A NOTE ON MEROMORPHIC FUNCTIONS IN THE UNIT CIRCLE

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(Received July 15, 1957)

Let $f(z)$ be meromorphic in $|z| < 1$. Let $n(r, a)$ be the number of 'a-points' of $f(z)$ in $|z| \leqq r < 1$ $(0 \leqq |a| \leqq \infty)$ and let $\bar{n}(r, a)$ be the number of 'a-points' counted once only.

Let

$$N(r, a) = \int_{r_0}^{r} \frac{n(t, a)}{t}\, dt \qquad\qquad (0 < r < 1)$$

$$\bar{N}(r, a) = \int_{r_0}^{r} \frac{\bar{n}(t, a)}{t}\, dt$$

and

$$\limsup_{r \to 1} \frac{T(r)}{\log \frac{1}{1-r}} = \alpha \qquad\qquad (0 \leqq \alpha \leqq \infty)$$

where $T(r) = T(r, f)$ is the Nevanlinna characteristic function of $f(z)$ we prove

THEOREM.

$$\liminf_{r \to 1} \sum_{1}^{q} \left(1 - \frac{\bar{N}(r, a_\nu)}{T(r, f)}\right) \leqq 2 + \frac{1}{\alpha} \tag{1}$$

PROOF. From Nevanlinna second theorem we have

$$(q-2)T(r) < \sum_{1}^{q} \bar{N}(r, a_\nu) + S(r)$$

where

$$S(r) < K + 4 \log^{+} \frac{1}{r} + 6 \log \frac{1}{\rho - r} + 8 \log T(\rho)$$

for $0 < r < \rho < 1$. The term $\log^{+} \frac{1}{r}$ can also be absorbed in the constant K because $\log^{+} \frac{1}{r} \leqq \log^{+} \frac{1}{r_0}$ for $r \geqq r_0$. Further, by putting $\rho = \frac{1+r}{2}$ and using the method of F. Nevanlnina one can easily remove the factor 6 from the expression $6 \log \frac{1}{\rho - r}$ see [1; 152.] Thus in the final form we have

$$(q-2)\; T(r) < \sum_{1}^{q} \bar{N}(r, a_\nu) + S(r) \tag{2}$$

where

$$S(r) < K' + \log \frac{1}{1-r} + 8 \log T(\rho).$$

From (2) we further get

$$q < \sum_{\nu} \frac{\overline{N}(r, a_\nu)}{T(r,f)} + 2 + \frac{K'}{T(r,f)} + \frac{\log \frac{1}{1-r}}{T(r,f)} + 8 \frac{\log T(\rho)}{T(r,f)}$$

and hence

$$\sum_{1}^{q} \left(1 - \frac{\overline{N}(r, a_\nu)}{T(r,f)}\right) < 2 + \frac{K'}{T(r,f)} + \frac{\log \frac{1}{1-r}}{T(r,f)} + 8 \frac{\log T(\rho)}{T(r,f)} .$$

Therefore,

$$\liminf_{r\to 1} \sum_{1}^{q} \left(1 - \frac{\overline{N}(r, a_\nu)}{T(r,f)}\right) \leqq 2 + \liminf_{r\to 1} \frac{\log \frac{1}{1-r}}{T(r,f)} + o(1)$$
$$= 2 + \frac{1}{\alpha} .$$

This proves the theorem.

We remark that for functions for which

$$\liminf_{r\to 1} \frac{\log \frac{1}{1-r}}{T(r,f)} = 0$$

we always have

$$\liminf_{r\to 1} \sum_{1}^{q} \left(1 - \frac{\overline{N}(r, a_\nu)}{T(r,f)}\right) \leqq 2.$$

From the theorem we can further deduce the following result of R. Nevanlinna [1, 158]

$$\sum_{\nu} \left(1 - \limsup \frac{\overline{n}(r, a_\nu)}{n(r, a_\nu)}\right) \leqq 2 + \frac{1}{\alpha} .$$

For,

$$N(r, a_\nu) \leqq T(r,f) + O(1)$$

so,

$$\liminf_{r\to 1} \left(1 - \frac{\overline{N}(r, a_\nu)}{N(r, a_\nu)}\right) \leqq \liminf_{r\to 1} \left(1 - \frac{\overline{N}(r, a_\nu)}{T(r,f)}\right)$$

Further

$$\liminf_{r\to 1} \left(1 - \frac{\overline{N}(r, a_\nu)}{N(r, a_\nu)}\right) = \left(1 - \limsup_{r\to 1} \frac{\overline{N}(r, a_\nu)}{N(r, a_\nu)}\right)$$

so,

$$\sum_{\nu}\left(1-\limsup_{r\to1}\frac{\overline{N}(r,a_\nu)}{N(r,a_\nu)}\right)\leqq\sum_{\nu}\liminf_{r\to1}\left(1-\frac{\overline{N}(r,a_\nu)}{T(r,f)}\right)$$

$$\leqq 2+\frac{1}{\alpha} \qquad \text{from (1).}$$

Now let

$$\limsup_{r\to1}\frac{\overline{n}(r,a_\nu)}{n(r,a_\nu)}=\mu \text{ (say)}$$

Then

$$\int_{r_0}^{r}\frac{\overline{n}(t,a_\nu)}{t}\,dt\leqq(\mu+\varepsilon)\int_{r_0}^{r}\frac{n(t,a_\nu)}{t}\,dt$$

Hence

$$\limsup_{r\to1}\frac{\overline{N}(r,a_\nu)}{N(r,a_\nu)}\leqq\mu$$

and the result follows.

REFERENCE

[1] R. NEVANLINNA, Le théorème de Picard-Borel et la théorie des fonctions Moromorphes, Paris (1929).

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