HOLOMORPHICALLY PROJECTIVE CHANGES AND THEIR GROUPS IN AN ALMOST COMPLEX MANIFOLD

SHIGERU ISHIHARA

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Recently, in an Hermitian manifold T. Ōtsuki and Y. Tashiro [10][1] have studied the holomorphically projective change of the Riemannian connection, i. e. a change which preserves the system of holomorphically planar curves, and have obtained interesting results. In an almost complex manifold Y. Tashiro [13] has also studied such a change of a symmetric affine connection with respect to which the almost complex structure is covariant constant. He has introduced the holomorphically projective curvature tensor which is invariant under holomorphically projective changes of the connection and has characterized the holomorphically projective flatness of the connection by the vanishing of its holomorphically projective curvature tensor. He has discussed also in [13] holomorphically projective correspondences of Kaehlerian manifolds. In the present paper we shall concern ourselves with the holomorphically projective changes of an affine connection of some type and the group of holomorphically projective transformations in an almost complex manifold.

In an almost complex manifold[2] we call an affine connection a ϕ-connection, if it preserves the almost complex structure. We consider a ϕ-connection Γ^{λ}_{ji}, said to be half-symmetric, which behaves as if it had the symmetry $\Gamma^{\lambda}_{\nu\mu}=\Gamma^{\lambda}_{\mu\nu}$ in complex coordinates. Restricting attention to half-symmetric ϕ-connection, we shall treat some problems concerning the holomorphically projective changes. The theory of such changes of a half-symmetric ϕ-connection is analogous to that of projective changes of an affine connection.[3]

M. Obata [9] has recently studied ϕ-connections in a manifold, almost complex, Hermitian or quaternion, and obtained many interesting and suggestive results, which will play fundamental roles in the present treatments. He has given some simple formulas characterizing completely a ϕ-connection in an almost complex manifold. It is also very useful for us that the torsion tensor of a ϕ-connection is completely characterized by its relations with tensors intrinsically defined by the structure of the manifold.

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1) The number in brackets refers to the Bibliography at the end of the paper.

2) In the present paper we shall restrict attention to manifolds which are of differentiable class C^∞ and satisfy the second axiom of countability. In such a manifold there always exists a Riemannian metric and consequently an affine connection (Cf. Steenrod [12]). We assume further in the paper that any geometric object, for example, any tensor field or any affine connection, is of class C^∞. We suppose for simplicity that the manifold is connected. In a complex manifold we consider only geometric objects which are analytic in real coordinates.

3) Cf. Weyl [16], Thomas [14], for instance.

In §1 we shall define the half-symmetry and also the semi-symmetry of a ϕ-connection. The existence of a half-symmetric or semi-symmetric ϕ-connection will be established. Further, some preliminary lemmas are given.

In §2 we shall define holomorphically projective, briefly, H-projective change of a half-symmetric ϕ-connection, i. e. a change of such a connection preserving the system of holomorphically planar curves, and characterize such a change by a formula analogous to that of a projective change of an affine connection. Other preliminary facts will be given by some lemmas. In §3 we shall study the H-projectively flat, half-symmetric ϕ-connection.

M. Obata has studied also in [9] the quaternion structure in an almost complex manifold. In §4, by using his results, we shall deal with H-projective changes of a connection with respect to which the quaternion structure is covariant constant.

In §5 we shall study a projective change which makes correspond a half-symmetric ϕ-connection to another[4].

In an almost complex manifold with a half-symmetric ϕ-connection we consider a transformation of the manifold, said to be holomorphically projective, which preserves the almost complex structure and the system of holomorphically planar curves. It might be interesting to study the group of holomorphically projective transformations. In §6 we shall study such a group, which is compact, or, which preserves the Ricci tensor, in an analogous way as in a previous paper [7].

In §7, we shall discuss some fundamental behaviors of the holomorphically projective, infinitesimal transformations of a half-symmetric ϕ-connection in an almost complex manifold. In §8, by using the results obtained in §7, we shall study a group of holomorphically projective transformations of order not less than $2(m^2+m+1)$ in a $2m$-dimensional almost complex manifold on a program analogous to that developed in [6] or in [15]. Then the following fact will be established: If an almost complex manifold with a half-symmetric ϕ-connection admits such a group, then the group is of the maximum order $2(m^2+2m)$, the connection is H-projectively flat, and the manifold is homeomorphic to the complex projective space.

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1. Affine connections in an almost complex manifold. In a differentiable manifold an *almost complex structure* is defined by assigning to the manifold a tensor field $\phi_i{}^h$ such that[5]

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4) Cf. Goldberg [4], Ōtsuki and Tashiro [10].

5) Indices take values as below: in real coordinates

$$a,b,c,\ldots\ldots,\ h,i,j,k,l,\ldots\ldots=1,2,\ldots\ldots,n;$$

and in complex coordinates

$$a,b,c,\ldots\ldots,\ h,i,j,k,l,\ldots\ldots=1,2,\ldots\ldots,m,\ \bar{1},\ \bar{2},\ldots\ldots,\bar{m};$$
$$\alpha,\beta,\gamma,\ldots\ldots,\ \lambda,\mu,\ \nu,\ \omega,\ldots\ldots=1,2,\ldots\ldots,m;$$
$$\bar{\alpha},\bar{\beta},\ \bar{\gamma},\ldots\ldots,\ \bar{\lambda},\ \bar{\mu},\ \bar{\nu},\ \bar{\omega},\ldots\ldots=\bar{1},\bar{2},\ldots\ldots,\ \bar{m}.\ (n=2m).$$

As to the notations, we follows Schouten [11] in principle.

$$\phi_i{}^a\phi_a{}^h = -\delta_i^h, \quad \delta_h^i = \begin{cases} 1, & \text{if } h = i, \\ 0, & \text{if } h \neq i. \end{cases}$$

Then an *almost complex manifold*, i.e. a manifold with an almost complex structure $\phi_i{}^h$, is necessarily of even dimension $n = 2m$.

The tensor $N_{ji}{}^h$ defined by

$$N_{ji}{}^h = \frac{1}{2}(\phi_{[j}{}^a \partial_{|a|}\phi_{i]}{}^h - \phi_{[j}{}^a \partial_{i]}\phi_a{}^h)$$

is called the *Nijenhuis tensor* of the almost complex structure $\phi_i{}^h$ or of the manifold. In a complex manifold $N_{ji}{}^h$ vanishes identically.

Let $Q_{\dot{i}}^{h}$ be a tensor in an almost complex manifold defined by $\phi_i{}^h$. M. Obata has introduced in [9] the following operators:

$$\Phi_1 Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h - \phi_i^b Q_{jb}{}^a\phi_a{}^h), \qquad \Phi_2 Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h + \phi_i^b Q_{jb}{}^a\phi_a{}^h),$$

$$\Phi_1^* Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h - \phi_j^c Q_{ci}{}^a\phi_b{}^h), \qquad \Phi_2^* Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h + \phi_{jc} Q_{ci}{}^a\phi^b{}_h),$$

$$\Phi_3 Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h - \phi_j{}^c\phi_i{}^b Q_{cb}{}^h), \qquad \Phi_4 Q_{ji}{}^h = \frac{1}{2}(Q_{ji}{}^h + \phi_j{}^c\phi_i{}^d Q_{cb}{}^h).$$

The operators Φ_3 and Φ_4 can apply also to a tensor Q_{ji} just as above. We have to recall some of formulas given in [9] for the later use as below:

$$\Phi_1 + \Phi_2 = \text{identity}, \quad \Phi_3 + \Phi_4 = \text{identity}; \quad \Phi_r\Phi_r = \Phi_r \ (r = 1, 2, 3, 4);$$

$$\Phi_1\Phi_2 = \Phi_2\Phi_1 = 0, \quad \Phi_3\Phi_4 = \Phi_4\Phi_3 = 0; \quad \Phi_s\Phi_r = \Phi_r\Phi_s \ (r, s = 1, 2, 3, 4);$$

$$\Phi_2\Phi_3 + \Phi_1\Phi_4 = \Phi_2^*. \tag{1.1}$$

In an almost complex manifold an affine connection Γ_{ji}^h is called a *ϕ-connection*, if the almost complex structure $\phi_i{}^h$ is covariant constant with respect to Γ_{ji}^h, i.e. if $\nabla_j\phi_i{}^h = 0$, where the covariant derivative of a vector field v^h is defined by

$$\nabla_j v^h = \partial_j v^h + \Gamma_{ja}^h v^a.$$

Let Γ_{ji}^h be an arbitrary affine connection. Then the affine connection $\Gamma_{ji}^h - \frac{1}{2}(\nabla_j\phi_i{}^a)\phi_a{}^h$ is denoted by $\Phi\Gamma_{ji}^h$. The following theorem is known:

THEOREM A.[6] *Let $\overset{1}{\Gamma}{}_{ji}^h$ be an arbitrary but fixed affine connection in an almost complex manifold defined by $\phi_i{}^h$. Then in order that an affine connection Γ_{ji}^h in the manifold be a ϕ-connection it is necessary and sufficient that Γ_{ji}^h be written in the form*

$$\Gamma_{ji}^h = \Phi\overset{1}{\Gamma}{}_{ji}^h + A_{ji}{}^h,$$

where $A_{ji}{}^h$ is a tensor field such that $\Phi_2 A_{ji}{}^h = 0$, or equivalently, there exists a tensor field $B_{ji}{}^h$ such as $A_{ji}{}^h = \Phi_1 B_{ji}{}^h$.

Theorem A implies that *in an almost complex manifold there always exists a ϕ-connection.* Let $S_{ji}{}^h$ be the torsion tensor of an arbitrary ϕ-con-

6) Cf. Theorem 7.1 in [9].

nection; then we have[7)]

(1.2) $$N_{ji}{}^{h} = 2\Phi_2 \Phi_3 S_{ji}{}^{h}.$$

From (1.2) it follows

(1.3) $$\Phi_1 N_{ji}{}^{h} = 0.$$

A ϕ-connection is said to be *half-symmetric* with respect to $\phi_i{}^h$ or, briefly, half-symmetric, if its torsion tensor $S_{ji}{}^{h}$ satisfies

(1.4) $$\Phi_1 \Phi_3 S_{ji}{}^{h} = 0.$$

Now we have

THEOREM. 1. *In an almost complex manifold there exists always a half-symmetric ϕ-connection.*

PROOF Let $\overset{1}{\Gamma}{}^{h}_{ji}$ be an arbitrary ϕ-connection and $\overset{1}{S}_{ji}{}^{h}$ its torsion tensor. Then, by virtue of Theorem A, the connection

$$\Gamma^{h}_{ji} = \overset{1}{\Gamma}{}^{h}_{ji} - \Phi_1 \Phi_3 \overset{1}{S}_{ji}{}^{h}$$

is a ϕ-connection. Since the tensor $\overset{1}{S}_{ji}{}^{h}$ is anti-symmetric in its covariant indices, we see easily that $\Phi_1\Phi_3 \overset{1}{S}_{ji}{}^{h}$ is also anti-symmetric. Hence, the torsion tensor $S_{ji}{}^{h}$ of Γ^{h}_{ji} is given by

$$S_{ji}{}^{h} = \overset{1}{S}_{ji}{}^{h} - \Phi_1 \Phi_3 \overset{1}{S}_{ji}{}^{h}.$$

Applying $\Phi_1\Phi_3$ to the both sides, we find $\Phi_1\Phi_3 S_{ji}{}^{h} = 0$ because $\Phi_1\Phi_3\Phi_1\Phi_3 = \Phi_1\Phi_3$. This shows that the Γ^{h}_{ji} is half-symmetric.

We see by means of (1.2) that in an almost complex manifold a ϕ-connection is half-symmetric if and only if its torsion tensor $S_{ji}{}^{h}$ satisfies

(1.5) $$N_{ji}{}^{h} = 2\ \Phi_3 S_{ji}{}^{h}.$$

It is known that for an arbitrary symmetric affine connection Γ^{h}_{ji} the ϕ-connection $\Phi\Gamma^{h}_{ji}$ has the torsion tensor $S_{ji}{}^{h}$ satisfying (1.5)[8)]. Thus we have

LEMMA 1.1. *Let Γ^{h}_{ji} be an arbitrary symmetric affine connection. Then the ϕ-connection $\Phi\,\Gamma^{h}_{ji}$ is half-symmetric.*

We assume that the torsion tensor $S_{ji}{}^{h}$ of a ϕ-connection $\Gamma_{ji}{}^{h}$ satisfies

(1.6) $$S_{ji}{}^{h} - \frac{4}{n}\,\Phi_4(S_{[j}\delta_{i]}{}^{h}) = 0, \qquad S_j = S_{ja}{}^{a}.$$

Then, the ϕ-connection Γ^{h}_{ji} is said to be *semi-symmetric* with respect to $\phi_i{}^h$ or, briefly, to be semi-symmetric. The torsion tensor $S_{ji}{}^{h}$ of a semi-symmetric ϕ-connection satisfies $\Phi_1\Phi_3 S_{ji}{}^{h} = 0$ because of $\Phi_3\Phi_4 = 0$. Thus we see that *any semi-symmetric ϕ-connection is half-symmetric.* We have now

THEOREM 2. *In order that in an almost complex manifold there exist a semi-symmetric ϕ-connection, it is necessary and sufficient that the Nijenhuis*

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7) Cf. Theorem 7.3 in [9].

8) Cf. Lemma 7.1 in [9].

tenor $N_{ji}{}^{h}$ *of the manifold vanish identically.*

PROOF. Let Γ_{ji}^{h} be a semi-symmetric ϕ-connection and $S_{ji}{}^{h}$ its torsion tensor. Then $\Phi_2\Phi_3 S_{ji}{}^{h} = 0$ holds good, since $\Phi_3\Phi_4 = 0$. This together with (1.2) implies $N_{ji}{}^{h} = 0$.

Conversely, if $N_{ji}{}^{h} = 0$, we see that in the manifold there exists a certain symmetric ϕ-connection. In fact, it is known that an almost complex manifold admits a symmetric ϕ-connection if and only if its Nijehuis tensor $N_{ji}{}^{h}$ vanishes identically[9]. It is obvious that any symmetric ϕ-connection is semi-symmetric. Thus there exists a semi-symmetric ϕ-connection in the manifold.

In a complex manifold any quantity, say $T_{ji}{}^{h}$, is said to be self-adjoint, if

$$T_{ji}{}^{h} = \overline{T_{\bar{j}\bar{i}}{}^{\bar{h}}}$$

in complex coordinates.[10] The self-adjoint quantity represents a real quantity in real coordinates and vice versa. The self-adjointness of a tensor is preserved by covariant differentiation with respect to a self-adjoint affine connection. We shall restrict ourselves to self-adjoint quantities.

In a complex manifold the complex structure $\phi_i{}^{h}$ has the numerical components

$$(\phi_i{}^{h}) = \begin{pmatrix} \sqrt{-1}\delta_\mu^\lambda & 0 \\ 0 & -\sqrt{-1}\delta_{\bar{\mu}}^{\bar{\lambda}} \end{pmatrix}$$

with respect to complex coordinates $(z^\lambda, \bar{z}^\lambda)$. Thus we have easily the following facts: In a complex manifold an affine connection is a ϕ-connection, if and only if

$$\Gamma_{j\bar{\mu}}^{\lambda} = 0, \qquad \Gamma_{j\mu}^{\bar{\lambda}} = 0$$

in complex coordinates.[11] It is easily seen that a ϕ-connection Γ_{ji}^{h} is half-symmetric if and only if

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}, \quad \text{conj.}; \quad \Gamma_{j\bar{\mu}}^{\lambda} = 0, \quad \text{conj.}$$

in complex coordinates,[12] and also that a ϕ-connection Γ_{ji}^{h} is semi-symmetric if and only if its torsion tensor $S_{ji}{}^{h}$ has the components

$$\text{(1.7)} \qquad \begin{aligned} &S_{\nu\mu}{}^{\lambda} = 0, \qquad S_{\bar{\nu}\bar{\mu}}{}^{\lambda} = 0, && \text{conj.};\\ &S_{\nu\bar{\mu}}{}^{\lambda} = -S_{\bar{\mu}\nu}{}^{\lambda} = A_{\bar{\nu}}\delta_{\mu}^{\lambda}, && \text{conj.,} \end{aligned}$$

where A_j is a certain vector field.

2. Holomorphically projective changes of affine connections. Let Γ_{ji}^{h}

9) Cf. Frölicher [3], Obata [9], for example.

10) $\bar{h}=\bar{\lambda}$, if $h=\lambda$; and $\bar{h}=\lambda$ if $h=\bar{\lambda}$.
The bar on the central letter denotes the complex conjugate.

11) Cf. Obata [9], for example.

12) The sign "conj." denotes the complex conjugate ohfe formulas already written.

be a half-symmetric ϕ-connection in an almost complex manifold. We consider in the manifold a curve defined by means of differential equations of the form:

$$\frac{d^2x^h}{dt^2}+\Gamma^h_{cb}\frac{dx^c}{dt}\frac{dx^b}{dt}=\alpha(t)\frac{dx^h}{dt}+\beta(t)\phi^h_a\frac{dx^a}{dt}, \tag{2.1}$$

where $\alpha(t)$ and $\beta(t)$ are certain functions of the parameter t. We call such a curve a *holomorphically planar curve.*[13] The set of all holomorphically planar curves is called the system of holomorphically planar curves. If the function $\beta(t)$ vanishes identically, the differential equation (2.1) defines the paths of the connection.

Consider a vector v^h at a point p of the almost complex manifold. Then the plane element at p spanned by the two vectors v^h and $\widetilde{v}^h=v^a\phi_a{}^h$ is called a holomorphic section containing the vector v^h. A curve is holomorphically planar, when and only when the holomorphic section containing the tangent vector of the curve is parallel along the curve itself.

For a vector u_j we denote by $\widetilde{u}_j$ the vector $\phi_j{}^b u_b$. We have now

LEMMA 2.1. *Let Γ^h_{ji} and $\overline{\Gamma}^h_{ji}$ be two half-symmetric ϕ-connections in an almost complex manifold. Then, the two connections have all holomorphically planar curves in common, when and only when*

$$\overline{\Gamma}^h_{ji}=\Gamma^h_{ji}+F_{(j}\delta^h_{i)}-\widetilde{F}_{(j}\phi_{i)}{}^h+T_j\delta^h_i+\widetilde{T}_j\phi_i{}^h \tag{2.2}$$

holds for certain vector fields F_j and T_j.

PROOF. When $\overline{\Gamma}^h_{ji}$ has the form given by (2.2), it is obvious that the two connections Γ^h_{ji} and $\overline{\Gamma}^h_{ji}$ have the common system of holomorphically planar curves. Conversely, we suppose that Γ^h_{ji} and $\overline{\Gamma}^h_{ji}$ have all holomorphically planar curves in common. Thus, on putting

$$A_{ji}{}^h=\overline{\Gamma}^h_{ji}-\Gamma^h_{ji},$$

we have

$$A_{ji}{}^h=U_{(j}\delta^h_{i)}+V_{(j}\phi_{i)}{}^h+P_{ji}{}^h, \tag{2.3}$$

where U_j and V_j denote certain vector fields and $P_{ji}{}^h=A_{[ji]}{}^h$.

Since the ϕ-connections Γ^h_{ji} and $\overline{\Gamma}^h_{ji}$ are half-symmetric, from Theorem A and the definition of half-symmetry it follows

$$\Phi_2A_{ji}{}^h=0,\qquad \Phi_1\Phi_3P_{ji}{}^h=0. \tag{2.4}$$

We have to note that $\Phi_2Q_{ji}{}^h+\Phi_2^*Q_{ij}{}^h=0$ holds good for any tensor $Q_{ji}{}^h$ such that $Q_{(ji)}{}^h=0$. Taking account of this fact, we find by virtue of (1.1)

$$\Phi_1\Phi_4P_{ji}{}^h=-\Phi_2P_{ij}{}^h-\Phi_2\Phi_3P_{ji}{}^h.$$

According to (2.4), this implies

$$P_{ji}{}^h=\Phi_2P_{ji}{}^h-\Phi_2P_{ij}{}^h-\Phi_2\Phi_3P_{ji}{}^h. \tag{2.5}$$

13) Cf. Ōtsuki and Tashiro [10], Tashiro [13].

On the other hand, applying Φ_2 to the both sides of (2.3), we have by virtue of (2.4)

$$\Phi_2 P_{ji}{}^h = -\frac{1}{2}\{\delta_j^h(U_i - \widetilde{V}_i) + \phi_j{}^h(\widetilde{U}_i + V_i)\}.$$

If we substitute this in (2.5), we find

$$P_{ji}{}^h = U_{[j}\delta_{i]}^h + \widetilde{U}_{[j}\phi_{i]}{}^h - \widetilde{V}_{[j}\delta_{i]}^h - V_{[j}\phi_{i]}{}^h.$$

Thus from (2.3) it follows

$$A_{ji}{}^h = F_{(j}\delta_{i)}^h - \widetilde{F}_{(j}\phi_{i)}{}^h + T_j\delta_i^h + \widetilde{T}_j\phi_i{}^h,$$

where we have put

$$F_j = U_j + \widetilde{V}_j, \qquad T_j = U_j - \widetilde{V}_j.$$

Lemma 2.1 is thereby proved completely.

We have the following lemma as an immediate consequence of Lemma 2.1.

LEMMA 2.2.[14] *Two symmetric ϕ-connections Γ_{ji}^h and $\overline{\Gamma}_{ji}^h$ have all holomorphically planar curves in common, when and only when*

$$\overline{\Gamma}_{ji}^h = \Gamma_{ji}^h + F_{(j}\delta_{i)}^h - \widetilde{F}_{(j}\phi_{i)}{}^h \tag{2.6}$$

holds for a certain vector field F_j.

Let Γ_{ji}^h and $\overline{\Gamma}_{ji}^h$ be two half-symmetric ϕ-connections satisfying (2.2) for certain vector fields F_j and T_j. Then the correspondence $\Gamma_{ji}^h \to \overline{\Gamma}_{ji}^h$ is called a *holomorphically projective change*[15] or, briefly, an *H-projective change* of Γ_{ji}^h. Such two half-symmetric ϕ-connections are said to be *H-projectively related* to each other.

If we take the anti-symmetric parts of the both sides of (2.2) with respect to the covariant indices, we have

$$\overline{S}_{ji}{}^h = S_{ji}{}^h + T_{[j}\delta_{i]}^h + \widetilde{T}_{[j}\phi_{i]}{}^h,$$

where $S_{ji}{}^h$ and $\overline{S}_{ji}{}^h$ are the torsion tensors of Γ_{ji}^h and $\overline{\Gamma}_{ji}^h$ respectively. Contracting indices h and i, we find

$$T_j = \frac{2}{n}(\overline{S}_{ja}{}^a - S_{ja}{}^a).$$

If we substitute this in the right-hand side of the above relation, we obtain easily

$$\overline{S}_{ji}{}^h - \frac{4}{n}\Phi_4(\overline{S}_{[j}\delta_{i]}^h) = S_{ji}{}^h - \frac{4}{n}\Phi_4(S_{[j}\delta_{i]}^h),$$

14) Cf. Tashiro [13].

15) Such a change has been said to be holomorphically projective by Ōtsuki and Tashiro [10] in an Hermitian manifold. In an almost complex manifold Tashiro [13] has called such a change holomorphically projective correspondence for symmetric connections. By Schouten and Struik such a change was called "Bahntreue Transformation".

where $\qquad S_j = S_{ja}{}^a, \qquad \bar{S}_j = \bar{S}_{ja}{}^a.$

Consequently, we have

LEMMA 2. 3. *Let $S_{ji}{}^h$ be the torsion tensor of a half-symmetric ϕ-connection in an almost complex manifold. Then the tensor*

$$S_{ji}{}^h - \frac{4}{n}\Phi_4(S_{[j}\delta^h_{i]})$$

is invariant under H-*projective changes of the connection, where $S_j = S_{ja}{}^a$.*

COROLLARY. *If a half-symmetric ϕ-connection is* H-*projectively related to a symmetric ϕ-connection, then it is semi-symmetric. Conversely, a semi-symmetric ϕ-connection is* H-*projectively related to a symmetric one.*

PROOF. The first part is an immediate consequence of Lemma 2. 3. Then we shall prove the second part. Let Γ^h_{ji} be a semi-symmetric ϕ-connection and $S_{ji}{}^h$ its torsion tensor. We consider an H-projectively related ϕ-connection Λ^h_{ji} defined by

$$\Lambda^h_{ji} = \Gamma^h_{ji} - \frac{4}{n}\Phi_4(S_j\delta^h_i), \tag{2.7}$$

where $S_j = S_{ja}{}^a$. Denoting by $T_{ji}{}^h$ the torsion tensor of Λ^h_{ji}, we find

$$T_{ji}{}^h = S_{ji}{}^h - \frac{4}{n}\Phi_4(S_{[j}\delta^h_{i]}). \tag{2.8}$$

The right-hand side vanishes identically, because the Γ^h_{ji} is semi-symmetric. Therefore, the ϕ-connection Λ^h_{ji} is symmetric.

Now we have easily

LEMMA 2. 4. *For a half-symmetric ϕ-connection Γ^h_{ji} the quantity*

$$\Pi^h_{ji} = \Gamma^h_{ji} - \frac{2}{n+2}(\Gamma^x_{a(j}\delta^h_{i)} - \Gamma^a_{ab}\phi_{(j}{}^b\phi_{i)}{}^h) - \frac{2}{n}(\Gamma^x_{[aj]}\delta^h_i + \phi_j{}^b\Gamma^x_{[ab]}\phi_i{}^h) \tag{2.9}$$

is invariant under H-*projective changes of Γ^h_{ji}. Conversely, if we have $\Pi^h_{ji} = \bar{\Pi}^h_{ji}$ for two half-symmetric ϕ-connections Γ^h_{ji} and $\bar{\Gamma}^h_{ji}$, then the two half-symmetric ϕ-connections are* H-*projectively related to each other.*

The Π^h_{ji} is not an affine connection, but it seems to be a quantity corresponding to the projective connection of T. Y. Thomas [14]. The Π^h_{ji} transforms like an affine connection under the transformation of coordinates whose Jacobian determinant is constant.

3. The holomorphically projective flatness. Let Γ^h_{ji} be a half-symmetric ϕ-connection in an almost complex manifold. Let us suppose that for any point of the manifold there exists a certain neighborhood of the point in which Γ^h_{ji} is H-projectively related to a flat ϕ-connection. Then the half-symmetric ϕ-connection Γ^h_{ji} is said to be H-*projectively flat.*

We consider the half-symmetric ϕ-connection Λ^h_{ji} formed from a half-symmetric ϕ-connection Γ^h_{ji} by means of (2.7). Then Λ^h_{ji} is H-projectively related to Γ^h_{ji}. If Γ^h_{ji} is supposed to be H-projectively flat, then Λ^h_{ji} is symmetric. In fact, the torsion tensor $T_{ji}{}^h$ of Λ^h_{ji} is given by (2.8). By virtue of Lemma 2.3 we see that T^h_{ji} vanishes identically because of the H-projective flatness of Γ^h_{ji}. Since Λ^h_{ji} is H-projectively related to Γ^h_{ji}, the H-projective flatness of Γ^h_{ji} implies that of Λ^h_{ji}. Thus we have

LEMMA 3.1. *In order that a half-symmetric ϕ-connection Γ^h_{ji} be* H-*projectively flat it is necessary and sufficient that there exist an* H-*projectively flat, symmetric ϕ-connection which is* H-*projectively related to Γ^h_{ji}.*

Lemma 3.1 implies together with Theorem 2 that *the Nijenhuis tensor of an almost complex manifold vanishes identically if the manifold admits an* H-*projectively flat, half-symmetric ϕ-connection.*

As to the symmetric ϕ-connection Γ^h_{ji} Y. Tashiro [13] has recently introduced the tensor

$$(3.1) \qquad P_{kji}{}^h = R_{kji}{}^h + \delta^h_{[k} P_{j]i} - P_{[kj]}\delta^h_i - \phi_{[k}{}^h P_{j]b}\phi_i{}^b + P_{[k|b|}\phi_{j]}{}^b \phi_i{}^h$$

in an almost complex manifold of dimension $n > 2$ and call it the *holomorphically projective* or, briefly, H-*projective curvature tensor* of Γ^h_{ji}, where $R_{kji}{}^h$ and $R_{ji} = R_{aji}{}^a$ are respectivey the curvature tensor and the Ricci tensor of Γ^h_{ji} and P_{ji} is defined by

$$(3.2) \qquad P_{ji} = -\frac{2}{n+2}\left\{R_{ji} + \frac{2}{n-2}\Phi_3(R_{ji} + R_{ij})\right\}.$$

He has proved that *the* H-*projective curvature tensor $P_{kji}{}^h$ is invariant under* H-*projective changes of* Γ^h_{ji} and that *in order that the* Γ^h_{ji} *be* H-*projectively flat in an almost complex manifold of dimension $n > 2$ it is necessary and sufficient that $P_{kji}{}^h$ vanish identically.*

The ϕ-connection Λ^h_{ji} defined by (2.7) is symmetric, if the ϕ-connection Γ^h_{ji} is semi-symmetric. Then we see that *the* H-*projective curvature tensor of Λ^h_{ji} is invariant under* H-*projective changes of Γ^h_{ji} if Γ^h_{ji} is semi-symmetric.*

From the argument above given it follows that *in an almost complex manifold of dimension $n > 2$ a half-symmetric ϕ-connection is* H-*projectively flat if and only if the connection is semi-symmetric and the* H-*projective curvature tensor of the connection Λ^h_{ji} defined by* (2.7) *corresponding to the given connection vanishes identically.*

The following theorem has been proved also in [13].

THEOREM B. *In order that a Kaehlerian manifold of dimension $n > 2$ be* H-*projectively flat*[16]*, it is necessary and sufficient that the manifold be of constant holomorphic sectional curvature.*

4. Quaternion manifolds. Let us consider an n-dimensional manifold admitting two almost complex structures $\phi_i{}^h$ and $\psi_i{}^h$ satisfying

$$(4.1) \qquad \phi_i{}^a\psi_a{}^h + \psi_i{}^a\phi_a{}^h = 0.$$

16) If in a Kaehlerian manifold the Riemannian connection is H-projectively flat, then the manifold is said to be so.

Such a manifold is called a *quaternion manifold* and the pair $(\phi_i{}^h, \psi_i{}^h)$ of two tensors satisfying (4.1) is called a *quaternion structure.*[17] Then it is known that any quaternion manifold is of dimension $n = 4p$.

Now, if we put $K_i{}^h = \phi_i{}^a\psi_a{}^h$, we see that $K_i{}^h$ is also an almost complex structure, i.e. $K_i{}^aK_a{}^h = -\delta_i^h$. If $(\phi_i{}^h, \psi_i{}^h)$ is a quaternion structure, $(\psi_i{}^h, K_i{}^h)$ and $(K_i{}^h, \phi_i{}^h)$ give the same quaternion structure to the manifold [9].

In a quaternion manifold an affine connection is called a *(ϕ, ψ)-connection,* if the two almost complex structures $\phi_i{}^h$ and $\psi_i{}^h$ are covariant constant with respect to the connection. Then, the almost complex structure $K_i{}^h$ is also covariant constant with respect to any (ϕ, ψ)-connection.

Now, let us consider a complex manifold with a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$ of class C^ω, where $\phi_i{}^h$ represents the complex analytic structure. Let $(z^\lambda, \bar{z}^\lambda)$ be a system of complex coordinates with respect to $\phi_i{}^h$. Then, in $(z^\lambda, \bar{z}^\lambda)$ $\phi_i{}^h$, $\psi_i{}^h$ and $K_i{}^h$ take the following components respectively[18]:

$$(\phi_i{}^h) = \begin{pmatrix} \sqrt{-1}\,\delta_\mu^\lambda & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar\mu}^{\bar\lambda} \end{pmatrix}, \quad (\psi_i{}^h) = \begin{pmatrix} 0 & \psi_\mu{}^{\bar\lambda} \\ \psi_{\bar\mu}{}^{\lambda} & 0. \end{pmatrix}, \quad (K_i{}^h) = \begin{pmatrix} 0 & \sqrt{-1}\psi_\mu{}^{\bar\lambda} \\ -\sqrt{-1}\,\psi_{\bar\mu}{}^{\lambda} & 0 \end{pmatrix}$$

It is known that *in a complex manifold with a quaternion structure $(\phi_i{}^h, \psi_j{}^h)$, where $\phi_j{}^h$ gives the complex analytic structure, the (ϕ, ψ)-connection Γ_{ji}^h is determined if a tensor field $T_{ji}{}^h$ of type in $(1,2)$: is given the manifold*

$$\Gamma_{\nu\mu}^\lambda = -(\partial_\nu\psi_\mu{}^{\bar\alpha})\psi_{\bar\alpha}{}^\lambda - \psi_\mu{}^{\bar\beta}T_{\nu\bar\beta}{}^{\bar\alpha}\,\psi_{\bar\alpha}{}^\lambda, \quad \text{conj.};$$
$$\Gamma_{\nu\bar\mu}^\lambda = T_{\bar\mu\lambda}{}^\nu \quad , \quad \text{conj.}, \tag{4.2}$$

the others being zero.[19]

Let us suppose that a (ϕ, ψ)-connection Γ_{ji}^h is semi-symmetric with respect to the complex analytic structure $\phi_i{}^h$. Then, by means of (1.7) the torsion tensor $S_{ji}{}^h$ of Γ_{ij}^h has the components:

$$S_{\bar\nu\mu}{}^\lambda = -S_{\mu\bar\nu}{}^\lambda = A_{\bar\nu}\delta_\mu^\lambda, \qquad S_{\nu\bar\mu}{}^{\bar\lambda} = -S_{\bar\mu\nu}{}^{\bar\lambda} = A_\nu\delta_{\bar\mu}^{\bar\lambda},$$

the others being zero, where A_j is a certain vector field. Thus, taking account of (4.2), we find

$$A_\mu = \frac{4}{n-2}(\partial_{[\mu}\psi_{\beta]}{}^{\bar\alpha})\,\psi_{\bar\alpha}{}^\beta.$$

Consequently, in a complex analytic manifold with a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$ where $\phi_i{}^h$ gives the complex analytic structure, there exists a unique (ϕ, ψ)-connection which is semi-symmetric with respect to $\phi_i{}^h$.

It is known[20] that in a complex manifold with a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$, where $\phi_i{}^h$ gives the complex analytic structure, there exists a symmetric (ϕ, ψ)-connection if and only if the Nijenhuis tensor $N_{ji}{}^h(\psi)$ of $\psi_i{}^h$ vanishes identically. We note here that $N_{ji}{}^h(\psi) = 0$ if and only if $\partial_{[\nu}\psi_{\mu]}{}^{\bar\lambda}$

17) Cf. Ehresmann [2], Libermann [8], Obata [9].
18) Cf. §2, Chap. I of [9].
19) Cf. Theorem 11.1 in [9].
20) Cf. Corollary 2 to Theorem 11.1 in [9].

$= 0$. Hence, the vector A_j above obtained is necessarily zero, if $N_{ji}{}^h(\psi) = 0$. Thus, any (ϕ, ψ)-connection is symmetric in such a manifold, if $N_{ji}{}^h(\psi) = 0$ and the connection is semi-symmetric with respect to $\phi_i{}^h$. Further, the unique symmetric (ϕ, ψ)-connection is given by

$$\Gamma^{\lambda}_{\nu\mu} = -(\partial_\nu \psi_\mu{}^{\bar\alpha})\psi_{\bar\alpha}{}^\lambda, \quad \text{conj.}; \quad \Gamma^{\lambda}_{\bar\nu\mu} = 0, \quad \text{conj.} \tag{4.3}$$

The curvature tensor $R_{kji}{}^h$ of the symmetric (ϕ, ψ)-connection given by (4.3) has the components

$$R_{\bar\omega\nu\mu}{}^\lambda = -R_{\nu\bar\omega\mu}{}^\lambda = -\partial_{\bar\omega}((\partial_\nu\psi_\mu{}^{\bar\alpha})\psi_{\bar\alpha}{}^\lambda),$$

the others being zero[21]. Thus the Ricci tensor R_{ji} vanishes identically. In fact, we see

$$R_{\bar\nu\mu} = R_{\alpha\bar\nu\mu}{}^\alpha = \partial_{\bar\nu}((\partial_\alpha\psi_\mu{}^{\bar\beta})\psi_{\bar\beta}{}^\alpha) = 0.$$

As the Ricci tensor R_{ji} vanishes identically, the H-projective curvature tensor with respect to $\phi_i{}^h$ coincides with the curvature tensor for the symmetric (ϕ, ψ)-connection. Thus, we have

THEOREM 3. *Let Γ^h_{ji} be a half-symmetric (ϕ, ψ)-connection with respect to $\phi_i{}^h$ in a complex manifold with a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$, where $\phi_i{}^h$ gives the complex analytic structure. If Γ^h_{ji} is* H*-projectively flat with respect to $\phi_i{}^h$, and, if $N_{ji}{}^h(\psi) = 0$, then Γ^h_{ji} is of zero curvature.*

Now we assume that in a Kaehlerian manifold defined by $(g_{ji}, \phi_i{}^h)$ there is given another almost complex structure $\psi_i{}^h$ which constitutes a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$ together with $\phi_i{}^h$. We further assume that the Riemannian connection leaves $\psi_i{}^h$ invariant, i.e. $\overset{0}{\nabla}_j\psi_i{}^h = 0$. Then, we say that the given manifold is a *Kaehlerian manifold with a quaternion structure.* It is known that the Ricci tensor of a Kaehlerian manifold vanishes identically if the manifold has a quaternion structure.[22] Thus we have

THEOREM 4. *If a Kaehlerian manifold defined by $(g_{ji}, \phi_i{}^h)$ has a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$, and, if it is* H*-projectively flat with respect to $\phi_i{}^h$, then it is of zero curvature.*

5. **ϕ-projective changes of half-symmetric ϕ-connections.** We assume that in an almost complex manifold two half-symmetric ϕ-connections Γ^h_{ji} and $\bar\Gamma^h_{ji}$ have all paths in common. Then, as is easily seen, on putting $A_{ji}{}^h = \bar\Gamma^h_{ji} - \Gamma^h_{ji}$, we have

$$A_{ji}{}^h = F_{(j}\delta^h_{i)} + P_{ji}{}^h,$$

where F_j is a certain vector field and $P_{ji}{}^h = A_{[ji]}{}^h$.

Now we shall determine the tensor $P_{ji}{}^h$ and then the tensor $A_{ji}{}^h$. The connections Γ^h_{ji} and $\bar\Gamma^h_{ji}$ being ϕ-connections, from Theorem A it follows $\Phi_2 A_{ji}{}^h = 0$. Thus we see

$$2\Phi_2 P^h_{ji} + \Phi_2(F_{(j}\delta^h_{i)}) = 0. \tag{5.1}$$

21) Cf. §11, Chap, III of [9].
22) Cf. Theorem 17.1 in [9].

Since these two connections are half-symmetric, we have $\Phi_1\Phi_3 P_{ji}{}^h = 0$. Combining this with (5.1), we find $\Phi_3 P_{ji}{}^h = 0$, since $\Phi_1 + \Phi_3 =$ identity, $\Phi_3\Phi_4 = 0$.

Since $P_{(ji)}{}^h = 0$, it is easily seen that $\Phi_2 P_{ji}{}^h + \Phi_2^* P_{ij}{}^h = 0$. By virtue of this and (1.1) we have

$$\Phi_2 P_{ji}{}^h = -\Phi_2\Phi_3 P_{ij}{}^h - \Phi_1\Phi_4 P_{ij}{}^h.$$

Taking account of $\Phi_3 P_{ji}{}^h = 0$, we find now

$$\Phi_1 P_{ji}{}^h + \Phi_2 P_{ij}{}^h = 0$$

because of $\Phi_3 + \Phi_4 =$ identity. Thus we obtain

$$P_{ij}{}^h = \Phi_2 P_{ji}{}^h - \Phi_2 P_{ij}{}^h,$$

from which we have, substituting (5.1),

$$P_{ji}{}^h = \Phi_4(F_{[j}\delta_{i]}^h).$$

We have therefore

LEMMA 5.1. *In order that two half-symmetric ϕ-connections Γ'^h_{ji} and Γ^h_{ji} have all paths in common, it is necessary and sufficient that*

$$\Gamma'^h_{ji} = \Gamma^h_{ji} + F_{(j}\delta_{i)}{}^h + \Phi_4(F_{[j}\delta_{i]}^h) \tag{5.2}$$

holds for a certain vector field F_j.

If we put $\Lambda_j = \dfrac{1}{2} F_j$, $T_j = F_j$, from (5.2) it follows

$$\overline{\Gamma}^h_{ji} = \Gamma^h_{ji} + \Lambda_{(j}\delta^h_{i)} - \widetilde{\Lambda}_{(j}\phi_{i)}{}^h + T_j\delta_i{}^h + \widetilde{T}_j\phi_i{}^h.$$

Thus *the change* (5.2) *of a half-symmetric ϕ-connection is an* H*-projective change.* Therefore, two connections Γ'^h_{ji} and Γ^h_{ji} have all holomorphically planar curves in common. Two half-symmetric ϕ-connections related by means of (5.2) are said to be *ϕ-projectively related* to each other. The correspondence $\Gamma'^h_{ji} \rightarrow \Gamma^h_{ji}$ defined by (5.4) is called a *ϕ-projective change* of the half-symmetric ϕ-connection Γ^h_{ji}. From the corollary to Lemma 2.3 we have now

LEMMA 5.2. *A half-symmetric ϕ-connection is semi-symmetric, if it is ϕ-projectively related to a symmetric ϕ-connection.*

LEMMA 5.3. *Let Γ^h_{ji} be a half-symmetric ϕ-connetion and $S_{ji}{}^h$ its torsion tensor. Then the half-symmetric ϕ-connection*

$$\Sigma^h_{ji} = \Gamma^h_{ji} - \frac{4}{n} S_{(j}\delta^h_{i)} - \frac{4}{n}\Phi_4(S_{[j}\delta^h_{i]}), \qquad S_j = S_{ja}{}^a \tag{5.3}$$

is invariant under ϕ-projective changes of Γ^h_{ji}. Conversely, if we have $\Sigma^h_{ji} = \overline{\Sigma}^h_{ji}$ for two half-symmetric ϕ-connections Γ^h_{ji} and $\overline{\Gamma}^h_{ji}$, then the two connections are ϕ-projectively related to each other.

PROOF. It is easily seen that the ϕ-connection Σ^h_{ji} is half-symmetric. Let $\overline{S}_{ji}{}^h$ be the torsion tensor of the ϕ-projectively related half-symmetric ϕ-connection Γ_{ji}^h given in (5.2). Taking the anti-symmetric parts of the both sides of (5.2) and contracting h and i, we find

$$F_j = \frac{4}{n}(\bar{S}_j - S_j),$$

where $\bar{S}_j = S_{ja}{}^a$. If we substitute this in (5.2), we have a relation which shows that the ϕ-connection Σ_{ji}^h defined by (5.3) is invariant under the ϕ-projective change (5.2). The first part of Lemma 5.3 is thereby proved. The second part will be proved easily.

COROLLARY. *If two symmetric ϕ-connections are projectively or, equivalently, ϕ-projectively related to each other, then they coincide with each other.*

Thus we have

THEOREM 5.[23] *In a complex manifold, if two Kaehlerian metrics have all geodesics in common, then their Riemannian connections coincide with each other.*

Keeping assumptions as in Theorem 5, we see that *the two metrics are homothetically related to each other if, moreover, at least one of the given Kaehlerian metrics is irreducible.*

Let Γ_{ji}^h be a half-symmetric ϕ-connection in an almost complex manifold. Let us suppose that for any point of the manifold there exists a certain neighborhood of the point in which Γ_{ji}^h is ϕ-projectively related to a flat ϕ-connection. Then the given half-symmetric ϕ-connection Γ_{ji}^h is said to be *ϕ-projectively flat*. If the Riemannian connection of a Kaehlerian manifold is projectively or, equivalently, ϕ-projectively flat, then the manifold is said to be projectively or ϕ-projectively flat. Now we have the following

COROLLARY. *If a Kaehlerian manifold is projectively flat, then the manifold is of zero curvature.*

6. Compact groups of H-projective transformations and groups of H-projective transformations preserving the Ricci tensor. In an almost complex manifold defined by $\phi_i{}^h$ we consider a transformation leaving the almost complex structure $\phi_i{}^h$ invariant and call it a *ϕ-transformation.* Let Γ_{ji}^h be a ϕ-connection in the manifold; then the affine connection $\bar{\Gamma}_{ji}^h$ induced[24] from Γ_{ji}^h by a ϕ-transformation is also a ϕ-connection. If Γ_{ji}^h is half-symmetric, then $\bar{\Gamma}_{ji}^h$ is also.

We assume that a ϕ-transformation s carries any holomorphically planar curve of a half-symmetric ϕ-connection Γ_{ji}^h into such a curve. Then, the ϕ-transformation s is said to be *holomorphically projective* or, briefly, H-*projective* with respect to Γ_{ji}^h. According to Lemma 2.1 we see that in this case the half-symmetric ϕ-connection $\bar{\Gamma}_{ji}^h$ induced from Γ_{ji}^h by s is related with Γ_{ji}^h by

23) This theorem has been proved by Bochner [1].

24) Cf. Ishihara [7], for example.

means of (2.2).

Let us consider a group[25] of H-projective transformations of a half-symmetric ϕ-connection Γ_{ji}^h in an almost complex manifold. If there exists in the manifold an H-projectively related half-symmetric ϕ-connection which is invariant under the group, then the group is said to be *essentially affine* with respect to the Γ_{ji}^h.

We have now the following theorems in an analogous way as in a previous paper.[26]

THEOREM 6. *A group of* H-*projective transformations of a half-symmetric ϕ-counection is essentially affine, if the group is compact.*

THEOREM 7. *A transitive group of* H-*projective transformations of a half-symmetric ϕ-connection is essentially affine, if its isotropy group is compact.*

Corresponding to Theorem 3 in [7] we have the following theorem, if we take account of the results which will be given in §7 and §8.

THEOREM 8. *Let G be a transitive group of* H-*projective transformations of a half-symmetric ϕ-connection, which is not* H-*projectively flat, in an almost complex manifold. If the identity component of the linear isotropy group of G at a point is irreducible, then G is essentially affine with respect to the given connection.*

Theorem 8 implies together with Theorem B the following

COROLLARY. *Let G be a transitive group of* H-*projective transformations of a Kaehlerian manifold whose holomorphic sectional curvature is not constant. If the identity component of the linear isotropy group of G at a point is irreducible, then G is essntially affine with respect to the Riemannian connection.*

In a complex manifold of dimension $n = 2m$, we consider a symmetric ϕ-connection Γ_{ji}^h and an H-projective transformation s of Γ_{ji}^h. Then the ϕ-connection $\bar{\Gamma}_{ji}^h$ induced from Γ_{ji}^h by s is related to Γ_{ji}^h by means of (2.2). Denoting by $R_{kji}{}^h$ and $\bar{R}_{kji}{}^h$ the curvature tensors of Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ respectively, we have by virtue of (2.2)

$$\bar{R}_{kji}{}^h = R_{kji}{}^h + \delta_{[k}^h F_{j]i} - F_{[kj]}\delta_i^h - \phi_{[k}^h F_{j]b}\phi_i^b + F_{[k|b|}\phi_{j]}{}^b\phi_i{}^h, \tag{6.1}$$

where F_{ji} is defined by

$$F_{ji} = \nabla_j F_i - \Phi_3(F_j F_i).$$

Contracting h and k in (6.1), we find

$$\bar{R}_{ji} = R_{ji} + \Phi_3(F_{ji} + F_{ij}) - \frac{n+2}{2} F_{ji}, \tag{6.2}$$

where R_{ji} and $\bar{R}_{ji}$ are the Ricci tensors of Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ respectively.

We suppose that the transformation s preserves the Ricci tensor, i.e., $\bar{R}_{ji} = R_{ji}$. Then from (6.2) it follows that

25) We restrict attention to Lie groups in the paper.
26) Cf. Theorems 1 and 2 in [7].

$$\Phi_3(F_{ji} + F_{ij}) - \frac{n+2}{2} F_{ji} = 0$$

which implies, provided $n > 2$, $F_{ji} = 0$. Thus, we have

LEMMA 6.1. *In a complex manifold of complex dimension $m > 1$, if an H-projective transformation of a symmetric ϕ-connection preserves the Ricci tensor, then the vector field F_j corresponding to the transformation satisfies*

$$\nabla_j F_i = \Phi_3 (F_j F_i). \tag{6.3}$$

It is easily seen from the above argument that an H-projective transformation of a symmetric ϕ-connection preserves the curvature tensor if and only if it leaves the Ricci tensor invariant. We have the following lemma as in a previous paper.[27]

LEMMA 6.2. *If in a complex manifold with a symmetric ϕ-connection there exists a non-trivial vector field F_j satisfying* (6.3), *then the homogeneous holonomy group of the manifold has an invariant hyperplane and the restricted homogeneous holonomy group of the manifold has an invariant covariant vector.*

In a complex manifold M with a half-symmetric ϕ-connection we denote by $HP(M)$ and $A(M)$ the group of all H-projective transformations and that of all affine transformations respectively. We shall denote by $HP^*(M)$ the group of all H-projective transformations preserving the Ricci tensor. In a Kaehlerian manifold we denote by $I(M)$ the group of all isometries and by $HP(M)$, $HP^*(M)$ and $A(M)$ respectively the corresponding groups with respect to the Riemannian connection. By virtue of Lemma 6.2, we have the following theorem as in a previous paper[28].

THEOREM 9. *Let M be a complex manifold of complex dimension $m > 1$ with a symmetric ϕ-connection. If the homogeneous holonomy group of M has no invariant hyperplane, or, if the restricted homogeneous holonomy group has no invariant covariant vector, then $HP^*(M) = A(M)$. If, moreover, the Ricci tensor of M vanishes identically, then $HP(M) = A(M)$.*

We can prove the following lemma as in a previous paper.[29]

LEMMA 6.3. *In a complex manifold with a symmetric ϕ-connection, if a covariant vector field F_j satisfying* (6.3) *has no singularity at any point, provided the manifold to be complete with respect to the ϕ-connection, then F_j vanishes identically.*

As a consequence of Lemma 6.3 we have the following theorem just as in a previous paper.[30]

THEOREM 10. *If a complex manifold M of complex dimension $m > 1$ is complete with respect to a symmetric ϕ-connection, then $HP^*(M) = A(M)$. If,*

27) Cf. Lemma 3 in [7].
28) Cf. Theorem 6 in [7].
29) Cf. Lemma 4 in [7].
30) Cf. Theorem 7 in [7].

moreover, the Ricci tensor of M vanishes identically, then $HP(M)=A(M)$.

We denote by $HP_0^*(M)$ the identity component of the group $HP^*(M)$ and use analogous notations for the other groups. We have now the following theorems as in a previous paper[31].

THEOREM 11. *If the restricted homogeneous holonomy group of a complete Kaehlerian manifold M has no invariant vector, and, if M is of complex dimension $m>1$, then $HP_0^*(M)=I_0(M)$. If, moreover, the Ricci tensor of M vanishes identically, then $HP_0(M)=I_0(M)$.*

THEOREM 12. *If M is a compact Kaehlerian manifold of complex dimension $m>1$, then $HP_0^*(M)=I_0(M)$. If, moreover, the Ricci tensor of M vanishes identically, then $HP_0(M)=I_0(M)$.*

7. **Infinitesimal H-projective transformations.** In an almost complex manifold defined by $\phi_i{}^h$ an infinitesimal transformation u^h is called an infinitesimal ϕ-transformation, if u^h preserves[32] the almost complex structure $\phi_i{}^h$, i. e., if

$$\underset{u}{\mathcal{L}}\, \phi_i{}^h = u^a\partial_a\phi_i{}^h - \phi_i{}^a\partial_a u^h + \phi_b{}^h\partial_i u^b = 0, \tag{7.1}$$

where $\underset{u}{\mathcal{L}}$ denotes the operator of Lie differentiation[33] with respect to u^h. Let Γ^h_{ji} be a half-symmetric ϕ-connection in the manifold. We call an infinitesimal ϕ-transformation u^h an *infinitesimal* H-*projective transformation* of Γ^h_{ji}, if we have

$$\underset{u}{\mathcal{L}}\, \Gamma^h_{ji} = F_{(j}\delta^h_{i)} - \widetilde{F}_{(j}\phi^h_{i)} + T_j\delta^h_i + \widetilde{T}_j\phi_i{}^h \tag{7.2}$$

for certain vector fields F_j and T_j. If the connection Γ^h_{ji} is symmetric, the condition (7.2) is reduced to

$$\underset{u}{\mathcal{L}}\, \Gamma^h_{ji} = F_{(j}\delta^h_{i)} - \widetilde{F}_{(j}\phi^h_{i)}. \tag{7.3}$$

Let u^h be an infinitesimal H-projective transformation of a half-symmetric ϕ-connection Γ^h_{ji} in an almost complex manifold. Contracting h and i in (7.2), we find

$$\underset{u}{\mathcal{L}}\, \Gamma^i_{ji} = \frac{n+2}{2}F_j + T_j.$$

Further, contracting h and j in (7.2), we have

$$\underset{u}{\mathcal{L}}\Gamma^i_{aj} = \frac{n+2}{2}F_j.$$

From these two relations it follows that

$$F_j = \frac{2}{n+2}\underset{u}{\mathcal{L}}\, \Gamma^i_{aj}, \qquad T_j = \frac{2}{n}\underset{u}{\mathcal{L}}\, \Gamma^i_{[ji]}\,.$$

31) Cf. Theorems 8 and 9 in [7].

32) If the Lie derivative of a geometric object vanishes with respect to an infinitesimal transformation u^h, then we say that u^h preserves the geometric object.

33) Cf. Yano [17].

If we substitute this in the right-hand side of (7.2), we obtain

$$\underset{u}{\pounds}\Pi^h_{ji} = 0, \tag{7.4}$$

where Π^h_{ji} is the quantity defined by (2.9) corresponding to Γ^h_{ji}. It is easily verified that the Lie derivative $\underset{u}{\pounds}\Pi^h_{ji}$ is a tensor. By using (7, 4) we have

LEMMA 7.1. *Let u^h be an infinitesimal* H-*projective transformation of a half-symmetric ϕ-connection Γ^h_{ji} in an almost complex manifold and p be a point of the manifold such that u^h does not vanish at p. Then, in a certain neighborhood of p there exists a system of coordinates (x^i) such that* $\partial_1 \Pi^h_{ji} = 0$.

This lemma implies together with Lemma 2.4

LEMMA 7.2. *Let G be a one-parameter group of ϕ-transformations in an almost complex manifold with a half-symmetric ϕ-connection Γ^h_{ji}. If the infinitesimal transformation induced by G is* H-*projective with respect to Γ^h_{ji}, then the group G is that of* H-*projective transformation of Γ^h_{ji}.*

By means of Lemma 7.2 the problems concerning groups of H-projective transformations is reduced to those of infinitesimal H-projective transformations, as far as connected groups are concerned.

Let us consider two infinitesimal H-projective transformations u^h and v^h. We define as usual the product w^h of u^h and v^h by

$$w^h = u^a\partial_a v^h - v^a\partial_a u^h.$$

Then, as is well known, we have[34)]

$$\underset{w}{\pounds}\Gamma^h_{ji} = \underset{u}{\pounds}(\underset{v}{\pounds}\Gamma^h_{ji}) - \underset{v}{\pounds}(\underset{u}{\pounds}\Gamma^h_{ji}). \tag{7.5}$$

In an almost complex manifold with a half-symmetric ϕ-connection Γ^h_{ji} we construct the half-symmetric ϕ-connection Λ^h_{ji} defined by (2.7) corresponding to Γ^h_{ji}. It is easily seen from (7.2) that for an infinitesimal H-projective transformation u^h of Γ^h_{ji}

$$\underset{u}{\pounds}\Lambda^h_{ji} = F_{(j}\delta^h_{i)} - \widetilde{F}_{(j}\phi_{i)}{}^h \tag{7.6}$$

holds. We suppose that $\underset{v}{\pounds}\Lambda^h_{ji} = G_{(j}\phi^h_{i)} - \widetilde{G}_{(j}\phi_{i)}{}^h$ holds good for another infinitesimal H-projective transformation v^h. Then from (7.5) it follows easily

$$\underset{w}{\pounds}\Lambda^h_{ji} = H_{(j}\delta^h_{i)} - \widetilde{H}_{(j}\phi_{i)}{}^h, \qquad H_j = \underset{u}{\pounds}G_j - \underset{v}{\pounds}F_j, \tag{7.7}$$

where w^h is the product of u^h and v^h. We see thereby that in an almost complex manifold with a half-symmetric ϕ-connection the set of all infinitesimal H-projective transformations of the ϕ-connection forms a Lie algebra.

Now we shall consider the integrability conditions of the differential equations (7.6). It is well known[35)]

$$\underset{u}{\pounds}R_{kji}{}^h = \nabla_k(\underset{u}{\pounds}\Lambda^h_{ji}) - \nabla_j(\underset{u}{\pounds}\Lambda^h_{ki}) - 2T_{kj}{}^a(\underset{u}{\pounds}\Lambda^h_{ai}),$$

34) Cf. Yano [17].
35) Cf. Yano [17].

∇_j denoting the operator of covariant differentiation with respect to λ_{ji}^h where $R_{kji}{}^h$ and $T_{ji}{}^h$ are respectively the curvature tensor and the torsion tensor of Λ_{ji}^h. Substituting (7.6) in the right-hand side of the above relation, we have

$$
\underset{u}{\pounds} R_{kji}{}^h = -\delta_{[k}^h\nabla_{j]}F_i + \nabla_{[k}F_{j]}\delta_i^h + \phi_{[k}{}^h\nabla_{j]}\widetilde{F}_i - \nabla_{[k}\widetilde{F}_{j]}\phi_i{}^h
$$
$$
- 2\,T_{kj}{}^a(F_{(a}\delta_{i)}^h - \widetilde{F}_{(a}\phi_{i)}{}^h). \tag{7.8}
$$

Contracting h and k, we find

$$
\underset{u}{\pounds} R_{ji} = \Phi_3(\nabla_jF_i + \nabla_iF_j) - \frac{n+2}{2}\nabla_jF_i - 2F_a\,\Phi_1T_{ji}{}^a \tag{7.9}
$$

because of $T_j{}^a = T_{jb}{}^a\phi_a^b = 0$, where R_{ji} denotes the Ricci tensor of Λ_{ji}^h. Now we have to note that for any infinitesimal ϕ-transformation u^h the operator $\underset{u}{\pounds}$ commutes with each of the operators Φ_r, i. e., $\underset{u}{\pounds}\,\Phi_r = \Phi_r\underset{u}{\pounds}\,(r = 1, 2, 3, 4)$. If we apply Φ_3 to the both sides of the above relation, we have, provided $n > 2$,

$$
\Phi_3(\nabla_jF_i + \nabla_iF_j) = -\frac{2}{n-2}\underset{u}{\pounds}\,\Phi_3(R_{ji} + R_{ij})
$$

because of $\Phi_1\Phi_3\,T_{ji}{}^h = 0$ and hence

$$
\nabla_jF_i = \underset{u}{\pounds}\,P_{ji} + \frac{2}{n+2}F_a\Phi_1T_{ji}{}^a, \tag{7.10}
$$

where P_{ji} is the tensor constructed formally from Λ_{ji}^h by means of (3.2).

By virtue of (7.6) and (7.10), if we put $u_i{}^h = \nabla_iu^h$, we see that u^h, F_j and $u_i{}^h$ form a system of solutions of the differential equations

$$
\nabla_ju^h = u_j{}^h,
$$
$$
\nabla_ju_i{}^h = 2\,u^a\nabla_jT_{ai}{}^h + 2\,u_j{}^aT_{ai}{}^h - u^aR_{aji}{}^h - F_{(j}\delta_{i)}^h + \widetilde{F}_{(j}\phi_{i)}{}^h,
$$
$$
\nabla_jF_i = u^a\nabla_aP_{ji} + P_{ai}u_j{}^a + P_{ja}u_i{}^a + \frac{n}{n+2}F_a\Phi_1T_{ji}{}^a.
$$

We have thereby

LEMMA 7.3. *Let u^h be an infinitesimal* H*-projective transformation of a half-symmetric ϕ-connection in an almost complex manifold of dimension $n > 2$ and F_j the vector field given in* (7.6) *corresponding to u^h. If all of u^h, F_j and ∇_iu^h vanishes at a point of the manifold, then u^h vanishes identically.*

We take a point p_0 of an almost complex manifold with a half-symmetric ϕ-connection Γ_{ji}^h. It is easily seen that the set of all infinitesimal H-projective transformations of Γ_{ji}^h vanishing at p_0 forms a subalgebra $\mathfrak{G}_0$ of the Lie algebra $\mathfrak{G}$ of all H-projective infinitesimal transformations of Γ_{ji}^h. The algebra $\mathfrak{G}_0$ is called the isotropy Lie algebra of $\mathfrak{G}$ at p_0. Let T_0 be the tangent space of the manifold at the point p_0. Then in T_0 there exists a system of complex coordinates $(\xi^\lambda, \xi^{\bar\lambda})$ in which the value $(\phi_i{}^h)_0$ of the almost complex structure $\phi_i{}^h$ at p_0 has the components

$$
{}_i{}^h) = \begin{pmatrix} \sqrt{-1}\delta_\mu^\lambda & 0 \\ 0 & -\sqrt{-1}\delta_{\bar\mu}^{\bar\lambda} \end{pmatrix}. \tag{7.11}
$$

respect to $\bar{\Gamma}^h_{ji}$. Thus the argument developed above concerning Λ^h_{ji} applies also to the flat ϕ-connection $\bar{\Gamma}^h_{ji}$ in U.

Let u^h be an infinitesimal H-projective transformation of Γ^h_{ji}. Denoting by $\bar{\nabla}_j$ the operator of covariant differentiation with respect to $\bar{\Gamma}^h_{ji}$, we see according to (7.2) that

$$\bar{\nabla}_j \bar{\nabla}_i u^h = F_{(j}\delta^h_{i)} - \tilde{F}_{(j}\phi_{i)}{}^h \tag{7.14}$$

holds in U for a vector field F_j, since $\bar{\Gamma}^h_{ji}$ is flat. From (7.10) it follows that

$$\bar{\nabla}_j F_i = 0 \tag{7.15}$$

holds in U if $n = 2m > 2$.

We consider another infinitesimal H-projective transformation v^h of Γ^h_{ji}. Denoting by G_j the vector field corresponding to v^h, we see from (7.7) that the vector field H_j corresponding to the product w^h of u^h and v^h is given in U by

$$H_j = G_a \bar{\nabla}_j u^a - F_a \bar{\nabla}_j v^a \tag{7.16}$$

because of (7.15). We see further on account of (7.14) that

$$\begin{aligned} \bar{\nabla}_i w^h = {} & \bar{\nabla}_i u^a \bar{\nabla}_a v^h - \bar{\nabla}_i v^a \bar{\nabla}_a v^h \\ & + u^a(G_{(i}\delta^h_{a)} - \tilde{G}_{(i}\phi_{a)}{}^h) - v^a(F_{(i}\delta^h_{a)} - \tilde{F}_{(i}\phi_{a)}{}^h) \end{aligned} \tag{7.17}$$

holds in U. From the definition we have in U

$$w^h = u^a \bar{\nabla}_a v^h - v^a \bar{\nabla}_a u^h. \tag{7.18}$$

The tensor $\bar{\nabla}_i u^h$ has self-adjoint components, such as given in (7.12), in complex coordinates $(\xi^\lambda, \bar{\xi}^\lambda)$ in the tangent space T_0 at p_0. If w^h is the product of two infinitesimal H-projective transformations u^h and v^h, denoting by $a^\lambda(u)$, $a_\mu(u)$ and $a_\mu{}^\lambda(u)$ respectively the values of u^h, F_j and $\bar{\nabla}_i u^h$ at a point p_0 with respect to $(\xi^\lambda, \bar{\xi}^\lambda)$, we have by virtue of (7.16), (7.17) and (7.18)

$$\begin{aligned} a^\lambda(w) &= a^\alpha(u)a_\alpha{}^\lambda(v) - a^\alpha(v)a_\alpha{}^\lambda(u), \\ a_\mu(w) &= a_\mu{}^\alpha(u)a_\alpha(v) - a_\mu{}^\alpha(v)a_\alpha(u), \\ a_\mu{}^\lambda(w) &= a_\mu{}^\alpha(u)a_\alpha{}^\lambda(v) - a_\mu{}^\alpha(v)a^\lambda{}_\alpha(u) \\ &\quad - (a_\mu(u)a^\lambda(v) - a_\mu(v)a^\lambda(u)) \\ &\quad - (a_\alpha(u)a^\alpha(v) - a_\alpha(v)a^\alpha(u))\,\delta^\lambda_\mu. \end{aligned}$$

For an infinitesimal H-projective transformation u^h we define as below:

$$b_0{}^0(u) = -\frac{1}{m+1}\, a_\alpha{}^\alpha(u), \qquad b_\mu{}^0(u) = a_\mu(u),$$

$$b_0{}^\lambda(u) = -a^\lambda(u), \qquad b_\mu{}^\lambda(u) = a_\mu{}^\lambda(u) - \frac{1}{m+1}\, a_\alpha{}^\alpha(u)\delta^\lambda_\mu.$$

Then the above relations can be witten as follows:

$$b_q{}^p(w) = \sum_{r=0}^{m} b_q{}^r(u)\, b_r{}^p(v) - \sum_{r=0}^{m} b_q{}^r(v)\, b_r{}^p(u), \tag{7.19}$$

Let u^h be an element of $\mathfrak{G}_0$ and F_j the vector field corresponding to it. Then, from (7.1) it follows

$$(\phi_i{}^a)_0(\nabla_a u^h)_0 = (\nabla_i u^a)_0(\phi_a{}^h)_0,$$

$(\nabla_i u^h)_0$ denoting the value of $\nabla_i u^h$ at p_0, since u^h vanishes at p_0. Thus, taking account of (7.11), we see that $(\nabla_i u^h)_0$ has self-adjoint components

$$(7.12) \qquad (\nabla_i u^h)_0 = \begin{pmatrix} a_\mu{}^\lambda(u) & 0 \\ 0 & a_{\bar\mu}{}^{\bar\lambda}(u) \end{pmatrix}$$

in complex coordinates $(\xi^\lambda, \bar\xi^\lambda)$ introduced in T_0. Denoting by $a_\mu(u)$ the components of F_j in $(\xi^\lambda, \bar\xi^\lambda)$, we shall define the complex $(m+1, m+1)$-matrix

$$(7.13) \qquad \alpha(u) = \begin{pmatrix} 0 & a_\mu(u) \\ 0 & a_\mu{}^\lambda(u) \end{pmatrix}$$

corresponding to u^h, where $n = 2m$.

Consider two elements u^h and v^h of $\mathfrak{G}_0$ and their product $w^h \in \mathfrak{G}_0$. Then, since $u^h = v^h = 0$ at p_0, from (7.7) it follows

$$\alpha(w) = [\alpha(u), \ \alpha(v)],$$

where the right-hand side denotes the commutator product of two matrices $\alpha(u)$ and $\alpha(v)$. If we denote by $\tilde{\mathfrak{L}}$ the set of all complex $(m+1, m+1)$-matrices of the form

$$A = \begin{pmatrix} 0 & a_\mu \\ 0 & a_\mu{}^\lambda \end{pmatrix},$$

then it forms a Lie algebra over the field of real numbers, which is denoted also by $\tilde{\mathfrak{L}}$. The correspondence $u^h \to \alpha(u)$ determines therefore a homomorphism α of $\mathfrak{G}_0$ into $\tilde{\mathfrak{L}}$. Thus, on account of Lemma 7.3 we have

LEMMA 7.4. *The isotropy Lie algebra $\mathfrak{G}_0$ is isomorphic to a subalgebra of $\tilde{\mathfrak{L}}$, i.e. the homomorphism α is an isomorphism, if the manifold is of dimension $n > 2$.*

We define for an element u^h of $\mathfrak{G}_0$ a complex (m, m)-matrix

$$\beta(u) = (a_\mu{}^\lambda(u)),$$

where $a_\mu{}^\lambda(u)$ is the coefficients of $\alpha(u)$. The set $\mathfrak{L}_m$ of all complex (m, m)-matrices forms a Lie algebra over the field of real numbers, which is denoted also by $\mathfrak{L}_m$. Then, the correspondence $u^h \to \beta(u)$ determines a homomorphism β of $\mathfrak{G}_0$ into $\mathfrak{L}_m$. The image $\beta(\mathfrak{G}_0)$ of $\mathfrak{G}_0$ by β is called the linear isotropy Lie algebra $\mathfrak{g}_0$ of $\mathfrak{G}$ at the point p_0.

Let us consider an almost complex manifold of dimension $2m$ which admits H-projectively flat half-symmetric ϕ-connection Γ_{ji}^h. Then the Nijenhuis tensor $N_{ji}{}^h$ of the manifold vanishes identically. We take a point p_0 of the manifold. There exists then a neighborhood U containing p_0 in which Γ_{ji}^h is H-projectively related to a flat ϕ-connection Γ_{ji}^h, because Γ_{ji}^h is H-projectively flat. Any infinitesimal H-projective transformation of Γ_{ji}^h is also H-projective in U with where $(p, q = 0, 1, \dots m)$.

Now we make correspond the complex matrix

$$B(u) = \begin{pmatrix} b_0{}^0(u) & b_\mu{}^0(u) \\ b_0{}^\lambda(u) & b_\mu{}^\lambda(u) \end{pmatrix} \tag{7.20}$$

to an infinitesimal H-projective transformation u^h of Γ_{ji}^h. Here we have to note that the trace of $B(u)$ vanishes. Let $S\,\mathfrak{L}_{m+1}$ be the set of all complex $(m+1, m+1)$-matrices whose trace vanishes. Then it forms a Lie algebra over the field of real numbers, which is denoted also by $S\,\mathfrak{L}_{m+1}$. The correspondence $u^h \to B(u)$ determines a homomorphism B of the Lie algebra $\mathfrak{G}$ of all infinitesimal H-projective transformations of Γ_{ji}^h into the Lie algebra $S\,\mathfrak{L}_{m+1}$. Further, from Lemma 7.3 it follows that the homomorphism B is an isomorphism if $n = 2m > 2$. By virtue of the argument above given we have

THEOREM 13. *Let $\mathfrak{G}$ be the Lie algebra of all infinitesimal* H*-projective transformations of an* H*-projectively flat, half-symmetric ϕ-connection in a complex manifold of complex dimension $m > 1$. Then $\mathfrak{G}$ is isomorphic to a subalgebra of the Lie algebra $S\,\mathfrak{L}_{m+1}$ i.e. the homomorphism B defined by* (7.20) *is an isomorphism.*

8. **Th group of H-projective transformations of sufficiently high order.** We shall now study a manifold admitting a group of H-projective transformations of sufficiently high order. For this purpose we shall give some preliminary lemmas. Let us consider an almost complex manifold with a half-symmetric ϕ-connection Γ_{ji}^h. We denote by $\mathfrak{G}$ the Lie algebra of all infinitesimal H-projective transformations of Γ_{ji}^h and by $\mathfrak{G}_0$ the isotropy Lie algebra of $\mathfrak{G}$ at a point p_0 of the manifold. The linear isotropy Lie algebra of $\mathfrak{G}$ at p_0 is denoted by $\mathfrak{g}_0$.

Let u^h be an infinitesimal H-projective transformation of Γ_{ji}^h and Λ_{ji}^h be the half-symmetric ϕ-connection defined by (2.7) corresponding to Γ_{ji}^h. It it easily seen from (7.6) that

$$F_j = \frac{2}{n+2} \underset{u}{\mathcal{L}} \Lambda_{aj}^a.$$

Substituting this in (7.10), since $\underset{u}{\mathcal{L}} T_{ii}{}^h = 0$, we have

$$\nabla_j F_i = \underset{u}{\mathcal{L}}\, \Pi_{ji}, \tag{8.1}$$

the quantity Π_{ji} being defined by

$$\Pi_{ji} = P_{ji} + \frac{4}{(n+2)^2} \Lambda_{ab}^a \Phi_1 T_{ji}{}^b,$$

where P_{ji} is the tensor defined by (3.2) and $T_{ji}{}^h$ is the torsion tensor of Λ_{ji}^h. The quantity Π_{ji} is not a tensor, but its Lie derivative $\underset{u}{\mathcal{L}}\, \Pi_{ji}$ is a tensor. If we substitute (8.1) in the right-hand side of (7.8), we find

$$\underset{u}{\mathcal{L}}\, \Pi_{kji}{}^h = -2\, T_{kj}{}^a (\widetilde{F}_{(a}\delta_{i)}{}^h - \widetilde{F}_{(a}\phi_{i)}{}^h), \tag{8.2}$$

where $\Pi_{kji}{}^h$ is defined by

$$\Pi_{kji}{}^h = R_{kji}{}^h + \delta^h_{[k}\Pi_{j]i} - \Pi_{[kj]}\delta^h_i - \phi_{k[}{}^h\,\Pi_{j]a}\,\phi_i{}^a + \Pi_{[k|a|}\,\phi_{j]}{}^a\,\phi_i{}^h.$$

The quantity $\Pi_{kji}{}^h$ is not a tensor, but its Lie derivative $\underset{u}{\mathcal{L}}\ \Pi_{kji}{}^h$ is a tensor.

LEMMA 8.1. *If the kernel of the homomorphisms $\beta : \mathfrak{G}_0 \to \mathfrak{g}_0$ is not the trivial subalgebra $\{0\}$, then the torsion tensor $T_{ji}{}^h$ of Λ^h_{ji}* vanishes at p_0.

PROOF From the condition of the lemma it follows that there exists an infinitesimal H-projective transformation u^h such that u^h and $\nabla_i u^h$ vanishes at the point p_0 but the vector field F_j corresponding to u^h does not vanish at p_0. We have to note that the Lie derivative $\underset{u}{\mathcal{L}}\ \Pi_{kji}{}^h$ is a linear combination of u^h and $\nabla_i u^h$. Thus $\underset{u}{\mathcal{L}}\ \Pi_{kji}{}^h$ vanishes at p_0. From (8.2) it follows that

$$T_{kj}{}^a\ (F_{(i}\delta^h_{a)} - \widetilde{F}_{(i}\phi_{a)}{}^h) = 0$$

holds at p_0. Contracting h and i, we obtain $T_{kj}{}^a F_a = 0$. Multiplying $\phi^i{}_h$ and contracting, we find $T_{kj}{}^a \widetilde{F_a} = 0$. Consequently, it follows $T_{kj}{}^h F_i - T_{kj}{}^a\,\phi_a{}^h \widetilde{F_i} = 0$, which becomes $T_{kj}{}^\lambda F_\mu = 0$ in complex coordinates $(\xi^\lambda, \bar{\xi}^\lambda)$ introduced in the tangent space T_0 at p_0. Thus we have $T_{kj}{}^h = 0$ at p_0, since the vector field F_j does not vanish at p_0. Lemma 8.1 is thereby proved.

Now we consider a semi-symmetric ϕ-connection Γ^h_{ji}. Then the connection Λ^h_{ji} defined by (2.7) corresponding to Γ^h_{ji} is symmetric. i.e. $T_{ji}{}^h = 0$. Let u^h be an infinitesimal H-projective transformation of Γ^h_{ji}. Taking account of $T_{ji}{}^h = 0$, from (7.8) and (7.10) we have

$$\underset{u}{\mathcal{L}}\ P_{kji}{}^h = 0, \tag{8.3}$$

where $P_{kji}{}^h$ is the H-projective curvature tensor of Λ^h_{ji}. As the integrability condition of (7.10), i.e. of $\nabla_j F_i = \underset{u}{\mathcal{L}}\ P_{ji}$, we have

$$\underset{u}{\mathcal{L}}\ P_{kji} = P_{kji}{}^a\ F_a, \tag{8.4}$$

where P_{kji} has been defined by $P_{kji} = 2\ \nabla_{[k}\ P_{j]i}$. The relation (8.4) being established, we have easily

LEMMA 8.2. *Let Γ^h_{ji} be a semi-symmetric ϕ-connection. If the kernel of the homomorphism $\beta : \mathfrak{G}_0 \to \mathfrak{g}_0$ is of the maximum dimension $2m$, then the* H-*projective curvature tensor of Λ^h_{ji} vanishes at p_0.*

The closed and connected subgroups of the group of all real (n, n)-matrices has been determined by H. C. Wang and K. Yano [15], if it is of dimension not less than $n^2 - 2n + 5$. Now we have Lemma 8.3 establishing a similar fact concerning subalgebras of the Lie algebra $\mathfrak{L}_m$. We shall use the following notations denoting the subalgebras of $\mathfrak{L}_m$:

$$S\mathfrak{L}_m = \{(a_\mu{}^\lambda) \mid a_\alpha{}^\alpha = 0\},$$
$$\mathfrak{M} \quad = \{(a_\mu{}^\lambda) \mid a_1{}^p = 0,\ p = 2, 3, \ldots, m\},$$
$$\mathfrak{M}' \quad = \{(a_\mu{}^\lambda) \mid a_q{}^1 = 0,\ q = 2, 3, \ldots, m\},$$

$\mathfrak{Z}(A)=\{(a_\mu{}^\lambda)\,|\,a_\mu{}^\lambda=At\,\delta_\mu{}^\lambda,\quad A:\text{ non-vanishing complex number}\}$, where t is a real variable. We see at once that dim $S\mathfrak{L}_m=2(m^2-1)$, dim $\mathfrak{M}=$ dim $\mathfrak{M}'=2(m^2-m+1)$, dim $\mathfrak{Z}(A)=1$. We denote by $\mathfrak{Z}(A)\times S\mathfrak{L}_m$ the Lie algebra generated by $\mathfrak{Z}(A)$ and $\mathfrak{S}\mathfrak{L}_m$.

LEMMA 8.3. *Each subalgebra of $\mathfrak{L}_m$ is, if its dimension is not less than $2(m^2-m+1)$, conjugate to one of the Lie algebras*: $\mathfrak{L}_m$, $\mathfrak{Z}(A)\times S\mathfrak{L}_m$, $S\mathfrak{L}_m$, $\mathfrak{M}$, $\mathfrak{M}'$.

By virtue of Lemma 8.3 we have

LEMMA 8.4. *Let $\mathfrak{H}$ be a subalgebra of the Lie algebra $\mathfrak{L}_m$. If* dim $\mathfrak{H}\geqq 2(m^2+1)$, *then the kernel of β in $\mathfrak{H}$ is of dimension $2m$ and the image $\beta(\mathfrak{H})$ is conjugate to one of the algebras indicated in Lemma* 8.3.

We consider an almost complex manifold of dimension $n=2m>2$ with a half-symmetric ϕ-connection Γ_{ji}^h which admits a group of H-projective transformations of order not less than $2(m^2+m+1)$. Let G be the group of all H-projective transformations in such a manifold; then we may assume that G is effective in the manifold. We denoted by $\mathfrak{G}$ the Lie algebra of all infinitesimal H-projective transformations induced in the manifold by G. Then we see dim $\mathfrak{G}\geqq 2(m^2+m+1)$, because G is effective in the manifold. Taking an arbitrary point p_0 of the manifold, we mean by $\mathfrak{G}_0$ and $\mathfrak{g}_0$ respectively the isotropy Lie algebra and the linear isotropy Lie algebra of $\mathfrak{G}$ at p_0. It is easily seen that dim $\mathfrak{G}_0\geqq$ dim $\mathfrak{G}-2m$, i. e.

$$\dim\,\mathfrak{G}_0\geqq 2(m^2+1).$$

Thus, from Lemmas 7.4 and 8.4 it follows that the kernel of β in $\mathfrak{G}_0$ is of dimension $2m$ and the linear isotropy Lie algebra $\mathfrak{g}_0=\beta(\mathfrak{G}_0)$ is conjugate to one of the Lie algebras indicated in Lemma 8.3.

Since the kernel of β in $\mathfrak{G}_0$ is of dimension $2m$, the point p_0 being taken arbitrary, Lemma 8.1 implies that the ϕ-connection Γ_{ji}^h is semi-symmetric. Thus, from Theorem 2 it follows that the Nijenhuis tensor $N_{ji}{}^h$ of the manifold vanishes identically if dim $G\geqq 2(m^2+m+1)$. Furthermore, from Lemma 8.2 it follows that the given half-symmetric ϕ-connection Γ_{ji}^h is H-projectively flat if dim $G\geqq 2(m^2+m+1)$. Hence, by virtue of Theorem 13 the Lie algebra $\mathfrak{G}$ is isomorphic to a subalgebra of $S\mathfrak{L}_{m+1}$. On the other hand, we have supposed dim $\mathfrak{G}\geqq 2(m^2+m+1)$. From Lemma 8.3 it follows thereby that $\mathfrak{G}$ is isomorphic to $S\,\mathfrak{L}_{m+1}$ itself. We see further that the group G is transitive in the manifold, since the manifold is connected.

The Lie algebra $S\,\mathfrak{L}_{m+1}$ contains the Lie algebra $\mathfrak{U}$ of all unitary matrices (b_q^p) such that $\bar{b}_q{}^p+b_q{}^p=0$ $(p,q=0,1,\ldots,m)$. The Lie algebra $\mathfrak{U}$ generates a compact group G' in the group G. We see easily that the orbit of G' is $2m$-dimensional in the manifold. Hence, the gronp G' being compact, it is transitive in the manifold and then the manifold admits a Kaehlerian metric with constant holomorphic sectional curvature. Therefore, the manifold is homeomorphic to the complex projective space of complex dimension m. [36] Summing

36) Cf. Igusa [5].

up the above arguments, we have

THEOREM 14. *Let G be an effective group of* H*-projective transformations in an almost complex manifold of complex dimension $n = 2m > 2$ with half-symmetric ϕ-connection. Suppose that the group G is of order not less than $2(m^2 + m + 1)$. Then the connection is* H*-projectively flat and the group G is of the maximum order $2(m^2 + 2m)$ and transitive in the manifold. The manifold is further homeomorphic to the complex projective space.*

We shall give without proof the following theorem which will be proved by using Lemmas 8.1 and 8.4.

THEOREM 15. *Let G be an effective group of* H*-projective transformations of a half-symmetric ϕ-connection in an almost complex manifold of dimension $n = 2m$. If the connection is not semi-symmetric, then* $\dim G \leqq 2m^2$ *for $m > 1$.*

COROLLLARY. *Let G be an effective group of* H*-projective transformations of a half-symmetric ϕ-connection in an almost complex manifold of dimension $n = 2m$. If the Nijenhuis tensor of the manifold does not vanish, then* $\dim G \leqq 2m^2$ *for $m > 1$.*

A similar fact holds good for the group of affine transformations of ϕ-connections. A ϕ-transformation is called an affine ϕ-transformation of a ϕ-connection, if the transformation leaves the ϕ-connection invariant. Thus, we have the following fact: *Let G be an effective group of affine ϕ-transformations of a ϕ-connection in an almost complex manifold of dimension $n = 2m$. If the ϕ-connection is not symmetric, or, if the Nijenhuis tensor of the manifold does not vanish, then* $\dim G \leqq 2m^2$ *for $m > 1$.*

A ϕ-transformation is called a ϕ-projective transformation of a ϕ-connection, if the transformation preserves the system of paths of the connection. Here we have easily the following facts by virtue of Lemma 5.3: *Any group of ϕ-projective transformations of a half-symmetric ϕ-connection is essentially affine. If the connection is symmetric, then the group leaves the connection invariant.*

BIBLIOGRAPHY

[1] S. BOCHNER, Curvature in Hermitian metric, Bull. Amer. Math. Soc. 53(1947), 179–195.

[2] C. EHRESMANN, Sur la théorie des espaces fibrès, Coll. Int. C.N.R. S. Top. Alg., Paris (1947), 3–35.

[3] A. FROLICHER, Zur Differentialgeometrie der komplexen Strukturen, Math. Ann, 129(1955), 50–95.

[4] S. I. GOLDBERG, Note on projectively Euclidean Hermitian manifolds, Proc. Nat. Acad. Sci. U.S.A, 42(1956), 128–130.

[5] J. IGUSA, On the structure of a certain class of Kaehler varieties, Amer. J. Math., 76(1954), 669–677.

[6] S. ISHIHARA, Groups of projective transformations on a projectively connected manifold, Jap. J. Math, 25(1955), 37–80.

[7] ———, Groups of projective transformations and groups of conformal transformations, J. Math. Soc. Jap, 9(1957), 195–227.

[8] P. LIBERMANN, Sur le problème d'équivalence de certaines structures infinité-

simales, Thése, (1953).
[9] M. OBATA, Affine connections on manifolds with almost complex, quaternion or Hermitian structure, Jap. J. Math., 26(1956), 43–77.
[10] T. ŌTSUKI AND Y. TASHIRO, On curves in Kaehlerian spaces, Math. J. Okayama Univ., 4(1954), 57–78.
[11] J. A. SCHOUTEN, Ricci Calculus, 2nd., Ed. Berlin, (1954).
[12] N. E. STEENROD, Topological method for construction of tensor functions, Ann. Math., 43(1942), 116–131.
[13] Y. TASHIRO, On a holomorphically projective correspondences in an almost complex space, Math. J. Okayama Univ., 6(1957), 147–152.
[14] T. Y. THOMAS, The differential invariants of generalized spaces, Cambridge Univ. Press, (1935).
[15] H. C. WANG AND K. YANO, A class of affinely connected spaces, Trans, Amer. Math. Soc., 80(1955), 72–92.
[16] H. WEYL, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung, Göttiger Nachr. (1921), 99–112.
[17] K. YANO, The theory of Lie derivatives and its applications, Amsterdam, (1956).

TOKYO METROPOLITAN UNIVERSITY.