THE CHARACTERISTIC ROOTS OF THE PRODUCT OF TWO MATRICES

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(Received January. 20, 1957)

1. Introduction and notations. Let A be a square matrix of order n with elements belonging to the field of complex numbers. Further, let $c(A)$ stand for an arbitrary characteristic root of A, whereas $\bar{c}(A)$ denotes the complex conjugate of $c(A)$.

In a recent paper [2], this author has found the upper bound for an arbitrary characteristic root $c(AB)$ of the product of two matrices A and B in terms of their elements. The purpose of this paper is to find the upper bounds for the real and imaginary parts of $c(AB)$ in terms of the elements of the associated Hermitian matrices $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$ and $(B-\bar{B}')/2i$. In what follows, $R_i(A)$ will denote the sum of the absolute values of the elements of an arbitrary matrix A in the i-th row, $T_i(A)$ will denote the sum of the absolute values of the elements of A in the i-th column, and $R(A)$, $T(A)$ will stand for the greatest of the $R_i(A)$ and $T_i(A)$ respectively.

2. Upper bounds for the real and imaginary parts of c(AB).

THEOREM. *Let A and B be two commuting n-square complex matrices. If $S'_r(A)$, $S''_r(A)$, $S'_r(B)$, $S''_r(B)$ are the sums of the absolute values of the elements in the r-th row of $(A+\bar{A}')/2$, $(A-\bar{A}')/2i$, $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$ respectively, and if $S'(A)$, $S''(A)$, $S'(B)$, $S''(B)$ are respectively the greatest of the $S'_r(A)$, $S''_r(A)$, $S'_r(B)$, $S''_r(B)$, then*

$$\left|\frac{c(AB)+\bar{c}(AB)}{2}\right| \leqq S'(A)S'(B)+S''(A)S''(B), \tag{1}$$

and

$$\left|\frac{c(AB)-\bar{c}(AB)}{2i}\right| \leqq S'(A)S''(B)+S''(A)S'(B). \tag{2}$$

PROOF. Any square matrix $A=\frac{A+\bar{A}'}{2}+i\frac{A-\bar{A}'}{2i}=P+iQ$, say, where $P=(p_{ij})$, $Q=(q_{ij})$ are Hermitian matrices; and any square matrix $B=\frac{B+\bar{B}'}{2}+i\frac{B-\bar{B}'}{2i}=U+iV$, say, where $U=(u_{ij})$ and $V=(v_{ij})$ are Hermitian matrices. Thus

$$AB=PU-QV+i(PV+QU), \tag{3}$$

and

$$\bar{A}'\bar{B}'=PU-QV-i(PV+QU). \tag{4}$$

Now, if λ is a characteristic root of AB, there exists a complex unit vector $x=(x_1, x_2, \ldots x_n)^T$, such that

$$\lambda x = ABx.$$

Premultiplying the above equation by $\bar{x}'$, we have

$$\lambda \bar{x}'x = \bar{x}'ABx,$$

or,

$$\lambda = \bar{x}'ABx. \tag{5}$$

Taking the conjugate transpose of (5) we have

$$\begin{aligned}\bar{\lambda} &= \bar{x}'(\bar{B}'\bar{A}')x \\ &= \bar{x}'(\bar{A}'\bar{B}')x,\end{aligned} \tag{6}$$

since $AB = BA$ implies $\bar{A}'\bar{B}' = \bar{B}'\bar{A}'$.

From (5) and (6) by addition and subtraction, we have

$$\frac{\lambda + \bar{\lambda}}{2} = \bar{x}'(PU - QV)x, \tag{7}$$

and

$$\frac{\lambda - \bar{\lambda}}{2i} = \bar{x}'(PV + QU)x. \tag{8}$$

From (7) and (8) we determine the upper bounds for $\left|\frac{\lambda + \bar{\lambda}}{2}\right|$ and $\left|\frac{\lambda - \bar{\lambda}}{2i}\right|$. Since these relations are identical in form, it is sufficient to carry the computation through one of them only.

Taking the absolute values in (7), we get

$$\begin{aligned}\left|\frac{\lambda + \bar{\lambda}}{2}\right| &= |\bar{x}'(PU - QV)x| \\ &= |\Sigma_{r,s}\, \alpha_{rs}\, \bar{x}_r x_s - \Sigma_{r,s}\, \beta_{rs}\, \bar{x}_r x_s|\end{aligned}$$

where α_{rs} and β_{rs} denote the elements of PU and QV, respectively, in the (r, s)-th position, or,

$$\left|\frac{c(AB) + \overline{c(AB)}}{2}\right| \leqq |\Sigma_{r,s}\, \alpha_{rs}\, \bar{x}_r x_s| + |\Sigma_{r,s}\, \beta_{rs}\, \bar{x}_r x_s|. \tag{9}$$

Let $\xi_r = |x_r|$, so that $\sum_r \xi_r^2 = 1$ and $\xi_r \xi_s \leqq 1/2\,(\xi_r^2 + \xi_s^2)$. Now, we consider the two terms on the right-hand side of (9) separately.

$$\begin{aligned}|\Sigma_{r,s}\, \alpha_{rs}\, \bar{x}_r x_s| &\leqq \Sigma_{r,s} |\alpha_{sr}| \xi_r \xi_s \\ &\leqq 1/2\, \Sigma_{r,s} |\alpha_{rs}| (\xi_r^2 + \xi_s^2) \\ &= 1/2 \left\{ \sum_r \xi_r^2 \sum_s |\alpha_{rs}| + \sum_s \xi_s^2 \sum_r |\alpha_{rs}| \right\} \\ &= 1/2 \left\{ \sum_r \xi_r^2\, R_r(PU) + \sum_s \xi_s^2 T_s(PU) \right\}.\end{aligned}$$

Supposing that $R_r(PU)$ and $T_s(PU)$ attain their maximum values respectively for $r = h$ and $s = k$, we have

$$\left| \sum_{rs} \alpha_{rs} \bar{x}_r x_s \right| \leqq 1/2\, \{R_h(PU) + T_k(PU)\}.$$

But, by definition,

$$R_h(PU) = \left|\sum_s p_{hs} u_{s1}\right| + \left|\sum_s p_{hs} u_{s2}\right| + \dots + \left|\sum_s p_{hs} u_{sn}\right|$$

$$\leqq \sum_s |p_{hs}|\,|u_{s1}| + \sum_s |p_{hs}|\,|u_{s2}| + \dots + \sum_s |p_{hs}|\,|u_{sn}|$$

$$= |p_{h1}| \sum_t |u_{1t}| + |p_{h2}| \sum_t |u_{2t}| + \dots + |p_{hn}| \sum_t |u_{nt}|$$

$$= |p_{h1}| R_1(U) + |p_{h2}| R_2(U) + \dots + |p_{hn}| R_n(U)$$

$$\leqq R(U)(|p_{h1}| + |p_{h2}| + \dots + |p_{hn}|).$$

$$= R_h(P)R(U)$$

$$\leqq R(P)R(U) = S'(A)S'(B); \qquad (10)$$

and

$$T_k(PU) = \left|\sum_s p_{1s} u_{sk}\right| + \left|\sum_s p_{2s} u_{sk}\right| + \dots + \left|\sum_s p_{ns} u_{sk}\right|$$

$$\leqq \sum_s |p_{1s}|\,|u_{sk}| + \sum_s |p_{2s}|\,|u_{sk}| + \dots + \sum_s |p_{ns}|\,|u_{sk}|$$

$$= |u_{1k}| \sum_t |p_{t1}| + |u_{2k}| \sum_t |p_{t2}| + \dots + |u_{nk}| \sum_t |p_{tn}|$$

$$= |u_{1k}| T_1(P) + |u_{2k}| T_2(P) + \dots + |u_{nk}| T_n(P)$$

$$\leqq T(P)T_k(U)$$

$$\leqq T(P)T(U) = S'(A)S'(B), \qquad (11)$$

since for any Hermitian matrix $H = (h_{rs})$, $T(H) = \max_s T_s(H) = \max_s \sum_r |h_{rs}|$

$= \max_s \sum_r |h_{sr}| = \max_s R_s(H) = R(H).$

The inequalities (10) and (11) give

$$|\Sigma \alpha_{rs} \bar{x}_r x_s| \leqq S'(A)S'(B). \qquad (12)$$

Similarly, taking $|\Sigma \beta_{rs} \bar{x}_r x_s|$ and proceeding as we did in establishing (12), we shall prove

$$|\Sigma \beta_{rs} \bar{x}_r x_s| \leqq R(Q)R(V) = S''(A)S''(B). \qquad (13)$$

Combining (12) and (13), we obtain

$$\left|\frac{c(AB) + \bar{c}(AB)}{2}\right| \leqq S'(A)S'(B) + S''(A)S''(B).$$

Similarly, starting with (8), we can establish the inequality (2).

This completes the proof of the Theorem.

The condition, that A and B commute, imposed on the matrices in the Theorem, is necessary as shown by the following example:

$$A = \begin{pmatrix} 0 & i \\ 2i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2i \\ -i & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \neq BA.$$

Here $S'(A) = S'(B) = 1/2$, $S''(A) = S''(B) = 3/2$, $c(AB) = 1, 4$, and 4 is not less than or equal to 5/2.

3. Some particular cases of (1) and (2). (i). Let A and B be commuting n-square Hermitian matrices, so that AB is also Hermitian and all $c(AB)$ are real. In this case $S'(A) = R(A)$, $S'(B) = R(B)$, and $S''(A) = S''(B) = 0$. Thus, for matrices A and B defined above, (1) reduces to

$$|c(AB)| \leqq R(A)R(B), \tag{14}$$

a result proved in [2].

(ii). Again, if A and B are commuting skew-Hermitian matrices of the same order, $A + \bar{A}' = B + \bar{B}' = 0$, $(A - \bar{A}')/2i = A/i$, and $(B - \bar{B}')/2i = B/i$. Also AB is Hermitian, so that all the characteristic roots of AB are real, $S'(A) = S'(B) = 0$ and $S''(A) = R(A/i) = R(A)$, and $S''(B) = R(B/i) = R(B)$. In this case also (10) reduces to

$$|c(AB)| \leqq R(A)R(B). \tag{15}$$

(iii). Let us put $B = I$, for which $S'(B) = 1$ and $S''(B) = 0$. In this case (1) and (2) reduce to

$$\left| \frac{c(A) + \bar{c}(A)}{2} \right| \leqq S'(A), \tag{16}$$

and

$$\left| \frac{c(A) - \bar{c}(A)}{2i} \right| \leqq S''(A), \tag{17}$$

results due to E. T. Browne [1], and W. V. Parker [3], giving the upper bounds for the real and imaginary parts of an arbitrary characteristic root of A in terms of the elements of the associated Hermitian matrices $(A + \bar{A}')/2$ and $(A - \bar{A}')/2i$.

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