beta(\theta)\,\mu^{-\alpha-1}\Big|^2,\end{aligned}$$

and using the above cited inequality if $\beta > \alpha > -1$, the last term is less than

$$C_{\alpha,\beta}\sum_{n=1}^{\infty}\frac{(A_n^\beta)^2\,|t_n^\beta(\theta)|^2\,n^{-2(\alpha+1)}}{n^{2(\beta-\alpha)+1}}$$

$$\leqq C_{\alpha,\beta}\sum_{n=1}^{\infty}\frac{|t_n^\beta(\theta)|^2}{n}.$$

For the case $-\infty < \alpha < -1$, we get

$$\sum_{n=1}^{\infty}\frac{n^{-2(\alpha+1)}}{n^{2(\beta-\alpha)+1}}\left|\sum_{\mu=1}^{n-1}A_\mu^\beta t_\mu^\beta(\theta)\right|^2$$

$$\leqq \sum_{n=1}^{\infty}\frac{1}{n^{2\beta+3}}\left|\sum_{\mu=1}^{n-1}A_\mu^\beta t_\mu^\beta(\theta)\right|^2$$

$$\leqq \sum_{n=1}^{\infty}\frac{n^2(A_n^\beta)^2|t_n^\beta(\theta)|^2}{n^{2\beta+3}} \leqq \sum_{n=1}^{\infty}\frac{|t_n^\beta(\theta)|^2}{n}.$$

Collecting the results of both cases, we find

$$\sum_{n=1}^{\infty}\frac{|\tau_n^\beta(\theta)|^2}{n^{-2(\alpha+1)}} \leqq D_{\alpha,\beta}\sum_{n=1}^{\infty}\frac{|t_n^\beta(\theta)|^2}{n}.$$

Since we have proved elsewhere [7],

$$\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}\frac{|t_n^\beta(\theta)|^2}{n}\right\}^{r/2}d\theta \leqq E_{\beta,r}\int_0^{2\pi}|u(\theta)|^r\,d\theta,$$

$\beta > 1/r$ when $1 < r \leqq 2$ and $\beta > 1/2$ when $r \geqq 2$, we can prove the theorem completely, when $1/2 < \beta \leqq 1$.

(3) For *the case* $\beta > 1$, we can proceed analogously, but we omit the details.

REMARK. In the above proof, if we see $n^{-\alpha}$ as a summability factor, and consider n^α for the place of $n^{-\alpha}$, then we have

$$g^*(0,\beta,\theta) \leqq A_{\alpha,\beta}\, g^*(\alpha,\beta,\theta)$$

for $-\beta < \alpha < \infty$. Since it is easy to verify that

$$g(\theta) \leqq B_\beta\, g^*(0,\beta\,;\theta), \qquad (\beta > -1),$$

the Littlewood-Paley theorem yields the following theorem.

THEOREM 2.2. *If* $-\beta < \alpha < \infty$ *and* $\beta > -1$, *then*

$$\int_0^{2\pi}\{g^*(\alpha,\,\beta\,;\theta)\}^r\,d\theta \leqq A\int_0^{2\pi}|u(\theta)|^r\,d\theta, \qquad (r > 1).$$

3. Let us write

$$\mu(\alpha,q\,;\theta) = \left\{\int_0^{2\pi}|u_\alpha(\theta+t) - u_\alpha(\theta-t)|^q\,t^{-q\alpha-1}\,dt\right\}^{1/q},$$

then we have the following theorem.

THEOREM 3.1. *If $q \geqq 2$, then*

$$\int_0^{2\pi} \{\mu(\alpha, q; \theta)\}^r d\theta \leqq A_{\alpha,q,r} \int_0^{2\pi} |u(\theta)|^r d\theta,$$

where $1 > \alpha > 0$ when $\infty > r \geqq q$ and $1 > \alpha > q/r - 1$ when $1 < r < q$.

Hirschman[3] [1] proved the cases $q = 2$ and $q = r$, and the author [8] proved the case $\alpha = 1$, but he modified $\mu(1, q; \theta)$ such as

$$\mu^*(1, q; \theta) \equiv \mu_q^*(\theta) = \left\{ \int_0^{2\pi} |u_1(\theta + t) - 2u_1(\theta) + u_1(\theta - t)|^q t^{-q-1} dt \right\}^{1/q}$$

and proved

$$\int_0^{2\pi} |\mu_q^*(\theta)|^r d\theta \leqq A_{q,r} \int_0^{2\pi} |u(\theta)|^r d\theta, \qquad (1 < r < \infty, \ q \geqq 2).$$

For the proof of the theorem, we need two lemmas.

If we put

$$g_q^*(\beta, \theta) \equiv g_q^*(0, \beta; \theta) = \left\{ \frac{1}{2\pi} \int_0^1 (1-\rho)^{2\beta+q-2} d\rho \int_0^{2\pi} \frac{|u_{-1}(\rho, \theta + t)|^q}{|1 - \rho e^{it}|^{2\beta}} dt \right\}^{1/q}$$

then, we have

LEMMA 3.1. *If $q \geqq 2$, then*

$$\int_0^{2\pi} \{g_q^*(\beta, \theta)\}^r d\theta \leqq A_{q,r} \int_0^{2\pi} |u(\theta)|^r d\theta, \qquad (1 < r < \infty),$$

where $2\beta > q/r$ when $r < q$ and $2\beta > 1$ when $r \geqq q$.

This is due to Koizumi [2].

For a fixed $\delta > 0$, let us put

$$S_q(\theta) = \left\{ \iint_{|t-\theta| \leqq \delta(1-\rho)} (1-\rho)^{q-2} |u_{-1}(\rho, t)|^q \rho \, d\rho \, dt \right\}^{1/q}$$

then we have

LEMMA 3.2. *If $q \geqq 2$, then*

$$\int_0^{2\pi} \{S_q(\theta)\}^r d\theta \leqq A_{q,r} \int_0^{2\pi} |u(\theta)|^r d\theta, \qquad (1 < r < \theta).$$

PROOF. In the domain $\Omega_\theta : |t - \theta| \leqq \delta(1-\rho)$, we have the estimation

$$|u_{-1}(\rho, t)| \leqq u^*(\theta)/(1-\rho),$$

3) The Theorem 1.2d of Hirschman's paper is valid only on the case $2\sigma > 1/p$ if $1 < p < 2$. Hence his Theorem 1.3 is established only on the case $\alpha > 2/p - 1$. The author thinks that the remaining case is false.

where $u^*\theta)$ is the maximum average of $u(\theta)$. Hence

$$\left[\int_0^{2\pi} \{S_1(\theta)\}^r d\theta\right]^{1/r}$$

$$= \left[\int_0^{2\pi} \left\{\int\int_{\Omega_\theta} |u_{-1}(\rho, t)|^2 \, |u_{-1}(\rho, t)|^{q-2} (1-\rho)^{rq-2} \rho \, d\rho \, dt\right\}^{r/q} d\theta\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi} \left\{\int\int_{\Omega_\theta} |u_{-1}(\rho, t)|^2 \rho \, d\rho \, dt. \, (u^*(\theta))^{q-2}\right\}^{q/r} d\theta\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi} \{(u^*(\theta))^{q-2} (S_2(\theta))^2\}^{r/q} d\theta\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi} (u^*(\theta))^{(1-2/q)r} S(_2(\theta))^{2r/q} d\theta\right]^{1/r}.$$

Applying Hölder's inequality, maximal theorem and generalized Lusin's theorem (Marcinkiewicz-Zygmund [6]) succesively, we have

$$\leqq \left\{\int_0^{2\pi} \{(u^*{}_\backslash\theta))^r d\theta\}^{(1-2/q)/r} \left\{\int_0^{2\pi} (S_2(\theta))^r d\theta\right\}^{2/qr}\right.$$

$$\leqq A_{q,r} \left\{\int_0^{2\pi} |u(\theta)|^r d\theta\right\}^{1/r}.$$

THE PROOF OF THEOREM. Since the method of proof is analogous to that of Hirschman, we sketch only its outline. Put $\rho_t = 1 - t/(4\pi)$, then we have by Cauchy's theorem,

$$u_\alpha(\theta + t) - u_\alpha(\theta - t)$$

$$= \frac{1}{2}\int_{\rho_t e^{i(\theta+t)}}^{e^{i(\theta+t)}} [z - e^{i(\theta+t)}]^2 u_\alpha'''(z) \, dz + \frac{1}{2}\int_{\rho_t e^{i\theta}}^{\rho_t e^{i(\theta+t)}} [z - e^{i(\theta+t)}]^2 u_\alpha'''(z) \, dz$$

$$+ \frac{1}{2} u_\alpha''(\rho_t, \theta)\{-[\rho_t e^{i\theta} - e^{i(\theta+t)}]^2 + [\rho_t e^{i\theta} - e^{i(\theta-t)}]^2\}$$

$$+ u_\alpha'(\rho_t e^{i\theta})\{[\rho_t e^{i\theta} - e^{i(\theta+t)}] - [\rho_t e^{i\theta} - e^{i(\theta-t)}]\}$$

$$- \frac{1}{2}\int_{\rho_t e^{i(\theta-t)}}^{e^{i(\theta-t)}} [z - e^{i(\theta-t)}]^2 u_\alpha'''(t) \, dt - \frac{1}{2}\int_{\rho_t e^{i(\theta-t)}}^{\rho_t e^{i\theta}} [z - e^{i(\theta-t)}]^2 u_\alpha'''(z) \, dz$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

say. Let $r = (1-1/q)(1-\alpha)$, then we have

$$|I_1|^q \leqq \int_{\rho_t}^1 |u_\alpha'''(\rho_t, \theta + t)|^q (1-\rho)^{q(2+r)} d\rho \left(\int_{\rho_t}^1 (1-\rho)^{-pr} d\rho \right)^{q/p}$$

$$\leqq A t^{\alpha q/p} \int_{\rho_t}^1 |u_\alpha'''(\rho_t, \theta + t)|^q (1-\rho)^{q(2+r)} d\rho.$$

Hence

$$\int_0^{2\pi} |I_1|^q t^{-1-q\alpha} dt$$

$$\leqq A \int_{1/2}^1 (1-\rho)^{q(2+r)} d\rho \int_0^{2\pi} |u_\alpha'''(\rho_t, \theta+t)|^q h_\rho(t) t^{-1-q\alpha+q\alpha/p} dt,$$

where $h_\rho(t)$ is the characteristic function of the interval $4\pi(1-\rho) \leqq t \leqq 2\pi$. On the other hand, since

$$|u_\alpha'''(\rho, \theta + t)| \leqq A \int_0^{2\pi} |u'(\rho^{1/2}, s)| \; |1 - \rho e^{i(t+\theta-s)} \; |^{-3+\alpha+2/q-2/q} ds,$$

it follows that

$$|u_\alpha'''(\rho, \theta+t)|^q \leqq A \left(\int_0^{2\pi} |u'(\rho^{1/2}, s)|^q |1 - \rho\, e^{i(t+\theta-s)}|^{-2} ds \right)$$

$$\cdot \left(\int_0^{2\pi} |1 - \rho\, e^{is}|^{(-3+\alpha+2/q)p} ds \right)^{q/p}$$

$$\leqq A(1-\rho)^{-2q+q\alpha+1} \int_0^{2\pi} |u'(\rho^{1/2}, s)|^q |1 - \rho\, e^{(t+\theta-s)}|^{-2} ds,$$

and

$$\int_0^{2\pi} |I_1|^q t^{-1-q\alpha} dt$$

$$\leqq A \int_{1/2}^1 (1-\rho)^{q(2+r)-2q+q\alpha+1} d\rho \int_0^{2\pi} |u'(\rho^{1/2}, s)|^q ds \int_0^{2\pi} |1 - \rho\, e^{i(t+\theta-s)}|^{-2}$$

$$\times h_\rho(t) t^{-1-q\alpha-q\alpha/p} dt$$

$$\leqq A \int_{1/2}^1 (1-\rho)^{q-2} d\rho \int_{|\theta-s| \leqq 1-\rho} |u'(\rho^{1/2}, s)|^q ds$$

$$+ A\int_{1/2}^{1} (1-\rho)^{q-1+\alpha}\, d\rho \int_{|\theta - s|\geqq 1-\rho} |u'(\rho^{1/2}, s)|^q |\theta - s|^{-1-\alpha}\, ds$$

$$\leqq A\{S_q(\theta)\}^q + A\{g_q^*(1+\alpha)/2\,;\,\theta)\}^q.$$

Using Lemma 3.1 and 3.2, if $r \leqq q$, when $1+\alpha > q/r$ (that is, $\alpha > q/r - 1$), and if $r > q$, when $1+\alpha > 1$, (that is, $\alpha > 0$) we have

$$\int_0^{2\pi} \left[\int_0^{2\pi} |I_1|^q t^{-q\alpha - 1}\, dt \right]^{r/q} d\theta \leqq A\|u\|_r^r .$$

Concerning with I_2, we have

$$|I_2|^q \leqq \int_\theta^{\theta+t} |u_\alpha'''(\rho_t, u)|\, du \left(\int_\theta^{\theta+t} u^{2p}\, du \right)^{q/p}$$

$$\leqq At^{(2p+1)q/p} \int_\theta^{\theta+t} |u_\alpha'''(\rho_t, u)|^q\, du$$

$$\leqq t^{(2p+1)q/p}\, t^{q+\alpha q} \int_0^{2\pi} |u'(\rho_t^{1/2}, s)|^q |1 - \rho_t^{1/2} e^{i(\theta+s)}|^{-2}\, ds.$$

Hence it follows that

$$\int_0^{2\pi} |I_2|^q t^{-1-q\alpha}\, dt$$

$$\leqq \int_0^{2\pi} t^q\, dt \int_0^{2\pi} |u'(\rho_t^{1/2}, s)|^q |1 - \rho_t e^{i(\theta+s)}|^{-2}\, ds$$

$$\leqq \int_0^1 (1-\rho)^q\, d\rho \int_0^{2\pi} \frac{|u'(\rho, \theta+s)|^q}{|1-\rho e^{i\theta}|^2}\, ds.$$

If $1 > \alpha > q/r - 1$ when $r \leqq q$, then we have $2 > q/r$, and by Lemma 3.1, it follows that

$$\int_0^{2\pi} \left\{ \int_0^{2\pi} |I_2|^q t^{-1-q\alpha}\, dt \right\}^{r/q} d\theta \leqq A\|u\|_r^r .$$

For I_3, we have

$$\int_0^{2\pi} t^{-1-q\alpha} |I_3|^q\, dt \leqq A\int_{1/2}^1 (1-\rho)^{2q-1-q\alpha} |u_\alpha''(\rho, \theta)|^q\, d\rho$$

$$\leqq A\int_{1/2}^1 (1-\rho)^{2q-1-q\alpha} \{ |u_{\alpha-1}(\rho, \theta)|^q + |u_{\alpha-2}(\rho, \theta)|^q \}\, d\rho$$

$$\leqq A \int_0^1 (1-\rho)^{q-1-q(\alpha-1)} |u_{\alpha-2}(\rho,\theta)|^q d\rho + A \int_0^1 (1-\rho)^{q-1-q\alpha} |u_{\alpha-1}(\rho,\theta)|^q d\rho$$

$$\leqq A\Delta(q, \alpha-1, u\,;\theta) + A\Delta(q,\alpha,u\,;\theta)$$

where $\Delta(q,\alpha,u,\theta)$ is defined in Hirschman's paper [1, p. 541]. Thus we have

$$\int_0^{2\pi} \left\{ \int_0^{2\pi} t^{-1-q\alpha} |I_3|^q dt \right\}^{r/q} d\theta \leqq A\|u\|_r^r .$$

For $k = 4, 5, 6$, we have similarly

$$\int_0^{2\pi} \left\{ \int_0^{2\pi} t^{-1-q\alpha} |I_k|^q dt \right\}^{r/q} d\theta \leqq A\|u\|_r^r ,$$

and the theorem is proved completely.

THEOREM 3.2. *If* $1 < p \leqq 2$, *and* $0 < \alpha < 1$. *then*

$$A_{p,\alpha,r} \int_0^{2\pi} \{\mu(p,\alpha\,;\theta)\}^r d\theta \geqq \int_0^{2\pi} |u(\theta)|^r d\theta, \qquad (1 < r < \infty).$$

The case $p = r$ is Theorem 4.2a of Hirschman.

The proof is similar to that of Hirschman [1, theorem 3.2]. We can show that

$$\mu(p,\alpha\,;\theta) > A\Delta(p,\alpha,u\,;\theta),$$

but we omit the details.

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