SOME REMARKS ON THE CAUCHY INDEX THEOREM

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1. Introduction. A classic problem in the geometry of the zeros of polynomials is the determination of the number of zeros of a polynomial in a given half-plane. This problem can easily be reduced to that of determining the number of zeros lying in the upper half-plane ($Im\, z > 0$). The classic Cauchy Index Theorem (see [1], p. 129, Th. 37, 1) solves the problem for $F(z) \equiv a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n \equiv f(z) + ig(z)$, where $f(z)$ and $g(z)$ are real polynomials, respectively of degree n and $\leqq n-1$, and where the highest degree term in $f(z)$ has the coefficient 1.

In the present paper we show that this theorem can easily be extended to the case in which $F(z)$ is of the form $f(z) + \lambda g(z)$, where λ is a non-real number, and $f(z)$ and $g(z)$ are real polynomials of any degree. Theorem 1 gives this extension. Theorem 7 is a converse of Theorem 1. In addition we derive some results from these two propositions (Theorems 2/6, Theorems 8/10).

2. Definitions. Let $f(z)$ and $g(x)$ be two real polynomials with no common real zeros. The combined real zeros of these polynomials can be arranged in order of increasing magnitude whereby multiple roots are counted according to their multiplicities. If no sequence of consecutive zeros of either polynomial is contained in this arrangement, we consider it final and call it S. In the other case we delete from each sequence of, say ξ, consecutive zeros of $f(z)$ or of $g(z)$ all ξ zeros if ξ is even and any set of $\xi - 1$ zeros if ξ is odd, repeating the process if necessary until an arrangement S is obtained in which no further deletions are possible. Let $m(\geqq 1)$ and $n(\geqq 1)$ be the degrees of $f(z)$ and $g(z)$ respectively. If $m \geqq n (m \leqq n)$ the number k of the zeros α_j of $f(z)$ (the zeros β_j of $g(z)$) in S is called the index of the polynomials $f(z)$ and $g(z)$. Of course $k \geqq 0$. S is an arrangement either of the form $\alpha_1 < \beta_1 < \ldots$, or $\beta_1 < \alpha_1 < \ldots$, or α_1, or β_1, or is empty.

REMARK. If $m = n$, then there are as many α_j as β_j in the final arrangement since $f(z)$ and $g(z)$ are real polynomials.

To show Theorem 1 we use the following

LEMMA. *Let L be a line on which a given polynomial $f(z)$ has no zeros. Let $\Delta_L \arg f(z)$ denote the net change in $\arg f(z)$ as point z traverses L in a specified direction, and accordingly let $p(q)$ denote the number of zeros of $f(z)$ to the left (right) of L. Then*

$$p - q = (1/\pi)\Delta_L \arg f(z).$$

See [1], p. 5, Th. 1, 6.

THEOREM 1. *Let* $f(z) \equiv A(z-a_1)(z-a_2)\ldots(z-a_m)$, $g(z) \equiv B(z-b_1)(z-b_2)\ldots(z-b_n)$ *be real polynomials with no real zero in common* $(AB \neq 0)$. *Let k be the index of the polynomials. Let* $\lambda = \lambda' + i\lambda''$ *be a non-real number. Let γ denote the number* $\lambda''(\alpha_1-\beta_1)g(\alpha_1)f(\beta_1)$ *if S contains at least α_1 and β_1, the number* $-\lambda''g(\alpha_1)$ *if S consists of α_1 only, the number* $\lambda''f(\beta)$ *if S consists of β_1 only, the number* 0 *if* $k=0$. *Let* $F(z) \equiv f(z) + \lambda g(z)$. *Furthermore let*

$$\mu_1 = \{\max(m,n) + (\operatorname{sgn}\gamma)k\}/2$$
$$\mu_2 = \{\max(m,n) - (\operatorname{sgn}\gamma)k\}/2$$

ASSERTION: *$F(z)$ has μ_1 zeros in the upper half-plane and μ_2 zeros in the lower half-plane.*

PROOF. Since λ is non-real, and $f(z)$ and $g(z)$ are real polynomials, with no common zeros it follows that $F(z)$ has imaginary zeros only. Now we investigate the behaviour of the point $w = F(z)$ when z runs through the real numbers from $-\infty$ to ∞. The curve described by W has an asymptotic direction $Rl\, z = 0$ if $m > n$, an asymptotic direction $Im\, z = 0$ if $m = n$, $A + \lambda'B = 0$, an asymptotic direction $(Im\, z)/(Rl\, z) = \lambda''/\lambda'$ if $m < n$, an asymptotic direction $(Im\, z)/(Rl\, z) = \lambda''B/(A+\lambda'B)$ if $m = n$, $A + \lambda'B \neq 0$. When z passes through a value a_j, then $f(z)$ vanishes, and the curve $W = F(z)$ cuts the line through the origin and λ at the point $\lambda g(a_j)$. When z passes through the value b_j, the curve cuts the real axis at the point $f(b_j)$. It depends on the signs of the factors of γ whether the origin is encircled clockwise or counterclockwise. The net change of $\arg F(z)$ is determined by the value of k and the sense in which the origin is encircled. Furthermore we have $\max(m,n) = \mu_1 + \mu_2$. The application of the lemma to all possible cases establishes the assertion.

3. Applications.

THEOREM 2. *If the real polynomials* $f(z) \equiv (z-a_1)(z-a_2)\ldots(z-a_n)$, $g(z) \equiv (z-b_1)(z-b_2)\ldots(z-b_n)$ *have real zeros with the property* $a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$, *then the polynomial* $F(z) \equiv f(z) + \lambda g(z)$, *where* $\lambda = \lambda' + i\lambda''$ *is a non-real number, has all its zeros either in the upper half-plane* $(\lambda'' > 0)$ *or in the lower half-plane* $(\lambda'' < 0)$.

PROOF. Here we have in each case $g(a_1)f(b_1) < 0$. Hence, if $\lambda'' > 0$, we have $\gamma > 0$ and thus $\mu_1 = n$, $\mu_2 = 0$; if $\lambda'' < 0$ we find $\mu_1 = 0$, $\mu_2 = n$.

THEOREM 3 (THE CAUCHY INDEX THEOREM ([1], p. 129)). *Let* $F(z) \equiv a_0 + a_1 z + \ldots + a_{n-1}z^{n-1} + z^n \equiv f(z) + ig(z)$, *where $f(z)$ and $g(z)$ are real polynomials with* $g(z) \not\equiv 0$. *Let z describe the real axis from $-\infty$ to ∞, let σ be the number of real zeros of $f(z)$ at which $f(z)/g(z)$ changes from $-$ to $+$, and τ the number of real zeros of $f(z)$ at which $f(z)/g(z)$ changes from $+$ to $-$. If $F(z)$ has no real zeros, μ_1 zeros in the upper half-plane and μ_2 zeros in the lower half-plane, then*

$$\mu_1 = \{n + (\tau - \sigma)\}/2, \quad \mu_2 = \{n - (\tau - \sigma)\}/2.$$

PROOF. Here k is the number of the reduced set of real zeros of $f(z)$ and equal to the absolute value of $\tau-\sigma$. Furthermore $\lambda''>0$. The assertion now follows from Theorem 1.

THEOREM 4. *Let* $F(z)\equiv a_0+a_1z+\ldots.+a_{n-1}z^{n-1}+z^n\equiv f(z)+ig(z)$. *If* $f(z)$ *has* n *real zeros* α_j, *and* $g(z)$ *has* $n-1$ *real zeros* β_j *with* $\alpha_1<\beta_1<\alpha_2<\ldots.<\beta_{n-1}<\alpha_n$, *then* $F(z)$ *has* n *zeros in the u pper half-plane if* $(-1)^n g(\alpha_1)>0$, *and* n *zeros in the lower half-plane if* $(-1)^n g(\alpha_1)<0$. See [1], p. 130, Ex. 3.

PROOF. Here $k=n$; $\lambda''>0, \alpha_1-\beta_1<0$.

THEOREM 5. *Let* $g(z)$ *be a real polynomial of degree* n *having* n *distinct zeros of which* p *are real. Let* $f(z)$ *be a real polynomial of degree* $n-1$ *which relative to* $g(z)$ *has the partial fraction development*

$$\frac{f(z)}{g(z)}\equiv\sum_{j=1}^{p}\frac{\gamma_j}{z-a_j}+\sum_{j=p+1}^{(n+p)/2}\left(\frac{\gamma_j}{z-a_j}+\frac{\overline{\gamma_j}}{z-\overline{a_j}}\right),$$

where $a_1, a_2, \ldots.. a_p$ *are real numbers,* $\gamma_j=m_j e^{i\mu_j}$ *with* $m_j>0$ $(j=1,2,\ldots.,n)$, $\mu_j=0$ $(j=1,2,\ \ldots.,p)$, μ_j *real and* $\neq 0$ *with* $|\mu_j|<\pi/2$ $(j=p+1,\ldots.,n)$. *Let* λ *be a non-real number* $\lambda'+i\lambda''$. *Let* $F(z)\equiv f(z)+\lambda g(z)$. *Let* γ *be the number* $\lambda''(\alpha_1-\beta_1)$, *where* α_1 *and* β_1 *have the same meaning as in* 2. *Then the polynomial* $F(z)$ *has* $\{n+(\operatorname{sgn}\gamma)p\}/2$ *zeros in the upper half-plane, and* $\{n-(\operatorname{sgn}\gamma)p\}/2$ *zeros in the lower half-plane.*

PROOF. According to Marden ([2], p. 92, Th. 2. 3) there lies an odd number of zeros of $f(z)$ between two consecutive zeros of $g(z)$. This implies that outside of the smallest closed interval containing all the real zeros of $g(z)$ there is an even number of zeros of $f(z)$ (or none). Furthermore we have $\operatorname{sgn} g(-\infty)\neq\operatorname{sgn} f(-\infty)$, because of our assumption concerning γ_j. The index of $f(z)$ and $g(z)$ is equal to p. In addition $f(\beta_1)g(\alpha_1)>0$. The application of Theorem 1 now yields the proof.

THEOREM 6. *Let* $f(z)$ *and* $g(z)$ *be the polynomials of the preceding theorem. Let* ω *be a point on the real axis distinct from the zeros of* $f(z)$ *and* $g(z)$, *and let* ω *be located next to* β_r $(r=1,2,\ldots.,p)$ *or possibly separated from the nearest* β_r *by an even number of real zeros of* $f(z)$. *Let* λ *be a non-real number. Let* γ *have the same meaning as in Theorem* 5.

Then $F(z)\equiv f(z)+\lambda(x-\omega)\,g(z)$ *has*

$\{n+(\operatorname{sgn}\gamma)|2r-p-1|\}/2$ *zeros in the upper half-plane, and*

$\{n-(\operatorname{sgn}\gamma)|2r-p-1|\}/2$ *zeros in the lower half-plane.*

PROOF. The index of the polynomial $f(z)$ and $(z-\omega)g(z)$ is $|2r-p-1|$. Furthermore, investigating all possible cases, we see that the sign of $(\alpha-\beta)f(\beta)g(\alpha)$ in this theorem is the same as that of the corresponding expression in the preceding theorem. Hence the assertion follows.

4. A converse of Theorem 1 is

THEOREM 7. *Let* $f(z)$ *and* $g(z)$ *be two real polynomials with no real zeros* *in common. Let* $F(z) \equiv f(z) + \lambda g(z)$, *where* λ *is a non-real number, have* μ_1 *zeros in the upper half-plane and* μ_2 *zeros in the lower half-plane. Let* $k = |\mu_1 - \mu_2|$. *Then one of the polynomials* $f(z)$ *and* $g(z)$, *namely the one with the highest degree, has at least* k *real zeros, while the other also has at least* k *real zeros if* $m - n \equiv (\mathrm{mod}\, 2)$, *and has* $k \pm 1$ *real zeros if* $m - n \equiv 1 \pmod 2$.

PROOF. The truth of the above assertion follows immediately from the relations concerning μ_1 and μ_2 of Theorem 1. For we have $\mu_1 - \mu_2 = k(\mathrm{sgn}\,\gamma)$.

5. Applications of Theorem 7.

THEOREM 8. *If* $f(z)$ *and* $g(z)$ *are real polynomials such that* $F(z) \equiv f(z) + ig(z)$ *has* q *roots in the lower-half-plane, then* $f(z)$ *has at most* q *pairs of conjugate complex zeros* (N. G. de Bruyn). See [3], p. 215, Lemma 2.

PROOF. We have $q = \{\max(m, n) - (\mathrm{sgn}\,\gamma)k\}/2$, where m, n, k and γ have the same meanings as in Theorem 1. From this equality it follows that the number of complex zeros of $f(z)$ is less than or equal to $2q + (\mathrm{sgn}\,\gamma)k - k'$, where k' is the number of real zeros of $f(z)$. Since $k \geqq k'$ according to the definition of k, the assertion of the theorem follows.

THEOREM 9: *If the real polynomial*

$$F(z) \equiv a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

has μ_1 *zeros in the left half-plane* ($Rl\, z < 0$) *and* μ_2 *zeros in the right half-plane* ($Rl\, z > 0$), *then the polynomials*

$$f(z) \equiv a_0 + a_2 z + a_4 z^2 + \dots + a_{2p} z^p \quad \left(p = \left[\frac{n}{2}\right]\right),$$

$$g(z) \equiv a_1 + a_3 z + a_5 z^2 + \dots + a_{2q+1} z^2 \quad \left(q = \left[\frac{n-1}{2}\right]\right)$$

have at least $\left[\dfrac{|\mu_1 - \mu_2| - 1}{2}\right]$ *negative zeros.*

PROOF. $F(iz) = f(-z^2) + izg(-z^2)$ has μ_1 zeros in the upper half-plane and μ_2 zeros in the lower half-plane. According to theorem 7 each of the polynomials $f(-z^2)$ and $zg(-z^2)$ has at least $|\mu_1 - \mu_2| - 1$ real zeros. When n is even, $f(-z^2)$ and $zg(-z^2)$ are of degrees n and $n-1$ respectively, and when n is odd, $f(-z^2)$ and $zg(-z^2)$ are of degrees $n-1$ and n respectively. This implies: for even n, $f(-z^2)$ has at least $|\mu_1 - \mu_2|$ real zeros and $g(-z^2)$ has at least $|\mu_1 - \mu_2| - 2$ real zeros; for odd n, $g(-z^2)$ has at least $|\mu_1 - \mu_2| - 1$ real zeros, and equally $f(-z^2)$. When $|\mu_1 - \mu_2|$ is odd, each of these numbers of real zeros must be augmented by 1. It follows that $f(-z)$ and $g(-z)$ have at least $\left[\dfrac{|\mu_1 - \mu_2| - 1}{2}\right]$ positive zeros, and that implies that $f(z)$ and $g(z)$ have at least $\left[\dfrac{|\mu_1 - \mu_2| - 1}{2}\right]$ negative zeros.

REMARK. As a special case of the preceding theorem we mention the following result which we proved recently in a different way (see [4], Th 4).

When the zeros of $a_0 + a_1 z + \dots + a_n z^n$ are all negative, the zeros of $f(z) \equiv a_0 + a_2 z + \dots + a_{2p} z^p$ $\left(p = \left[\frac{n}{2}\right]\right)$ and $g(z) \equiv a_1 + a_3 z + \dots + a_{2q+1} z^q$ $\left(q = \left[\frac{n-1}{2}\right]\right)$ are all negative.

6. LEMMA. *Let $f(z)$ and $g(z)$ be two real polynomials of degrees m and n respectively, and of index k. Then, if for some arbitrary real constants A and B the polynomial $F(z) \equiv Af(z) + Bg(z)$ has the degree* $\max(m, n)$, *$F(z)$ has at least k real zeros.*

PROOF. Let $m \geqq n$. Then $f(z)$ has at least k distinct real zeros such that between any two consecutive zeros lies at least one zero of $g(z)$. By using continuity properties it can be seen that in any such interval lies at least one zero of $F(z)$. So $F(z)$ has at least $k-1$ real zeros. But, since the degree of $F(z)$ is m, it follows that $F(z)$ has at least k real zeros.

By application of this lemma and theorem 7 we have

THEOREM 10. *If the polynomial*

$$F(z) \equiv (a_0 + ib_0) + (a_1 + ib_1)z + \dots + (a_n + ib_n)z^n \equiv f(z) + ig(z)$$

has μ_1 zeros in the upper half-plane and μ_2 zeros in the lower half-plane then the real polynomial $Af(z) + B(g(z)$ where A and B are real constants with $Aa_n + Bb_n \neq 0$ has at least $|\mu_1 - \mu_2|$ real zeros.

REMARK. Another application of the preceding lemma and theorem 7, due to several authors (see [1], p. 130, pr. 4) is the following.

If $F(z) \equiv a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n \equiv f(z) + ig(z)$ where $f(z)$ and $g(z)$ are real polynomials, has n zeros in the upper half-plane (or in the lower half-plane), then for real constants A and B the polynomial $Af(z) + Bg(z)$ has n distinct real zeros.

REFERENCES

[1] MORRIS MARDEN, The geometry of the zeros of a polynomial in a complex variable, Amer. Math. Soc., New York (1949).

[2] ———, The zeros of certain real rational and meromorphic functions, Duke Math. Journ. 16 (1949), 91–97.

[3] N. G. DE BRUYN, The roots of trigonometric integrals, Duke Math. Journ., 17(1950), 197–226.

[4] HAN KHWAT TIK AND L. KUIPERS, On a proof of the Lucas theorem, Proc. Kon. Ned. Ak. v. W., 58, (1955), 435–437.

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