CESÀRO SUMMABILITY OF WALSH-FOURIER SERIES

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1. It is well known that the trigonometrical Fourier series of an integrable function is (C,α) $(\alpha > 0)$ summable almost everywhere. Moreover, the maximal theorems for (C,α) means of the Fouier series are known (see, for example, [8; § 10.22, p. 248]).

Recently, N. J. Fine has proved that the Walsh-Fourier series of an integrable function is (C,α) $(\alpha > 0)$ summable almost everywhere. In this note, we prove that the maximal theorems for the (C,α) means of Walsh-Fourier series are also true. For functions in L^p, $p > 1$, proofs are given by Paley [4] and Sunouchi [5]. Our proof is completely different from their ones, and is based on the estimation for (C,α) kernels of Walsh functions and a lemma of Fine [3]. For notations and background materials, the reader is referred to the paper of Fine [2].

THEOREM. *Let $\sigma_n^{(\alpha)}(x) = \sigma_n^{(\alpha)}(x; f)$ denote the (C,α) mean of the Walsh-Fourier series of an integrable function $f(x)$. Then for $\alpha > 0$*

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)|^p \, dx \leqq A_{p,\alpha} \int_0^1 |f(x)|^p \, dx, \qquad p > 1, \tag{1}$$

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)| \, dx \leqq A_\alpha \int_0^1 |f(x)| \log^+ |f(x)| \, dx + B_\alpha, \tag{2}$$

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)|^r \, dx \leqq A_r \left\{ \int_0^1 |f(x)| \, dx \right\}^r, \qquad 0 < r < 1, \tag{3}$$

where the constants A, B with the subscripts are dependent only on the quantities indicated by subscripts.

For the proof of the theorem, we need the following lemma;

LEMMA. *Let E be a measurable subset of the interval $[0, 1]$, $D(x) = \rho(x, E)$, the distance from x to E, and $\{h_n\}$, $1 \geqq h_0 \geqq h_1 \geqq h_2 \geqq \dots \geqq 0$, be a sequence satisfying*

$$\sum_{h_j \leqq \delta} h_j \leqq M\delta,$$

$$\sum_{h_j > \delta} \frac{1}{h_j} \leqq \frac{M}{\delta}$$

for a constant M and for every $\delta > 0$. Let us set

$$\varphi(x) = \sum_{j=0}^{\infty} \frac{D(x \pm h_j)}{h_j}.$$

Then for any $R > 0$ *and for any choice of* ± 1, *we have*

$$\text{meas}\, \{x \in E: \varphi(x) > R\} \leqq \frac{AM}{R} \text{ meas } E^c,$$

where E^c *denote the complement of the set* E *with respect to* $[0, 1]$ *and* A *is an absolute constant.*

This lemma is due to Fine [3]. Although the lemma is not stated there in this form, a tedious inspection for his proof gives the above formulation. In the following, we apply this lemma to $h_j = 2^{-j}$.

To prove the theorem, we may confine ourselves to the case $\alpha = 1$; the general case $\alpha > 0$ can be deduced easily from the case $\alpha = 1$ (see [6], [7]). Moreover, the theorem for $\alpha = 1$ can be obtained from the following inequalities:

$$\int_0^1 \sup_n |\sigma_{2^n}(x)|^p \, dx \leqq A_p \int_0^1 |f(x)|^p \, dx, \qquad p > 1, \tag{4}$$

$$\int_0^1 \sup |\sigma_{2^n}(x)| \, dx \leqq A \int_0^1 |f(x)| \log^+ |f(x)| \, dx + B, \tag{5}$$

$$\int_0^1 \sup |\sigma_{2^n}(x)|^r \, dx \leqq A_r \left\{ \int_0^1 |f(x)| \, dx \right\}^r, \qquad 0 < r < 1, \tag{6}$$

(see, for example, [5], [6]).

Now let us set[1)]

$$f^*(x) = \sup_{0<|h|\leqq 1} \frac{1}{h} \int_x^{x+h} |f(t)| \, dt,$$

$$\sigma(x) = \sup_n |\sigma_{2^n}(x; f)|,$$

then the inequalities (4)-(6) are immediate consequences of the following inequality

$$\text{meas}\, \{x: \sigma(x) > aR\} \leqq K \text{ meas } \{x: f^*(x) > R\}, \quad R > 0, \tag{7}$$

where a, K are positive constants independent of $f(x)$ and R; for we get by (7)

$$\int_0^1 \{\sigma(x)\}^s \, dx \leqq A_s \int_0^1 \{f^*(x)\}^s \, dx \qquad (s > 0)$$

and so the maximal theorems of Hardy and Littlewood gives the desired inequalities (see, for example, [8; pp. 241-245]).

1) The functions are supposed to be periodic with period 1.
2) The sets are supposed to lie in the interval [0, 1].

To prove the inequality (7), we may suppose that $f(x) \geqq 0$. Let $\alpha_n(x) = r2^{-n} \leqq x < (r+1)\,2^{-n} = \beta_n(x)$. Then, as is known (see [6], [7]),

$$\text{(8)} \qquad \sigma_{2^n}(x) = \frac{2^n+1}{2}\int_{\alpha_n(x)}^{\beta_n(x)} f(t)\,dt + \sum_{j=1}^{n} 2^{j-2}\int_{\alpha_n(x\dot{+}2^{-j})}^{\beta_n(x\dot{+}2^{-j})} f(t)\,dt.$$

Since $f^*(x)$ is lower semicontinuous, the set

$$E_R = \{x : f^*(x) \leqq R\}$$

is closed.

Define $y_j = y_j(x)$ $(j \geqq 0)$ be a point of E_R closest to $z_j = z_j(x) = x \dot{+} 2^{-j}$, and denote the distances from y_j to $\alpha_n(z_j)$ and to $\beta_n(z_j)$ by $\rho_j(x)$ and $\rho'_j(x)$ respectively. (We interprete that $z_j = x$ for $j = 0$). Then by the definition of the set E_R we have

$$\int_{\alpha_n(z_j)}^{\beta_n(z_j)} f(t)\,dt = \int_{\alpha_n(z_j)}^{y_j} f(t)\,dt + \int_{y_j}^{\beta_n(z_j)} f(t)\,dt$$

$$= \rho_j(x)\frac{1}{\rho_j(x)}\int_{\alpha_n(z_j)}^{y_j} f(t)\,dt + \rho'(x)\frac{1}{\rho'_j(x)}\int_{y_j}^{\beta_n(z_j)} f(t)\,dt$$

$$\leqq \{\rho_j(x) + \rho'_j(x)\}R.$$

Since

$$\rho_j(x) \text{ and } \rho'_j(x) \leqq \rho(z_j, E_R) + 2^{-n},$$

it follows that

$$\int_{\alpha_n(z_j)}^{\beta_n(z_j)} f(t)\,dt \leqq 2\,\{\rho(z_j, E_R) + 2^{-n}\}\,R,$$

and we have by (8)

$$\sigma(x) \leqq 4R \sup_n \sum_{j=0}^{n} \{\rho(z_j, E_R) + 2^{-n}\}\,2^j$$

$$\leqq 4R\left\{\sum_{j=0}^{\infty} \rho(z_j, E_R)2^j + 2\right\}.$$

Hence, for x belonging to the set

$$F = \left\{x : \sum_{j=0}^{\infty} \rho(x \dot{+} 2^{-j}, E_R)2^j \leqq 1\right\},$$

we have

$$\sigma(x) \leqq 12R.$$

Now[3)]

$$\{x : \sigma(x) > 12R\} \subset F^c \subset \left\{x \in E_R : \sum_{j=0}^{\infty} \rho(x \dot{+} 2^{-j}, E_R)2^j > 1\right\} \cup E_R^c$$

3) E^c denotes the complement of the set E with respect to the interval $[0,1]$.

and it follows from the lemma that

$$\text{meas}\left\{x \in E_R : \sum_{j=0}^{\infty} \rho(x \dotplus 2^{-j},\ E_R)2^j > 1\right\} \leqq K \text{ meas } E_R^c.$$

Consequently we obtain

$$\begin{aligned}\text{meas}\{x : \sigma(x) > 12R\} &\leqq K \text{ meas } E_R^c \\ &= K \text{ meas}\{x : f^*(x) > R\},\end{aligned}$$

which proves the inequality (7).

2. In this section we treat the problem of Cesàro summability of generalized Walsh-Fourier series.

Let $\{\psi_n(x)\}$, $n = 0, 1, 2, \ldots$, be the generalized Walsh functions of order α. For the definition, notation and backgroud materials, the reader is referred to the paper of Chrestenson [1].

The Cesàro summability of ordinary Walsh-Fourier series is based on the estimation for Fejér kernel of Walsh-Fourier series, and the circumstance is quite same for generalized Walsh-Fourier series. Thus we shall first prove the following lemma.

LEMMA. *Let $D_n(t)$, $K_n(t)$ denote the Dirichlet and Fejér kernels, respectively, for generalized Walsh-Fourier series of order α. Then for $n \geqq 0$,*

$$K_{\alpha^n}(t) = \left(\frac{1}{2} + \frac{1}{2\alpha^n}\right)D_{\alpha^n}(t) + \frac{1}{\alpha^2}\sum_{j=1}^{n} \alpha^{j-n} Q_{j-1}(t) \sum_{k=1}^{\alpha-1} D_{\alpha^n}(t \dotplus k\alpha^{-j}), \tag{1}$$

where

$$Q_j(t) = \begin{cases} \alpha(\alpha-1)/2 & \text{if } \varphi_j(t) = 1 \\ \dfrac{\alpha}{1-\varphi_j(t)} & \text{otherwise,} \end{cases} \tag{2}$$

and $\varphi_j(t)$ is the generalized Rademacher function of order α.

PROOF. For $n = 0$, (1) is easily verified. Suppose that (1) holds for an $n \geqq 0$. Then we use the identity (cf. [1; (5.2)-(5.4)])

$$K_{\alpha^{n+1}}(t) = \frac{1}{\alpha} R_n(t) K_{\alpha^n}(t) + \frac{1}{\alpha} Q_n(t) D_{\alpha^n}(t), \tag{3}$$

where

$$Q_n(t) = \begin{cases} \alpha(\alpha-1)/2 & \text{if } \varphi_n(t) = 1, \\ \alpha/(1-\varphi_n(t)) & \text{otherwise,} \end{cases} \tag{4}$$

and

$$R_n(t) = \begin{cases} \alpha & \text{if } \varphi_n(t) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Substituting (1) into (3) and observing that $D_{\alpha^n}(t) = 0$ for $\alpha^{-n} \leqq t < 1$, we have

$$K_{\alpha^{n+1}}(t) = \frac{1}{\alpha}\left[Q_n(t) + R_n(t)\left(\frac{1}{2} + \frac{1}{2\alpha^n}\right)\right]D_{\alpha^n}(t)$$

$$+\frac{1}{\alpha^3}R_n(t)\sum_{j=1}^{n}\alpha^{j-n}Q_{j-1}(t)\sum_{k=1}^{\alpha-1}D_{\alpha^n}(t\dotplus k\alpha^{-j})$$
$$=S_n+T_n,$$

say.

Since

$$D_{\alpha^n}(t)=\frac{1}{\alpha}\sum_{k=0}^{\alpha-1}D_{\alpha^{n+1}}(t\dotplus k\alpha^{-n-1})$$

and $D_{\alpha^{n+1}}(t\dotplus k\alpha^{-n-1})$ vanishes outside the interval $0\leqq t<\alpha^{-n-1}$ for $k=0$, and vanishes outside the interval $(\alpha-k)\alpha^{-n-1}\leqq t<(\alpha-k+1)\alpha^{-n-1}$ for $1\leqq k\leqq\alpha-1$, it follows from the definition of $Q_n(t)$ and $R_n(t)$ that

$$S_n=\frac{1}{\alpha^2}\left[Q_n(t)+R_n(t)\left(\frac{1}{2}+\frac{1}{2\alpha^n}\right)\right]\sum_{k=0}^{\alpha-1}D_{\alpha^{n+1}}(t\dotplus k\alpha^{-n-1})$$

$$=\frac{1}{\alpha^2}\left[\frac{\alpha(\alpha-1)}{2}+\alpha\left(\frac{1}{2}+\frac{1}{2\alpha^n}\right)\right]D_{\alpha^{n+1}}(t)+\frac{1}{\alpha^2}Q_n(t)\sum_{k=1}^{\alpha-1}D_{\alpha^{n+1}}(t\dotplus k\alpha^{-n-1})$$

(6)
$$=\left(\frac{1}{2}+\frac{1}{2\alpha^{n+1}}\right)D_{\alpha^{n+1}}(t)+\frac{1}{\alpha^2}Q_n(t)\sum_{k=1}^{\alpha-1}D^{\alpha^{n+1}}(t\dotplus k\alpha^{-n-1}).$$

By a similar reasoning we get

(7)
$$T_n=\frac{1}{\alpha^3}R_n(t)\sum_{j=1}^{n}\alpha^{j-n}Q_{j-1}(t)\sum_{k=1}^{\alpha-1}D_{\alpha^n}(t\dotplus k\alpha^{-j})$$
$$=\frac{1}{\alpha^3}R_n(t)\sum_{j=1}^{n}\alpha^{j-n-1}Q_{j-1}(t)\sum_{k=1}^{\alpha-1}\sum_{i=0}^{\alpha-1}D_{\alpha^{n+1}}(t\dotplus i\,\alpha^{-n-1}\dotplus k\alpha^{-j}).$$

Combining (6) and (7) it is shown that (1) holds for $n+1$, so that the lemma is proved.

If we use this lemma, the proofs of Cesàro summability of ordinary Walsh-Fourier series [2, 3, 6] and the problem of approximation by Walsh function [7] can be carried over almost word for word to the corresponding results for the generalized Walsh Fourier series, so that we do not enter into the details.

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