A SUFFICIENT CONDITION FOR THE ABSOLUTE RIESZ SUMMABILITY OF A FOURIER SERIES

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1. We suppose that $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$, and we write

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t),$$

$$\phi(t) = \frac{1}{2} \{f(t+x) + f(x-t)\},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du \ (\alpha > 0), \quad \phi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t) \ (\alpha > 0),$$

Concerning the absolute Riesz summability $|R, \lambda(w), k|$, where the type $\lambda(w)$ is equal to $\exp\{(\log w)^\Delta\}$, $(\Delta > 1)$, the following theorems are known.

THEOREM. Mohanty and Misra [1], Kinukawa [2]. *If* $\phi_\alpha(t) \log (k/t)$, *where* $k > \pi e^2$, *is of bounded variation in* $(0, \pi)$, *then the series* $\sum_{2}^{\infty} A_n(x)$ *is summable* $|R, \exp\{(\log w)^\Delta\}, 1|$, *where* $0 < \alpha < 1$ *and* $\Delta = 1 + 1/\alpha$.

THEOREM. Pati [3]. *If* α *is an integer* $\geqq 1$, *and* $\phi_\alpha(t) \log (k/t)$, $(k > \pi e^{\alpha+2})$ *is of bounded variation in* $(0, \pi)$, *then the Fouries series of* $f(t)$, *at* $t = x$, *is summable* $|R, \exp\{(\log w)^{1+1/\alpha}\}, 1+\alpha|$.

We shall prove here the following

THEOREM[1]. *If* $\phi_\alpha(t) (\log k/t)^{\alpha(\Delta-1)}$, $(k > \pi e^{\alpha(\Delta-1)+1})$, *is of bounded variation in* $(0, \pi)$, *then* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* $|R, \exp\{(\log w)^\Delta\}, \beta|$, *where* $\beta > \alpha > 0$ *and* $\Delta \geqq 1$.

This theorem is an improvement of the above two theorems, and when $\Delta = 1$ this theorem reduces to a theorem on $|C, k|$ summability proved by L. S. Bosanquet [4], further this theorem shows that the summability $|R, \exp\{(\log w)^\Delta\}, \beta|$, $\beta > 1$, of a Fourier series is a local property of the generating function.

For the proof of the theorem, it suffices to show that, when $\Delta > 1$,

1). The $(R, \exp\{(\log w)^\Delta\}, \beta)$-analogue has already been proved by K. Kanno On the Riesz summability of Fourier series, Tôhoku Math. Journ., 8(1956), in somewhat weak form. But we can complete the Kanno theorem, applying the method of this paper.

$$I = \int_2^\infty \frac{\beta\Delta(\log w)^{\Delta-1}}{w \exp\{\beta(\log w)^\Delta\}} \left| \sum_{n<w} [\exp\{(\log w)^\Delta\} - \exp\{(\log n)^\Delta\}]^{\beta-1} \exp\{(\log n)^\Delta\} A_n(x) \right| dw < \infty.$$

To simplify the proof we use the following notations throughout the paper.

$e(w) = \exp\{(\log w)^\Delta\}, E^{(\rho)}(w, t) = \dfrac{\partial^\rho}{\partial t^\rho} E(w,t), \quad \{F(t,n)\}_\rho = \dfrac{\partial^\rho}{\partial t^\rho} F(t,n),$

$E(w,t) = \sum_{n<w} \{e(w) - e(n)\}^{\beta-1} e(n) \cos nt,$

$F(w,t,\rho,s) = \sum_{n<w} \{e(w) - e(n)\}^{\beta-1} e(n) n^s (\cos nt)_\rho,$

$S_n^k(t,\rho)$: n-th Cesàro sums of order k of $\frac{1}{2} + \sum_1^\infty \cos nt, (\rho = 0)$; or of $\sum_1^\infty (\cos nt)_\rho \ (\rho \geqq 1)$.

$\mathfrak{S}^k(x; t, \rho) = \frac{1}{2} x^k + \sum_{n<x} (x-n)^k \cos nt, (\rho = 0)$; or, $= \sum_{n<x} (x-n)^k (\cos nt)_\rho, (\rho \geqq 1)$,

$\Gamma(1 + [\alpha] - \alpha) g(w,u) = \int_u^\pi (t-u)^{[\alpha]-\alpha} E^{[\alpha]}(w,t)\, dt,$

$\Gamma(\alpha+1) G(w,u) = \int_0^u \frac{v^\alpha}{(\log k/v)^{\alpha(\Delta-1)}} \frac{d}{dv} g(w,v)\, dv,$

$\Gamma(\alpha+1) H(w,u) = \int_u^\pi \frac{v^\alpha}{(\log k/v)^{\alpha(\Delta-1)}} \frac{d}{dv} g(w,v)\, dv.$

Then we have

(1) $$I = \int_2^\infty \frac{\beta\Delta(\log w)^{\Delta-1}}{w\, e^\beta(w)} \left| \int_0^\pi \phi(t) E(w,t)\, dt \right| dw$$

$$= \int_2^\infty \frac{\beta\Delta(\log w)^{\Delta-1}}{w\, e^\beta(w)} \left| \left[\sum_1^{[\alpha]} (-1)^{\rho-1} \Phi_\rho(t) E^{(\rho-1)}(w,t) \right]_0^\pi + (-1)^{[\alpha]} \int_0^\pi \Phi_{[\alpha]}(t) E^{([\alpha])}(w,t)\, dt \right| dw,$$

also

(2) $$\int_0^\pi \Phi_{[\alpha]}(t) E^{([\alpha])}(w,t)\, dt = \frac{1}{\Gamma(1+[\alpha]-\alpha)} \int_0^\pi E^{([\alpha])}(w,t)\, dt \int_0^t (t-u)^{[\alpha]-\alpha}\, d\Phi_\alpha(u)$$

$$= \frac{1}{\Gamma(1+[\alpha]-\alpha)} \int_0^\pi d\,\Phi_\alpha(u) \int_u^\pi (t-u)^{[\alpha]-\alpha} E^{([\alpha])}(w, t)\, dt = \int_0^\pi g(w, u)\, d\Phi_\alpha(u)$$

$$= \Big[\Phi_\alpha(u) g(w, u) \Big]_0^\pi - \int_0^\pi \Phi_\alpha(u) \frac{d}{du} g(w, u)\, du,$$

further

$$(3) \qquad \Gamma(\alpha+1) \int_0^\pi \Phi_\alpha(u) \frac{d}{du} g(w, u)\, du$$

$$= \int_0^\pi \phi_\alpha(u) (\log k/u)^{\alpha(\Delta-1)} \frac{u_\alpha}{(\log k/u)^{\alpha(\Delta-1)}} \frac{d}{du} g(w, u)\, du$$

$$= \Big[\phi_\alpha(u) (\log k/u)^{\alpha(\Delta-1)} G(w, u) \Big]_0^\pi - \int_0^\pi d\{\phi_\alpha(u) (\log k/u)^{\alpha(\Delta-1)}\} G(w, u).$$

From (1), (2) and (3) it will suffice to show that

$$(4) \qquad J = \int_2^\infty \frac{(\log w)^{\Delta-1}}{w\, e^\beta(w)} |E^{(\rho)}(w, \pi)| dw < \infty, \ 0 \leqq \rho \leqq [\alpha]-1, \text{ when } [\alpha] \geqq 1;$$

$$(5) \qquad K = \int_2^\infty \frac{(\log w)^{\Delta-1}}{w\, e^\beta(w)} |g(w, \pi)| dw < \infty\,;$$

$$(6) \qquad L = \int_2^\infty \frac{(\log w)^{\Delta-1}}{w\, e^\beta(w)} |G(w, \pi)| dw < \infty\,;$$

and finally

$$(7) \qquad M = \int_2^\infty \frac{(\log w)^{\Delta-1}}{w\, e^\beta(w)} |G(w, u)| dw = O(1) \qquad (0 < u < \pi),$$

since $\phi_\alpha(u) (\log k/u)^{\alpha(\Delta-1)}$ is of bounded variation in $(0, \pi)$.

2. Lemmas.

LEMMA 1. [3] $S_n^k(t, \rho) = \{\Omega(n, t, k)\}_\rho + \{W(n, t, k)\}_\rho, \ (\rho = 0, 1, 2, \ldots.)$
where

$$\Omega(n, t, k) = \frac{\sin\{(n + (k+1)/2)t - k\pi/2\}}{(2 \sin t/2)^{k+1}},$$

and

$$\{W(n, t, k)\}_\rho = \begin{cases} 0 & (k = 0) \\ O(n^{k-1} t^{-\rho-2}) & (k = 1, 2, 3, \ldots.). \end{cases}$$

LEMMA 2. *We have, when* $[\beta] = 0$,

$$(8)\quad F(w,t,\rho,s)=\sum_{l=0}^{\rho} O\left\{\frac{1}{t^{\beta+\rho-l}}\, e^{\beta}(w)w^{s-\beta+l+1}(\log w)^{(\beta-1)(\Delta-1)}\right\}$$

$$+\sum_{l=0}^{\rho} O\left\{\frac{1}{t^{\beta+\rho-l+1}}\, e^{\beta}(w)w^{s-\beta+l}(\log w)^{\beta(\Delta-1)}\right\}$$

$$+\,O\left\{\frac{1}{t^{\beta}}\, e^{\beta}(w)w^{-\beta}\left(w-\frac{1}{t}\right)^{s+\rho+1}(\log w)^{\beta(\Delta-1)}\left(\log\left(w-\frac{1}{t}\right)\right)^{-(\Delta-1)}\right\},$$

and we have, when $\beta>1$,

$$(9)\quad F(w,t,\rho,s)$$

$$=O\left(\frac{1}{t^{[\beta]+\rho}}+\frac{1}{t^{\rho+2}}\right)e^{\beta-1}(w)+\sum_{l=0}^{\rho} O\left\{\frac{1}{t^{[\beta]+\rho-l+1}}e^{\beta}(w)w^{s-[\beta]+l}(\log w)^{[\beta](\Delta-1)}\right\}$$

$$+\sum_{l=0}^{\rho} O\left\{\frac{1}{t^{\beta+\rho-l}}\, e^{\beta}(w)w^{s-\beta+l+1}(\log w)^{(\beta-1)(\Delta-1)}\right\}$$

$$+\,O\left\{\frac{1}{t^{\rho+2}}\, e^{\beta}(w)w^{s-1}(\log w)^{([\beta]-1)(\Delta-1)}\right\}.$$

PROOF. We may assume $e(x)=x,\ (0\leqq x\leqq 1)$.
When $[\beta]=0$.

$$(10)\quad F(w,t,\rho,s)=\left(\sum_{0}^{[w-1/t]}+\sum_{[w-1/t]+1}^{[w]}\right)\left\{e(w)-e(n)\right\}^{\beta-1} e(n)n^{s}(\cos nt)_{\rho}$$

$$=\sum_{i=1}^{2} D_i(w,t,\rho,s),\ \text{say}.$$

$$(11)\quad -D_1(w,t,\rho,s)=\int_0^{w-1/t}\mathfrak{S}^0(x,t,\rho)\frac{d}{dx}\left[\{e(w)-e(x)\}^{\beta-1}e(x)x^{s}\right]dx$$

$$=\Delta(\beta-1)\int_0^{w-1/t}\mathfrak{S}^0(x;t,\rho)\{e(w)-e(x)\}^{\beta-2}e^2(x)x^{s-1}(\log x)^{\Delta-1}dx$$

$$+\Delta\int_0^{w-1/t}\mathfrak{S}^0(x;t,\rho)\{e(w)-e(x)\}^{\beta-1}e(x)x^{s-1}(\log x)^{\Delta-1}dx$$

$$+s\int_0^{w-1/t}\mathfrak{S}^0(x;t,\rho)\{e(w)-e(x)\}^{\beta-1}e(x)x^{s-1}dx$$

$$=\sum_{i=1}^{3} C_i D_{1i}(w,t,\rho,s),\ \text{say, where } C_i \text{ are constants.}$$

$$(12)\quad D_{11}(w,t,\rho,s)=\int_0^{w-1/t}\left(\frac{\sin([x]+1/2)t}{2\sin t/2}\right)_{\rho}\{e(w)-e(x)\}^{\beta-2}e^2(x)x^{s-1}(\log x)^{(\Delta-1)}dx$$

$$= \sum_{l=0}^{\rho} C \int_0^{w-1/t} \frac{([x]+1/2)^l \frac{\sin}{\cos}([x]+1/2)t(\cos t/2)^n}{(2\sin(t/2))^{m+1}} \{e(w)-e(x)\}^{\beta-2} e^2(x) x^{s-1} (\log x)^{\Delta-1}\, dx$$

$$(0 \leqq m, n \leqq \rho - l)$$

$$= \sum_{l=0}^{\rho} O\Big[\frac{1}{t^{1+m}} \{e(w)-e(w-1/t)\}^{\beta-2} e^2(w-1/t)(w-1/t)^{s-1+l}$$

$$\cdot \Big\{ \log (w-1/t)^{(\Delta-1)} \int_{\xi}^{w-1/t} \frac{\sin}{\cos}([x]+1/2)t\, dx \qquad (0 \leqq \xi \leqq w - 1/t)$$

$$= \sum_{l=0}^{\rho} O\Big\{ \frac{1}{t^{\beta+\rho-l}} e^{\beta}(w) w^{s+1+l-\beta} (\log w)^{(\beta-1)(\Delta-1)} \Big\},$$

where, and in the following, C's are some constants which differ in different occurrences.

(13) $D_{12}(w, t, \rho, s)$

$$= \int_0^{w-1/t} \left(\frac{\sin([x]+1/2)t}{2\sin(t/2)} \right)_{\rho} \{e(w)-e(x)\}^{\beta-1} e(x) x^{s-1} (\log x)^{(\Delta-1)}\, dx$$

$$= \sum_{l=0}^{\rho} O\Big\{ \frac{1}{t^{m+1}} (e(w)-e(w-1/t))^{\beta-1} e(w-1/t)(w-1/t)^{s-1+l} (\log (w-1/t))^{\Delta-1}$$

$$\int_{\eta}^{w-1/t} \frac{\sin}{\cos}([x]+1/2)t\, dx \Big\}$$

$$= \sum_{l=0}^{\rho} O\Big\{ \frac{1}{t^{\beta+\rho-l+1}} e^{\beta}(w) w^{s+l-\beta} (\log w)^{\beta(\Delta-1)} \Big\}, \quad (0 \leqq \eta \leqq w - 1/t,\ 0 \leqq m, n \leqq \rho - l).$$

(14) The order of D_{13} is less than that of D_{12}.

$$(15) \quad D_2(w, t, \rho, s) = O\Big[\int_{[w-1/t]}^{[w]} \{e(w)-e(x)\}^{\beta-1} e(x) x^{s+\rho}\, dx \Big]$$

$$= O\Big[[x^{s+\rho+1} (\log x)^{-\Delta+1} \{e(w)-e(x)\}^{\beta}]_{[w-1/t]}^{[w]} + \int_{[w]-1/t}^{[w]} \{e(w)-e(x)\}^{\beta} x^{s+\rho} (\log x)^{-(\Delta-1)}\, dx \Big]$$

$$= O\{ t^{-\beta} e^{\beta}(w) w^{-\beta} (w-1/t)^{s+\rho+1} (\log w)^{\beta(\Delta-1)} \log (w-1/t)^{-(\Delta-1)} \}.$$

Lemma 2 for the case $[\beta] = 0$ follows from (8) and (10), ...(15). When $\beta > 1$.

(16) $F(w, t, \rho, s)$

$$= -\int_0^w \mathfrak{S}^0(x; t, \rho) \frac{d}{dx}\Big[\{e(w)-e(x)\}^{\beta-1} e(x) x^s \Big] dx$$

$$= \sum_{k=1}^{[\beta]-1} \left[\frac{(-1)^k}{k!} \mathfrak{S}^k(x;t,\rho) \left(\frac{d}{dx}\right)^k \left[\{e(w)-e(x)\}^{\beta-1} e(x) x^s \right] \right]_0^w$$

$$+ \frac{(-1)^{[\beta]}}{([\beta]-1)!} \int_0^w \mathfrak{S}^{[\beta]-1}(x;t,\rho) \left(\frac{d}{dx}\right)^{[\beta]} \left[\{e(w)-e(x)\}^{\beta-1} e(x)\, x^s \right] dx$$

$$= \sum_{k=1}^{[\beta]-1} \frac{(-1)^k}{k!} \left[J_{1k}(x;t,\rho) \right]_0^w + \frac{(-1)^{[\beta]}}{([\beta]-1)!} J_2(w,t,\rho), \text{ say.}$$

Now, by Lemma 1 we have

(17) $$\mathfrak{S}^k(x;t,\rho) = \sum_{n=1}^{[x]-k-1} \Delta^{k+1}(x-n)^k S_n^k(t,\rho) + \Delta^k(x-([x]-k))^k S^k_{[x]-k}(t,\rho) + \dots$$
$$+\Delta^2(x-([x]-2))^k S^2_{[x]-2}(t,\rho) + \Delta(x-([x]-1))^k S^1_{[x]-1}(t,\rho) + (x-[x])^k S^0_{[x]}(t,\rho)$$
$$= \sum_{j=0}^{k} C\left(\frac{\sin\{([x]-j+i/2+1/2)t-\pi j/2\}}{(2\sin(t/2))^{j+1}} \right)_\rho + O\left\{ \sum_{j=1}^{k} ([x]-j)^{j-1} t^{-\rho-2} \right\}$$
$$= C\sum_{j=0}^{k} C \sum_{l=0}^{\rho} C \left(\frac{\left([x]-\frac{j}{2}+\frac{1}{2}\right)^l \begin{matrix}\sin\\ \cos\end{matrix}\left\{\left([x]-\frac{j}{2}+\frac{1}{2}\right)t-\frac{\pi j}{2}\right\}\left(\cos\frac{1}{2}t\right)^n}{(2\sin(t/2))^{j+1+m}} \right)$$
$$+ O\left(\frac{x^{k-1}}{t^{\rho+2}}\right), \text{ where } 0 \leqq m, n \leqq \rho - l. \text{ Hence}$$

(18) $$\mathfrak{S}^k(x;t,\rho) = O\left(\frac{x^l}{t^{k+1+\rho-l}} + \frac{x^{k-1}}{t^{\rho+2}}\right), \quad (0 \leqq l \leqq \rho).$$

On the other hand

(19) $$\left(\frac{d}{dx}\right)^k \left[\{e(w)-e(x)\}^{\beta-1} e(x) x^s \right]$$
$$= \sum_{i=0}^{k} C \left[\{e(w)-e(x)\}^{\beta-1-i} e^{1+i}(x) x^{s-k} (\log x)^{q(\Delta-1)} \right]$$

where $0 \leqq q \leqq i \leqq k$. Using (18) and (19) we have, in (16),

(20) $$\left[J_{1k}(x;t,\rho) \right]_0^w = O\left(\frac{1}{t^{k+1+\rho}} + \frac{1}{t^{\rho+2}} \right) e^{\beta-1}(w), \quad 1 \leqq k \leqq [\beta]-1.$$

For the estimation of $J_2(x;t,\rho)$, it is easily seen that we may use on the terms of $\mathfrak{S}^k(x;t,\rho)$: the term $j=[\beta]-1$; then

(21) $J_2(w;t,\rho)$
$$= \sum_{l=0}^{\rho} O\left(\sum_{i=0}^{[\beta]} \int_0^w \frac{\left([x]-\frac{[\beta]}{2}+1\right)^l \begin{matrix}\sin\\ \cos\end{matrix}\left\{\left(([x]-\frac{[\beta]}{2}+1\right)t-\frac{\pi}{2}([\beta]-1)\right\}\left(\cos\frac{1}{2}t\right)^n}{(2\sin(t/2))^{[\beta]+\rho-l}} \right.$$
$$\left. \cdot \left\{e(w)-e(x)\right\}^{\beta-1-i} e^{1+i}(x) x^{s-[\beta]} (\log x)^{[\beta](\Delta-1)} dx \right)$$
$$+ O\left(\sum_{i=0}^{[\beta]} \int_0^w \frac{x^{[\beta]-2}}{t^{\rho+2}} \{e(w)-e(x)\}^{\beta-1-i} e^{1+i}(x) x^{s-[\beta]} (\log x)^{[\beta](\Delta-1)} dx \right)$$

$$= \sum_{l=0}^{\rho} O\Big[\sum_{i=0}^{[\beta]} J_{21i}(w\,;t,\rho)\Big] + O\Big[\sum_{i=0}^{[\beta]} J_{22i}(w\,;t,\rho)\Big].$$

When $i < [\beta]$,

(22) $J_{21i}(w : t, \rho)$

$$= \sum_{l=0}^{\rho} O\Big(\frac{1}{t^{[\beta]+\rho-l}}\int_0^w \{e(w)-e(x)\}^{\beta-1-i}\, e^{1+i}(x) x^{s-[\beta]+l} (\log x)^{[\beta](\Delta-1)}$$

$$\cdot \frac{\sin}{\cos}\Big\{\Big([x]-\frac{[\beta]}{2}+1\Big)t - \frac{\pi}{2}([\beta]-1)\Big\}\,dx\Big)$$

$$= \sum_{l=0}^{\rho} O\Big\{\frac{1}{t^{[\beta]+\rho-l+1}}\, e^{\beta}(w) w^{s-[\beta]+l} (\log w)^{[\beta](\Delta-1)}\Big\}.$$

When $i = [\beta]$,

$$(23)\quad J_{21[\beta]}(w\,;t,\rho) = \int_0^{w-1/t} + \int_{w-1/t}^{w} = J_{21[\beta]1}(w\,;t,\rho) + J_{21[\beta]2}(w\,;t,\rho), \text{ say.}$$

(24) $J_{21[\beta]1}(w\,;t,\rho)$

$$= \sum_{l=0}^{\rho} O\Big(\frac{1}{t^{[\beta]+\rho-l}}\Big\{e(w)-e(w-1/t)\Big\}^{\beta-1-[\beta]} e^{1+[\beta]}(w-1/t)$$

$$\cdot (w-1/t)^{s-[\beta]+l} (\log w)^{[\beta](\Delta-1)} \int_{\sigma}^{w-1/t} \frac{\sin}{\cos}\Big\{\Big([x]-\frac{[\beta]}{2}+1\Big)t - \frac{\pi}{2}([\beta]-1)\Big\}\,dx\Big)$$

$$= \sum_{l=0}^{\rho} O\Big\{\frac{1}{t^{\beta+\rho-l}}\, e^{\beta}(w) w^{s-\beta+l+1} (\log w)^{(\beta-1)(\Delta-1)}\Big\}, \quad (0 \leqq \sigma \leqq w-1/t).$$

(25) $J_{21[\beta]2}(w\,;t,\rho)$

$$= \sum_{l=0}^{\rho} O\Big(\frac{1}{t^{[\beta]+\rho-l}}\int_{w-1/t}^{w} \{e(w)-e(x)\}^{\beta-1-[\beta]}\, e^{1+[\beta]}(x)\, x^{s-[\beta]+l} (\log x)^{[\beta](\Delta-1)}\,dx\Big)$$

$$= \sum_{l=0}^{\rho} O\Big(\frac{e^{[\beta]}(w) w^{s-[\beta]+l+1} (\log w)^{([\beta]-1)(\Delta-1)}}{t^{[\beta]+\rho-l}}\Big[\{e(w)-e(x)\}^{\beta-[\beta]}\Big]_{w-1/t}^{w}\Big)$$

$$= \sum_{l=0}^{\rho} O\Big\{\frac{1}{t^{\beta+\rho-l}}\, e^{\beta}(w) w^{s-\beta+l+1} (\log w)^{(\beta-1)(\Delta-1)}\Big\}.$$

For the part J_{22i}, when $i = 0$,

$$(26)\quad J_{220}(w\,;t,\rho) = O\Big\{\frac{1}{t^{\rho+2}} e^{\beta-1}(w)\int_{\eta}^{\xi} \frac{e(x)(\log x)^{(\Delta-1)}}{x}\, x^{s-1} (\log x)^{([\beta]-1)(\Delta-1)}\,dx\Big\}$$

$$= O\Big\{\frac{1}{t^{\rho+2}}\, e^{\beta}(w) w^{s-1} (\log w)^{([\beta]-1)(\Delta-1)}\Big\}, \quad (0 \leqq \xi, \eta \leqq w),$$

and when $i > 0$,

$$(27)\quad J_{22i}(w\,;t,\rho)=O\left(\frac{1}{t^{\rho+2}}e^{i}(w)w^{s-1}(\log w)^{([\beta]-1)(\Delta-1)}\left[\{e(w)-e(x)\}^{\beta-i}\right]_0^w\right)$$
$$=O\left\{\frac{1}{t^{\rho+2}}e^{\beta}(w)w^{s-1}(\log w)^{([\beta]-1)(\Delta-1)}\right\}.$$

We have the order of $J_2(w\,;t,\rho)$ from (21), ...(27).

Lemma 2 for the case $\beta>1$ follows from (16), (20) and (21).

LEMMA 3. *If $\beta>1$; or if $\beta<1$ and $s+\rho+1>0$; then we have* $F(w,t,\rho,s)=O\left\{\{e^{\beta}(w)w^{s+\rho+1}(\log w)^{-(\Delta-1)}\right\}.$

PROOF.

$$F(w,t,\rho,s)=O\left(\int_0^w\{e(w)-e(x)\}^{\beta-1}\frac{e(x)\,(\log x)^{\Delta-1}}{x}x^{s+\rho+1}(\log x)^{-(\Delta-1)}\,dx\right),$$

hence Lemma 3 follows immediately.

3. Proof of Theorem.

PROOF OF (4). By Lemma 2 we have, when $\beta>[\alpha]\geqq 1$, and $[\alpha]-1\geqq\rho$, $E^{(\rho)}(w,\pi)=F(w,\pi,\rho,0)$

$$=O\{e^{\beta-1}(w)\}+\sum_{l=0}^{\rho}O\{e^{\beta}(w)w^{-[\beta]+l}\}+\sum_{l=0}^{\rho}O\{e^{\beta}(w)w^{-\beta+l+1}\}+O\{e^{\beta}(w)w^{-1}\}.$$

Since $-\beta+l+1<0$, we have

$$J=\int_2^\infty\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|E^{(\rho)}(w,\pi)|dw<\infty.$$

PROOF OF (5). Using the 1st and the 2nd mean value theorem we have

$$\Gamma(1+[\alpha]-\alpha)g(w,u)=\left(\int_u^{u+1/n}+\int_{u+1/n}^{\pi}\right)(t-u)^{[\alpha]-\alpha}E^{([\alpha])}(w,t)\,dt$$

$$=\sum_{n<w}\{e(w)-e(n)\}^{\beta-1}e(n)\left\{(\cos n\,\theta)_{[\alpha]}\int_u^{u+1/n}(t-u)^{[\alpha]-\alpha}dt+n^{\alpha-[\alpha]}\int_{u+1/n}^{\zeta}(\cos nt)_{[\alpha]}dt\right\}$$
$$(u+n^{-1}\leqq\zeta\leqq\pi)$$

$$=\sum_{n<w}\{e(w)-e(n)\}^{\beta-1}e(n)\left[n^{\alpha-[\alpha]-1}(\cos n\,\theta)_{[\alpha]}+n^{\alpha-[\alpha]}\begin{Bmatrix}n^{-1}(\cos nv)_{[\alpha]}\\ n^{-2}(\cos nv)_{[\alpha]+1}\end{Bmatrix}\right]$$
$$(u+n^{-1}\leqq\theta,v\leqq\pi)$$

$$=O\{F(w,u,[\alpha],\alpha-[\alpha]-1)+F(w,u,[\alpha]+1,\alpha-[\alpha]-2)\}.$$

Hence, by Lemma 2, we have, when $[\beta]=0$,

$$g(w,\pi)=O\{e^{\beta}(w)w^{\alpha-\beta}(\log w)^{(\beta-1)(\Delta-1)}\}+O\{e^{\beta}(w)w^{\alpha-1-\beta}(\log w)^{\beta(\Delta-1)}\}$$

$$+\sum_{m=0}^{1} O\{e^{\beta}(w)w^{\alpha-1-\beta+m}(\log w)^{(\beta-1)(\Delta-1)}\}+\sum_{m=0}^{1} O\{e^{\beta}(w)w^{\alpha-2-\beta+m}(\log w)^{\beta(\Delta-1)}\},$$

and when $\beta > 1$,

$$g(w,\pi)=O\{e^{\beta-1}(w)\}+\sum_{l=0}^{[\alpha]} O\{e^{\beta}(w)w^{\alpha-2[\alpha]-1+l}(\log w)^{[\beta](\Delta-1)}\}$$

$$+\sum_{l=0}^{[\alpha]} O\{e^{\beta}(w)w^{\alpha-[\alpha]-\beta+l}(\log w)^{(\beta-1)(\Delta-1)}\}+O\{e^{\beta}(w)w^{\alpha-[\alpha]-2}(\log w)^{([\beta]-1)(\Delta-1)}\}$$

$$+\sum_{m=0}^{[\alpha]+1} O\{e^{\beta}(w)w^{\alpha-2[\alpha]+m-2}(\log w)^{[\beta](\Delta-1)}\}+\sum_{m=0}^{[\alpha]+1} O\{e^{\beta}(w)w^{\alpha-[\alpha]-\beta+m-1}(\log w)^{(\beta-1)(\Delta-1)}\}$$

$$+O\{e^{\beta}(w)w^{\alpha-[\alpha]-3}(\log w)^{([\beta]-1)(\Delta-1)}\}.$$

Hence we have

$$K=\int_2^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}\,|g(w,\pi)|\,dw<\infty.$$

PROOF OF (6).

$$\Gamma(\alpha+1)G(w,\pi)=\left[\frac{v^{\alpha}}{(\log k/v)^{\alpha(\Delta-1)}}g(w,v)\right]_0^{\pi}$$

$$-\alpha\int_0^{\pi}\frac{v^{\alpha-1}}{(\log k/v)^{\alpha(\Delta-1)}}\int_v^{\pi}(t-v)^{[\alpha]-\alpha}\,E^{([\alpha])}(w,t)\,dt$$

$$=Cg(w,\pi)+C\int_0^{\pi}E^{([\alpha])}(w,t)t^{[\alpha]}\int_0^{1}\frac{s^{\alpha-1}(1-s)^{[\alpha]-\alpha}}{(\log k/ts)^{\alpha(\Delta-1)}}\,ds\,dt$$

$$=Cg(w,\pi)+C\int_0^{\pi}E^{([\alpha])}(w,t)\frac{t^{[\alpha]}}{(\log k/t)^{\alpha(\Delta-1)}}\,dt$$

$$=Cg(w,\pi)+C\sum_{r=1}^{[\alpha]}\left[E^{([\alpha]-r)}(w,t)\frac{t^{[\alpha]-r+1}}{(\log k/t)^{\alpha(\Delta-1)}}\right]_0^{\pi}$$

$$+\int_0^{\pi}\frac{E(w,t)}{(\log k/t)^{\alpha(\Delta-1)}}\,dt.$$

We substitute this for $G(w,\pi)$ in (6), then by (4) and (5) we have

$$L=O(1)+O\left\{\int_0^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}\,|E(w,\pi)|\,dw\right\}$$

$$+O\left\{\int_0^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}\left|\sum_{n<w}\{e(w)-e(n)\}^{\beta-1}e'(n)\int_0^{\pi}\frac{\cos nt}{(\log k/t)^{\alpha(\Delta-1)}}\,dt\right|dw\right\}.$$

The second term of the right hand is finite, since this term occurs when

$\beta > 1$, and then we have by Lemma 2 for the case $\beta > 1$,

$$E(w,\pi) = F(w,\pi,0,0)$$
$$= O\{e^{\beta-1}(w)\} + O\{e^{\beta}(w)w^{-[\beta]}\} + O\{e^{\beta}(w)w^{-\beta+1}\} + O\{e^{\beta}(w)w^{-1}\}.$$

The last term is finite since

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\frac{\cos nt}{(\log k/t)^{\alpha(\Delta-1)}}\,dt\right| < \infty,\ \ where\ k > \pi\, e^{\alpha(\Delta-1)+1}.$$

Thus we have

$$L = \int_2^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|G(w,\pi)|\,dw < \infty.$$

PROOF OF (7). We put $\tau = (\log k/u)^{(\Delta-1)}/u$, then using (6), we have

$$(28)\quad M = \int_2^{\tau}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|G(w,u)|\,dw + \int_{\tau}^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|G(w,\pi) - H(w,u)|\,dw$$

$$\leqq O(1) + \int_2^{\tau}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|G(w,u)|\,dw + \int_{\tau}^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|H(w,u)|\,dw.$$

By Lemma 3, similarly as in the proof of (5), we have

$$g(w,u) = O\{F(w,u,[\alpha],\alpha-[\alpha]-1) + F(w,u,[\alpha]+1,\alpha-[\alpha]-2)\}$$
$$= O\{e^{\beta}(w)w^{\alpha}(\log w)^{-(\Delta-1)}\}.$$

Hence

$$G(w,u) = O\left\{\frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}}e^{\beta}(w)w^{\alpha}(\log w)^{-(\Delta-1)}\right\},$$

and, since $\alpha > 0$ we have

$$(29)\quad \int_2^{\tau}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}|G(w,u)|\,dw$$

$$= O\left\{\frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}}\int_2^{\tau}w^{\alpha-1}dw\right\} = O(1),\ \ (0 < u < \pi).$$

On the other hand

$$\Gamma(\alpha+1)H(w,u) = Cg(w,\pi)$$
$$+\frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}}g(w,u) + C\int_u^{\pi}\frac{v^{\alpha}}{(\log k/v)^{\alpha(\Delta-1)}}g(w,v)\,dv$$
$$+ C\int_u^{\pi}\frac{v^{\alpha-1}}{(\log k/v)^{\alpha(\Delta-1)+1}}g(w,v)\,dv.$$

Since we may write $g(w,v) = O\{p(w)v^{-q}\}$, $q > \beta$, we have

$$\int_u^x \frac{v^{\alpha-1}}{(\log k/v)^{\alpha(\Delta-1)}} g(w, v)\, dv = O\left\{ p(w) \int_u^x \frac{v^{\alpha-1-q}}{(\log k/v)^{\alpha(\Delta-1)}} dv \right\}$$

$$= O\left\{ p(w) \frac{u^{\alpha-q}}{(\log k/u)^{\alpha(\Delta-1)}} \right\} = O\left\{ \frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}} g(w, u) \right\}.$$

Hence

$$(30)\quad \int_\tau^\infty \frac{(\log w)^{\Delta-1}}{we^{\beta}(w)} |H(w, u)| dw$$

$$= O(1) + O\left\{ \frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}} \int_\tau^\infty \frac{(\log w)^{\Delta-1}}{we^{\beta}(w)} |g(w, u)|\, dw \right\}.$$

By Lemma 2 and from the first expression of the proof of (5), we have, when $[\beta] = 0$,

$$g(w, u) = O\{F(w, u, 0, \alpha - 1)\} + O\{F(w, u, 1, \alpha - 2)\}$$

$$= O\left\{ \frac{1}{u^{\beta}} e^{\beta}(w) w^{\alpha-\beta} (\log w)^{(\beta-1)(\Delta-1)} \right\} + O\left\{ \frac{1}{u^{\beta+1}} e^{\beta}(w) w^{\alpha-1-\beta} (\log w)^{\beta(\Delta-1)} \right\}$$

$$+ O\left\{ \frac{1}{u^{\beta}} e^{\beta}(w) w^{-\beta} (\log w)^{\beta(\Delta-1)} \left(w - \frac{1}{u}\right)^{\alpha} \left(\log\left(w - \frac{1}{u}\right)\right)^{-(\Delta-1)} \right\}$$

$$+ \sum_{l=0}^{1} O\left\{ \frac{1}{u^{\beta+1-l}} e^{\beta}(w) w^{\alpha-1-\beta+l} (\log w)^{(\beta-1)(\Delta-1)} \right\}$$

$$+ \sum_{l=0}^{1} O\left\{ \frac{1}{u^{\beta+2-l}} e^{\beta}(w) w^{\alpha-2-\beta+l} (\log w)^{\beta(\Delta-l)} \right\},$$

and when $\beta > 1$,

$$g(w, u) = O\{F(w, u, [\alpha], \alpha - [\alpha] - 1)\} + O\{F(w, u, [\alpha] + 1, \alpha - [\alpha] - 2)\}$$

$$= O\left(\frac{1}{u^{2[\alpha]+1}} + \frac{1}{u^{[\alpha]+3}} \right) e^{\beta-1}(w) + \sum_{l=0}^{[\alpha]} O\left\{ \frac{1}{u^{2[\alpha]-l+1}} e^{\beta}(w) w^{\alpha-2[\alpha]-1+l} (\log w)^{[\beta](\Delta-1)} \right\}$$

$$+ \sum_{l=0}^{[\alpha]} O\left\{ \frac{1}{u^{\beta+[\alpha]-l}} e^{\beta}(w) w^{\alpha-[\alpha]-\beta+l} (\log w)^{(\beta-1)(\Delta-1)} \right\}$$

$$+ O\left\{ \frac{1}{u^{[\alpha]+2}} e^{\beta}(w) w^{\alpha-[\alpha]-2} (\log w)^{([\beta]-1)(\Delta-1)} \right\}$$

$$+ \sum_{m=0}^{[\alpha]+1} O\left\{ \frac{1}{u^{2[\alpha]-m+2}} e^{\beta}(w) w^{\alpha-2[\alpha]-2+m} (\log w)^{[\beta](\Delta-1)} \right\}$$

$$+ \sum_{m=0}^{[\alpha]+1} O\left\{ \frac{1}{u^{\beta+[\alpha]+1-m}} e^{\beta}(w) w^{\alpha-[\alpha]-1+m-\beta} (\log w)^{(\beta-1)(\Delta-1)} \right\}$$

$$+ O\left\{ \frac{1}{u^{[\alpha]+3}} e^{\beta}(w) w^{\alpha-[\alpha]-3} (\log w)^{([\beta]-1)(\Delta-1)} \right\}.$$

Substituting each of these values for $g(w, u)$, we have

$$(31)\quad \frac{u^{\alpha}}{(\log k/u)^{\alpha(\Delta-1)}}\int_{\tau}^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}\,|g(w,u)|\,dw = O(1),\ \ (0<u<\pi),$$

From (28), ...(31) we have

$$M=\int_{2}^{\infty}\frac{(\log w)^{\Delta-1}}{we^{\beta}(w)}\,|G(w,u)|\,dw = O(1),\ \ (0<u<\pi).$$

Thus the theorem is completely proved.

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