ON THE CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION I

HARI SHANKAR

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1. Introduction. Let $f(z)$ be meromorphic for $|z| < \infty$ and

$$T(r) = \int_0^r \frac{S(t)}{t}\, dt$$

where

$$S(r) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \left(\frac{|f'(te^{i\theta})|}{1 + |f(te^{i\theta})|^2} \right)^2 t\, dt\, d\theta$$

be its Nevanlinna characteristic function in "Spherical Normal" form [2; p. 177] and

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = \rho \ (0 \leqq \rho \leqq \infty)$$

be its order. If $0 < \rho < \infty$

$$\begin{matrix}\alpha \\ \beta\end{matrix} = \lim_{r\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \frac{T(r)}{r^\rho}\ ; \qquad \begin{matrix}\gamma \\ \delta\end{matrix} = \lim_{r\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \frac{S(r)}{r^\rho}\ .$$

S. K. Singh [3; p. 10, Pt 2.] has established the following results

(i) $\delta \leqq \rho\beta \leqq \delta\left(1 + \log \frac{\gamma}{\delta}\right)$

(ii) $\delta \leqq \frac{\gamma}{e} e^{\delta/\gamma} \leqq \rho\alpha \leqq \gamma$

(iii) $\gamma + \delta \leqq e\rho\alpha.$

The object of this note is to establish results which include the above as special cases. I compare the growth function with a more general function $r^\rho L(r)$ where $L(x)$ is a "slowly changing" function; i. e. $L(x) > 0$ and continuous for $x \geqq x_0$ and $L(cx) \sim L(x)$ as $x \to \infty$, for every constant $c > 0$. ($L(x)$ need not tend to infinity.) The following results for such functions are worth mentioning [1] which are employed here too.

(a) For any $\lambda > 0, x \to \infty$

$x^{-\lambda} L(x) \to 0\ ;\ x^\lambda L(x) \to \infty$

(b) $\int_1^u x^{\lambda-1} L(x)\, dx \sim L(u) \frac{u^\lambda}{\lambda}$

$$\text{(c)} \qquad \int_u^\infty x^{-\lambda-1} L(x)\,dx \sim L(u)\frac{u^{-\lambda}}{\lambda}$$

2. We set

$$\begin{matrix}\tau \\ t\end{matrix} = \lim_{r\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \frac{T(r)}{r^\rho L(r)}; \qquad \begin{matrix}\mu \\ \nu\end{matrix} = \lim_{r\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \frac{S(r)}{r^\rho L(r)}$$

and obtain the following results.

THEOREM. *If* $0 < \rho < \infty$

$$\text{(i)} \qquad \nu \leqq \rho t \leqq \nu\left(1 + \log\frac{\mu}{\nu}\right) \leqq \mu;\ \nu \neq 0$$

$$\text{(ii)} \qquad \nu \leqq \frac{\mu}{e} e^{\nu/\mu} \leqq \rho\tau \leqq \mu \leqq e\rho\tau$$

and in particular $\mu + \nu \leqq e\rho\tau$. *Obviously if there is equality in* $\mu \leqq e\rho\tau$; $\nu = 0$.

COROLLARY. $\mu = \nu$ *if and only if* $\nu = \rho\tau$. *Consequently equality can not hold simultaneously in* $\nu \leqq \rho\tau$ *and* $\mu + \nu \leqq e\rho\tau$ *and hence a fortiori it can never hold simultaneously in* $\mu \leqq e\rho\tau$ *and* $\nu \leqq \rho\tau$ *if* $\tau > 0$.

PROOF. (i) Let $R = rk^{1/\rho}$ where $k \geqq 1$ is an arbitrary number. If $\nu < \infty$ then

$$T(R) = K_1 + \left(\int_{r_0}^{r} + \int_r^R\right)\frac{S(t)}{t}\,dt \tag{2.1}$$

$$> K_1 + (\nu - \varepsilon)\int_{r_0}^{r} t^{\rho-1} L(t)\,dt + S(r)\int_r^R \frac{dt}{t}$$

$$\sim (\nu - \varepsilon)\frac{L(r)}{\rho} r^\rho + \frac{S(r)}{\rho}\log k.$$

Therefore

$$k\frac{T(R)}{R^\rho L(R)} > \frac{(\nu - \varepsilon)}{\rho} + \frac{S(r)}{r^\rho L(r)} \cdot \frac{\log k}{\rho}.$$

Hence we get

$$\rho_k \tau \geqq \nu + \mu \log k \tag{2.2}$$

and

$$\rho_k t \geqq \nu(1 + \log k). \tag{2.3}$$

On the other hand (2.1) gives for $\mu < \infty$

$$T(R) < K_1 + (\mu + \varepsilon)\int_{r_0}^{r} t^{\rho-1} L(t)\,dt + S(R)\int_r^R \frac{dt}{t}.$$

Whence we get

$$\rho_k \tau \leqq \mu(1 + k\log k) \tag{2.4}$$

and

(2.5) $$\rho_k t \leqq \mu + \nu_k \log k.$$

Which also hold when $\mu = \infty$.

Divide (2.5) by k, then the right hand side of the new inequality has a minimum when $k = \mu/\nu$, $(\nu \neq 0)$ and we get

$$\rho t \leqq \nu\left(1 + \log \frac{\mu}{\nu}\right).$$

Taking $k = 1$ in (2.3) we get $\nu \leqq \rho t$.

Further, since $1 + \log x \leqq x$ for all x we obtain finally

$$\nu \leqq \rho t \leqq \nu\left(1 + \log \frac{\mu}{\nu}\right) \leqq \nu \frac{\mu}{\nu} = \mu.$$

PROOF (ii). Take $k = \exp\left(1 - \frac{\nu}{\mu}\right)$ in (2.2), then

$$\mu \leqq \rho\tau \exp\left(1 - \frac{\nu}{\mu}\right).$$

Again, as $e^x \geqq ex$ for all x we have

$$\mu e \frac{\nu}{\mu} \leqq \mu e^{\nu/\mu} \leqq e\rho\tau.$$

Or

(2.7) $$\nu \leqq \frac{\mu}{e} e^{\nu/\mu} \leqq \rho\pi.$$

Taking $k = 1$ in (2.4) we get $\rho\tau \leqq \mu$. From the right hand inequality of (2.7) we get

$$e\rho\tau \geqq \mu e^{\nu/\mu} \geqq \mu\left\{1 + \frac{\nu}{\mu}\right\}$$

and finally we obtain

$$\nu \leqq \frac{\mu}{e} e^{\nu/\mu} \leqq \rho\tau \leqq \mu \leqq e\rho\tau$$

and $$\nu + \mu \leqq e\rho\tau.$$

PROOF OF COROLLARY. If $\nu = \rho\tau$ from (2.7) we have $\mu e^{\nu/\mu} \leqq e\nu$.

Or $$e^{\nu/\mu} \leqq e \frac{\nu}{\mu}$$

and since $e^x \geqq ex$ for all x and the equality holds only if $x = 1$ $e^{\nu/\mu} < e \frac{\nu}{\mu}$ is not possible. Hence $e^{\nu/\mu} = e \frac{\nu}{\mu}$, i. e., $\nu = \mu$. If $\mu = \nu$ again from (2.7) we get

$$e\mu = e\nu \leqq e\rho\tau,$$

While $$\mu \geqq \rho\tau.$$

Hence $$\mu = \nu = \rho\tau.$$

Next, if $\nu = \rho\tau$ which implies $\mu = \rho\tau$. So $\mu + \nu = 2\rho\tau < e\rho\tau$. Now, let

$\mu + \nu = e\rho\tau$ then ν will be less than $\rho\tau$ for if it were equal to $\rho\tau$ then $\mu + \nu$ will have to be less than $e\rho\tau$. $\therefore$ Contradiction. Hence the result.

3. We remark that:

(i) If $\mu = 0$ then $\tau = 0$ and conversely.

(ii) If $\nu = \infty$ then $t = \infty$.

(iii) If $t = \infty$ then $\mu = \infty$.

(iv) If $\nu = 0$, $\mu < \infty$ then $t = 0$.

(v) If $t = 0$ then $\nu = 0$.

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HINDU COLLEGE MORADABAD, INDIA.