ON THE REALIZATION OF THE STIEFEL-WHITNEY CHARACTERISTIC CLASSES BY SUBMANIFOLDS

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Introduction

We know several results on the realization of cohomology classes by submanifolds in a compact differentiable manifold [2, 3]. A fundamental theorem by R. Thom [3] shows that the realizability of cohomology classes can be

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reduced to existence of a mapping with certain properties (see section 1).

It is quite natural to ask whether the Stiefel-Whitney classes are realizable by submanifolds. There are two ways to attack this problem. The first one is to use *Schubert varieties* in a Grassmann manifold. It gives rather general information about the problem in vector bundles. The second one is to find directly a map satisfying the requirements of Thom's fundamental theorem. It can be applied to the Stiefel-Whitney classes of any vector bundles and it depends on the study of a homotopy type of a cell complex $M(O(k))$. Thus we can use this method successfully for low dimensional classes.

In Chapter I, we define *induced Schubert subvarieties* and obtain a series of necessary conditions for realizability of the Stiefel-Whitney classes of vector bundles over a compact differentiable manifold, calculating the cohomology class of a singular locus. If the dimension of the manifold is equal to the codimension of the singular locus, then a sufficient condition for the classes to be realizable is stated as follows: The cohomology class of the singular locus with respect to integer coefficients vanishes.

In Chapter II, we discuss the realization of the Stiefel-Whitney classes of vector bundles over a compact differentiable manifold, using the canonical isomorphism from cohomology group of base space onto that of total space and the Steenrod Square operations. We compute the second **k**-invariant of $M(O(2))$ and obtain a rather strong sufficient condition in order that W_2 of a vector bundle over V_6 is realizable by a submanifold. In particular, any W_2 of a vector bundle of an orientable manifold V_6 is realizable.

In the last Chapter, we consider complete intersections of non-singular hypersurfaces, in which any W_i is realizable by a submanifold.

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CHAPTER I

REALIZATION OF THE STIEFEL-WHITNEY CLASSES BY INDUCED SUBMANIFOLDS

1. Preliminaries.

Let $\mathfrak{S}^n$ be an n-vector bundle over a finite cell complex with any closed subgroup G of the orthogonal group $O(n)$ as its structural group. It is induced from an N-universal bundle $A_{G,n}$ over a classifying space $B_{G,n}$, for instance, a Grassmann manifold $G_{n,N}$ for a sufficiently large integer N (see Steenrod [1]). Suppose $S_{G,n}$ be an associated $(n-1)$-sphere bundle to $A_{G,n}$. Combining $S_{G,n}$ and $A_{G,n}$, one can make an associated closed n-cell bundle $\bar{A}_{G,n}$ where $S_{G,n}$ is the boundary. Shrinking $S_{G,n}$ into a point, we get a *cell complex* $M(G, n)$ *corresponding to the subgroup* G *of* $O(n)$.

Since $B_{G,n}$ is a differentiable manifold, it has a simplicial subdivision. We can assume that the diameter of each simplex is so small that it is contained in a coordinate neighborhood. Let b be an n-cell of fiber in the fiber bundle $\bar{A}_{G,n}$. We take all cells of the form $\sigma \times b$ for any simplex σ in $B_{G,n}$. They give a cellular subdivision of $\bar{A}_{G,n}$ up to its boundary $S_{G,n}$. Define a cochain isomorphism $\varphi_{G,n}$ of $C^i(B_{G,n}; Z_2)$ onto $C^{n+i}(\bar{A}_{G,n}, S_{G,n}; Z_2)$ by the formula,

$$\varphi_{G,n}(c)(\sigma \times b) = c(\sigma)$$

for any cochain $c \in C^i(B_{G,n}; Z_2)$ and for $i \geqq 0$. This induces the *canonical* isomorphism $\varphi^*_{G,n}$: $H^i(B_{G,n}; Z_2) \approx H^{n+i}(A_{G,n}, B_{G,n}; Z_2)$. Let $1_{G,n}$ be the unit class of $H^*(B_{G,n}; Z_2)$. $H^n(M(G,n); Z_2)$ is generated by $\varphi^*_{G,n}(1_{G,n}) = U_{G,n}$ which is called the *fundamental class of* $M(G,n)$.

If $G = O(n)$, then we denote $A_{G,n}$, $B_{G.n}$ and $M(G,n)$ by $A_{O(n)}$, $B_{O(n)}$ and $M(O(n))$ respectively.

Let K be a topological space and let u be an element of $H^n(X; Z_2)$. We say that u is *realizable* for $G \subset O(n)$, if there is a mapping f: $K \rightarrow M(G,n)$ such that $u = f^* U_{G,n}$. Suppose F_r is a submanifold of dimension r in a compact differentiable manifold M of dimension $m \geqq r$ and of class C^∞. Let i be the imbedding $F_r \subset M$. If an element z of $H_r(M; Z_2)$ is the image of the fundamental class of F_r, then we say that z is *realized by the submanifold* F_r.

FUNDANENTAL THEOREM (**THOM**). *A cohomology class u of $H^n(M; Z_2)$ is realizable for the group $G \subset O(n)$ if and only if the dual homology class z of u is realized by a submanifold F_r of dimension r and the fiber bundle of normal vectors on F_r in M has the group G as its structural group* (see [2]).

A sum of two realizable classes is not necessarily realizable. *Their cup-product*, however, *is realizable* (see [2, 3]). All the above statements are valid for integer coefficients if M is orientable.

It is well known that the Grassmann manifold has a cellular subdivision by the *Schubert varieties*, where *variety* means a set defined by a system of algebraic equations, which may have singular locus. The Stiefel-Whitney class W_j of dimension j is defined as a cohomology class with coefficients in Z_2, determined by the Schubert class

$$\{0, \ldots, 0, \underbrace{1, \ldots, 1}_{j}\}.$$

It coincides with the class of obstruction cocycle of a field of $(n-j+1)$-frames over the j-skeleton of $G_{n,N}$. We can see that any 1-dimensional cohomology class in a manifold is realizable. Hence W_1 is necessarily realizable.

Now we mention the following important relation due to Thom [4], between W_j of the N-universal bundle over $B_{G,n}$ and the Steenrod square operation Sq^i;

$$Sq^j U_{G,n} = W_j U_{G,n}$$

$$= \varphi^*_{G,n} W_j \text{ for } 0 \leqq j \leqq n. \qquad (1.1)$$

Let f be a mapping of a finite cell complex L into $B_{G,n}$ which induces an n-vector bundle $\mathfrak{S}^n$ over L. The Stiefel-Whitney class $W_j(\mathfrak{S}^n)$ is given by

$$f^* W_j = W_j(\mathfrak{S}^n).$$

Let $\varphi^*_{\mathfrak{S}^n}$ be the canonical isomorphism of $\mathfrak{S}^n$ defined in the same way as $\varphi^*_{G,n}$. Suppose B_n be an associated $(n-1)$-sphere bundle to $\mathfrak{S}^n$, which can be regarded as the boundary of an associated n-cell bundle $\mathfrak{S}^n$. $\varphi^*_{\mathfrak{S}^n}$ is the isomorphism of $H^i(L; Z_2)$ onto $H^{n+i}(\mathfrak{S}^n, B_n; Z_2)$. Putting $f^* U_{G,n} = U_{\mathfrak{S}^n} = W_n(\mathfrak{S}^n)$, (1.1) leads immediately to the relation,

$$\begin{aligned} Sq^j U_{\mathfrak{S}^n} &= W_j(\mathfrak{S}^n)\, U_{\mathfrak{S}^n} \\ &= \varphi^*_{\mathfrak{S}^n} W_j(\mathfrak{S}^n), \end{aligned} \qquad (1.2)$$

for $0 \leqq j \leqq n$.

Suppose F_r be a subvariety of a compact differentiable manifold M_{n+r} with a singular subvariety F_{r_1} of dimension $r_1 < r$, F_{r_2} denotes a singular subvariety in F_{r_1} of dimension $r_2 < r_1$ and so on. The sequence $F_{r_1} \supset F_{r_2} \supset \ldots$ ends by F_{r_i} after finite repetitions. The transversality theorem[1] says that *for any differentiable mapping f of a compact differentiable manifold V_n to M_{n+r}, there exists a mapping which is homotopic and arbitrarily near to f and also transversally regular with respect to $F_r \supset F_{r_1} \supset F_{r_2} \supset \ldots \supset F_{r_i}$* (see Thom [5, 17]).

2. Subvarieties Corresponding to W_i.

Suppose V_n be a compact differentiable manifold of dimension n, and suppose $\mathfrak{E}^m$ be an m-vector bundle over V_n. Then we have a mapping f of V_n into a Grassmann manifold $G_{m,N}$ such that the induced bundle is $\mathfrak{E}^m$.

W_i in $G_{m,N}$ is realized by the Schubert variety $[\underbrace{N-1, N-1, \ldots, N-1}_{i}, N, \ldots, N] = F_i$. By the transversality theorem, there exists a differentiable mapping f_1 which is homotopic to f and transversally regular with respect to singular subvarieties of $[N-1, \ldots, N-1, N, \ldots, N]$. Therefore $W_i(\mathfrak{E}^m)$ is realized by the subvariety[2] $f_1^{-1}(F_i)$, which we call an *induced Schubert variety.* It has a singular subvariety S_1 which is a realization of $f^*\{0, \ldots, 0, \underbrace{2, \ldots, 2}_{i+1}\} = S_1^*$ and S_1 has a singular subvariety $\{S_1\}^2$ which corresponds to $f^*\{0, \ldots, 0, \underbrace{3, \ldots, 3}_{i+2}\} = [\{S_1\}^2]^*$ and so on.[3] Thus we can say that S_1^* is the first obstruction to the realization of $W_i(\mathfrak{E}^m)$ by an induced Schubert variety. $[\{S_1\}^2]^*$ is the second one. Hence we get the idea of higher obstructions.

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1) When F_r has no singularity, the transversality theorem is given in [2, Theorem I. 6]. Its proof in general case is found in [5, Chap. II, Theorem 1] and [17].

2) We use the term of subvariety for a subset defined by a system of algebraic equations, which may have singularities, and also its inverse image by a differentiable map.

3) See [5, Chap. I].

If S_1^* vanishes then the Schubert variety becomes an actual manifold. This idea is the main tool of this section and section 4.

According to Chern's paper [6], we have several relations for multiplication of the Schubert classes. For the sake of brevity, we denote $\{0, \ldots, 0, a_k, \ldots, a_n\}$ by $\{a_k, \ldots, a_n\}$. We have $\{0\} = 1$. Put $\{a\} = 0$ if $a < 0$. Then the following formula holds:

$$\{a_k, \ldots, a_n\}\{b\} = \Sigma\{a_k + b_k, \ldots, a_n + b_n\} \tag{2.1}$$

where the sum extends over all partitions of b satisfying the conditions that $a_j + b_j \leqq a_{j+1}$, $\sum_{j=k}^{n} b_j = b$. We have also the relation,

$$\{a_1, \ldots, a_n\} = \begin{vmatrix} \{a_1\}, & \{a_1 - 1\}, \ldots & \{a_1 - n + 1\} \\ \{a_2 + 1\} & \{a_2\} \quad \ldots & \\ \ldots & \quad\quad \ldots & \\ \{a_1 + n - 1\} & \quad\quad \ldots & \{a_n\} \end{vmatrix}. \tag{2.2}$$

Put $\{j\} = \overline{W}_j$. Then (2.1) leads to

$$\sum_{0 \leqq j \leqq k} W_j \overline{W}_{k-j} = 0 \qquad 1 \leqq k \leqq n. \tag{2.3}$$

(2.3) shows that $\overline{W}_j$ can be solved in W_j. Using (2.2), it can be seen that any Schubert classes are polynomials in W_j, since we have

$$\begin{aligned} \overline{W}_1 &= W_1 \\ \overline{W}_2 &= W_1^2 + W_2 \\ \overline{W}_3 &= W_1^3 + W_3 \\ \overline{W}_4 &= W_1^4 + W_2W_1^2 + W_2^2 + W_4 \\ \overline{W}_5 &= W_1^5 + W_2^2W_1 + W_3W_1^2 + W_5, \end{aligned} \tag{2.4}$$

and so on.

Now we shall consider the realization of W_2 by the induced Schubert variety without singularity which gives a method to solve realizability of W_2. The first obstruction is the class $\{0, \ldots 0, 2, 2, 2\}$. Using (2.2), we obtain that

$$\{0, \ldots, 0, 2, 2, 2\} = \begin{vmatrix} \{2\} & \{1\} & \{0\} \\ \{3\} & \{2\} & \{1\} \\ \{4\} & \{3\} & \{2\} \end{vmatrix}. \tag{2.5}$$

Substitute (2.4) in (2.5), we get the relation,

$$\{0, \ldots, 0, 2, 2, 2\} = W_2W_4 + W_3^2.$$

If an induced Schubert variety is a submanifold, then its singularity vanishes. Hence we get the result:

THEOREM 2.1. *If $W_2(\mathfrak{G}^m)$ is realizable by the induced Schubert submanifold, then we have*

$$W_2(\mathfrak{G}^m)\, W_4(\mathfrak{G}^m) = (W_3(\mathfrak{G}^m))^2. \tag{2.6}$$

In the same way, the first obstruction of realization of $W_3(\mathfrak{E}^m) = f^*\{0, \ldots, 0, 1, 1, 1\}$ by the Schubert manifold is given by the following formula

$$\{0, \ldots, 0, 2, 2, 2, 2\} = P_8$$

which is the Pontrjagin class of dimension 8 and is a cohomology class with integer coefficients. Using (2.2) we have

$$P_8 = \begin{vmatrix} \{2\} & \{1\} & 1 & 0 \\ \{3\} & \{2\} & \{1\} & 1 \\ \{4\} & \{3\} & \{2\} & \{1\} \\ \{5\} & \{4\} & \{3\} & \{2\} \end{vmatrix}. \tag{2.7}$$

Substituting (2.4) in (2.7), we obtain

$$P_8 = W_3 W_5 + W_4^2, \qquad \text{mod } 2.$$

Thus, in order that $W_3(\mathfrak{E}^m)$ is realizable by the induced Schubert submanifold, it is necessary that

$$W_3(\mathfrak{E}^m)\, W_5(\mathfrak{E}^m) = (W_4(\mathfrak{E}^m))^2. \tag{2.8}$$

This result can be generalized for any Stiefel-Whitney class $W_{2j+1}(\mathfrak{E}^m)$ of odd dimension.

THEOREM 2.2. *If $W_{2j+1}(\mathfrak{E}^m)$ is realizable by the induced Schubert submanifold, then we have*

$$P_{4(j+1)}(\mathfrak{E}^m) = 0 \qquad (\textit{integer coefficients}) \tag{2.9}$$

and

$$W_{2(j+1)-1}(\mathfrak{E}^m)\, W_{2(j+1)+1}(\mathfrak{E}^m) = (W_{4(j+1)}(\mathfrak{E}^m))^2 \qquad \textit{mod } 2.$$

PROOF. By definition we have $\{0, \ldots, 0, 2, \ldots, 2\} = P_{4(j+1)}$ and $P_{4(j+1)}(\mathfrak{E}^n) = f^* P_{4(j+1)}$ which vanishes. Thus the first part of Theorem follows immediately.

Let $\mathbf{1}$ be a canonical mapping of the real Grassmann manifold $G_{m,N}$ into the complex Grassmann manifold $C_{m,N}$ and let C_{2k} be the Chern class of dimension $2k$.

W. T. Wu [7] proves that

$$\mathbf{1}^* C_{2k} = (W_k)^2 \qquad \text{mod } 2$$

and

$$\mathbf{1}^* C_{2k} = (-1)^{k/2} P_{2k} + (1/2)\,\delta U_{2k-1}$$

where $k = 2(j+1)$ and $U_{4j+3} = \sum_{i=0}^{2j+1} W_i\, W_{4j-i+3}$. It follows that

$$(1/2)\delta U_{2k-1} = (1/2)\delta\Big(\sum_{i=0}^{2j+1} W_i\, W_{4j-i+3}\Big) \tag{2.10}$$

$$= (1/2)\delta\,(W_{4j+3} + W_1 W_{4j+2} + \ldots + W_{2j+1} W_{2j+1}).$$

We have the relations (Wu [9]),

$$Sq^1 W_i = W_1 W_i + \left\{ \begin{matrix} i+1 \\ 1 \end{matrix} \right\} W_{i+1},$$
$$Sq^1 W_{4j+3} = W_1 W_{4j+3},$$
$$Sq^1 (W_1 W_{4j+2}) = W_1 W_{4j+3},$$
$$Sq^1 (W_2 W_{4j+1}) = W_3 W_{4j+1},$$
$$Sq^1 (W_3 W_{4j}) = W_3 W_{4j+1},$$
$$\cdots\cdots$$

Substituting these formulae into (2.10), we obtain

$$(1/2)\delta U_{2k-1} = W_{2j+1} W_{2j+3} \qquad \text{mod } 2,$$

which leads to the second part of our theorem.

Theorem 2.2 might be generalized for W_{2k}, but we have no general formula to compute it. If $n < 6$, then the both sides of (2.6) vanish. Hence it holds necessarily. Similarly (2.9) holds necessarily if $n < 4(k+1)$.

3. Examples.

$P(i)$ denotes an i-dimensional real projective space. The *cobordism group* $\mathfrak{N}^6$ *mod* 2 of real compact manifolds of dimension 6 admits as generators (see Thom [2]),

(i) $P(6)$,

(ii) $P(4) \times P(2)$,

(iii) $P(2) \times P(2) \times P(2)$.

THEOREM 3.1. *The relation*

$$W_3^2 + W_2 W_4 = 0 \qquad \textit{mod } 2 \tag{3.1}$$

holds for manifolds of type (i), *and not for manifolds of types* (ii), (iii).

PROOF. We denote by $W_j(\mathrm{i}), \ldots,$ the j-th Stiefel-Whitney classes of manifolds of types (i), ... It is well known that the total Stiefel-Whitney Class of $P(i)$ is given by

$$W(P(i)) = (1+h)^{i+1}$$

where h is the generator of the cohomology ring $H^*(P(i); Z_2)$.

In the case (i), we have

$$W_2(\mathrm{i}) = \binom{7}{2} h^2 = h^2,$$

$$W_3(\mathrm{i}) = \binom{7}{3} h^3 = h^3,$$

$$W_4(\mathrm{i}) = \binom{7}{4} h^4 = h^4.$$

Therefore it follows that

$$(W_3(\mathrm{i}))^2 = h^6 = h^2 h^4$$
$$= W_2(\mathrm{i}) W_4(\mathrm{i}).$$

In the case (ii), we denote by h_1 and h_2 the generators of cohomology ring $H^*(P(4); Z_2)$ and $H^*(P(2); Z_2)$ respectively. We have the total Stiefel-Whitney classes,

$$W(P(4)) = 1 + h_1 + h_1^4,$$
$$W(P(2)) = 1 + h_2 + h_2^2.$$

It follows that

$$W_2(\text{ii}) = h_2^2 + h_1h_2,$$
$$W_3(\text{ii}) = h_1h_2^2,$$
$$W_4(\text{ii}) = h_1^4.$$

Thus we get

$$(W_3(\text{ii}))^2 = h_1^2h_2^4,$$
$$W_2(\text{ii})W_4(\text{ii}) = (h_2^2 + h_1h_2)h_1^4$$
$$= h_1^4h_2^2 + h_1^5h_2.$$

Therefore we have

$$(W_3(\text{ii}))^2 \neq W_2(\text{ii})\, W_4(\text{ii}).$$

In the case (iii), let h_1, h_2 and h_3 be generators of cohomology rings of first, second and third factors in $P(2) \times P(2) \times P(2)$. We have the total Stiefel-Whitney classes

$$W(P(2)) = 1 + h_i + h_i^2.$$

Thus it follows that

$$W_2(\text{iii}) = \sum_1^3 h_i^2 + \sum_{i \neq j} h_ih_j,$$

$$W_3(\text{iii}) = \sum_{i \neq j} h_i^2h_j + h_1h_2h_3,$$

$$W_4(\text{iii}) = \sum_{i \neq j} h_i^2h_j^2 + \sum_{(i,j,k)} h_i^2h_jh_k.$$

Thus we get

$$(W_3(\text{iii}))^2 = \sum_{i \neq j} h_i^4h_j^2 + h_1^2h_2^2h_3^2$$
$$= h_1^2h_2^2h_3^2,$$
$$W_2(\text{iii})\, W_4(\text{iii}) = 6h_1^2h_2^2h_3^2 = 0.$$

Hence, we obtain the result,

$$(W_3(\text{iii}))^2 \neq W_2(\text{iii})\, W_4(\text{iii}).$$

Any other manifold else belongs to the trivial type, for which the theorem always holds.

The *cobordism group* $\mathfrak{N}^8$ *mod* 2 of real compact manifolds of dimension 8 admits as generators,

(i) $P(8)$,

(ii) $P(6) \times P(2)$,

(iii) $P(4) \times P(4)$,

(iv) $P(4) \times P(2) \times P(2)$,

(v) $P(2) \times P(2) \times P(2) \times P(2)$.

The first class of singularity of W_3 is the Pontrjagin class $P_8 = 0$, since Pontrjagin classes are multiplicative and since they are trivial in any real projective space. Thus any cobordism class of real compact manifold of dimension 8 contains a manifold in which the first class of singularity in an induced Schubert variety of W_3 vanishes. By the same argument any cobordism class of dimension $4(k+1)$ contains a manifold in which the first class of singularity in an induced Schubert variety of W_{2k+1} vanishes. (On the contrary, we don't have a corresdonding result for the Stiefel-Whitney class W_{2k} as it is easily seen in Theorem 3.1.)

REMARK. (1) Equivalence in the sense of cobordism does not conserve the realizability by the induced Schubert submanifold of cohomology classes. For example, a complex projective plane $PC(2)$ and $P(4)$ belong to the same cobordism type mod 2 because every corresponding Stiefel-Whitney numbers of both manifolds are equal. We have, however, $P_4(PC(2)) \neq 0$ and $P_4(P(4)) = 0$, therefore W_3 is realizable in $P(4)$ and is not in $PC(2)$ (see sec. 4). (2) Theorem 3.1 shows that the method of the induced Schubert manifold is negative for the cases (ii), (iii) of cobordism types mod 2 of dimension 6. For any differentiable map $V_6 \to R_5$ (5-dimensional Euclidean space), the critical variety has at least one singular point, if V_6 belongs to classes (ii), (iii).

4. A Sufficient Condition.

Let V_n and M_{n+r} be compact differentiable manifolds of dimensions n and $n+r$ respectively. Suppose F_r be a compact subvariety[4] in M_{n+r} which may have some singularities. Let f be a differentiable mapping of V_n into M_{n+r}. Using the transversality theorem and the assumption about dimensions of manifolds, we can take a mapping of V_n into M_{n+r} which is sufficiently near and homotopic to f, satisfying following conditions:

(1) It is a transversally regular mapping with respect to F_r and its singularities, that means, in particular:

(2) Its image intersects F_r in regular points.

(3) The inverse image of F_r is a set of isolated points.

Without loss of generality, we assume that f is such a mapping as far as the induced homomorphism of homology groups is concerned.

Using the above property (2), we construct *tubular sets* N_{r_k} with respect to singular loci F_{r_k} for $1 \leq k \leq i$ which do not contain at all the image points of f. N_{r_k} is defined as the set of all points of normal geodesics of F_{r_k} of length ρ_{r_k} which we call the *diameter* of N_{r_k}. Let N_{r_0} be a tubular set of F_{r_0} by means of normal geodesics of length ρ_{r_0} putting $r = r_0$. Define a *tubular neighborhood*

4) See footnote 2 of section 2.

$N(F_r)$ of F_r as a union of all N_{r_k} $(k = 0, \dots, i)$. We denote its boundary by $T(F_r)$. We can take ρ_{r_k} such that ρ_{r_k} is sufficiently small to $\rho_{r_{k+1}}$. It makes the cellular subdivision of $N(F_r)$ simple, namely the cellular subdivision stated in section 1 can be applied for $N(F_r)$ successively from lowest dimension. We can construct a neighborhood deformation retraction of $T(F_r)$ in $N(F_r)$ by an induction in k, using a deformation along normal geodesics in a neighborhood of $T(F_r)$. Put $A = M_{n+r} - N(F_r)$. Obviously A is a neighborhood deformation rectract in M_{n+r}. Using a triangular subdivision of M_{n+r}, we can construct a cellular subdivision of M_{n+r} compatible with that of $N(F_r)$.

We consider the problem to compress f into A in the sense of Spanier-Whitehead (see [8]). F_r denotes also the chain determined by the subvariety F_r and D denotes the homomorphism of chain groups to cochain groups by taking intersection numbers in integer coefficients.

LEMMA 4.1. *If we have $f^*DF_r = 0$ with respect to integer coefficients, then we get $f^*V_n \circ F_r = 0$, where $\circ$ means an intersection of chains.*

PROOF. It follows from the condition of our lemma that

$$\begin{aligned} DF_r \cap f_*V_n &= f_*(f^*DF_r \cap V_n) \\ &= f_*(0 \cap V_n) \\ &= 0, \end{aligned}$$

namely

$$\begin{aligned} F_r \circ f_*V_n &= DF_r \cap f_*V_n \\ &= 0. \end{aligned}$$

The main theorem in this section is the following:

THEOREM 4.1. *Suppose M_{n+r} be simply connected and F_r be a compact subvariety. Let f be a mapping of V_n into M_{n+r}. If we have $f^*DF_r = 0$ with respect to integer coefficients, then f is compressible into A.*

PROOF. Let M_i be the i-skeleton of M_{n+r}. The theory of compression by Spanier and Whitehead [8] tells us the following: Suppose $A \cup M_{i-1}$ is simply connected, $i \geqq 2$ and $\dim(M_i - A) = i$. Let f_i be a mapping of V_n into $(A \cup M_i, A)$. Let j_i be the inclusion map $(A \cup M_i, A) \subset (A \cup M_i, A \cup M_{i-1})$. Then we get the following diagram,

$$\pi^i(A \cup M_i, A \cup M_{i-1}) \xrightarrow{j_i^*} \pi^i(A \cup M_i, A) \xrightarrow{f_i^*} \pi^i(V_n).$$

The first obstruction to compressing f_i into $A \cup M_{i-1}$ is defined by

$$z_i(f_i) = f_i^* j_i^* \in Z_i(A \cup M_i,\ A;\ \pi^i(V_n)). \tag{4.1}$$

If $z_i(f_i) = 0$, then f_i is compressible into $A \cup M_{i-1}$.

We have $\pi^i(V_n) = 0$ for $n < i \leqq n + r$. Hence we get $z_i(f_i) = 0$ for such i. Therefore f can be compressed successively into $A \cup M_n$. $z_n(f_n)$ is a critical obstruction. On the other side, we can find a deformation d_t $(0 \leqq t \leqq 1)$ of $A \cup M_{n-1}$ in M_{n+r} leaving $T(F_r)$ fixed in such a way that the image of d_1 does not intersect at all with F_r. Therefore if f_n is compressed into $A \cup M_{n-1}$, $d_1 f_{n-1}$ which is homotopic to f is the required compression. We want to show $z_n(f_n) = 0$ under the assumption $f^*DF_r = 0$.

We know $\pi^n(V_n) \approx H^n(V_n)$ (the Hopf's mapping theorem). Excising the interior A^i of A from (M_{n+r}, A) because of the neighborhood retraction of A in M_{n+r}, we get the following result,

$$\begin{aligned} Z_n(A \cup M_n, A; \pi^n(V_n)) &= Z_n(M_{n+r}, A; H^n(V_n)) \\ &= Z_n(N(F_r), T(F_r); H^n(V_n)) \\ &\overset{\varphi_{N(Fr)}}{\approx} Z_0(F_r; L \otimes H^n(V_n)) \\ &= H_0(F_r; L \otimes H^n(V_n)) \\ &= L \otimes H^n(V_n), \end{aligned}$$

where L is a local system corresponding to the orientation of the normal bundle over F_r. All groups of L are isomorphic to Z. From (4.1) we have

$$\begin{aligned} z_n(f_n) &= f_n^* j_n^* \\ &= f^*(U_{N(F_r)}) \in H^n(V_n) \\ &= f^*(DF_r) \\ &= av_n, \end{aligned}$$

where v_n is the fundamental cocycle of V_n and a is an integer. It follows that

$$\begin{aligned} f^*(DF_r)\,(V_n) &= DF_r(f_* V_n) \\ &= I(f_*(V_n), F_r) \\ &= a, \end{aligned}$$

where $I(\ )$ denotes an intersection number. Lemma 4.1 shows that $a = 0$. Hence we get $z_n(f_n) = 0$.

Since M_{n+r} is simply connected, so is M_{n-1} if $n \geqq 3$, because a homotopy of a closed curve into a constant mapping can be compressed into M_{n-1}. By the same reason, $A \cup M_{n-1}$ is simply connected, if $n \geqq 3$. Moreover we can assume that in the cellular subdivision of $N(F_r)$, any 1-cell is in $T(F_r)$. Hence, $A \cup M_{n-1}$ is still simply connected if $n = 2$. Thus the conditions of the Spanier-Whitehead's theorem about compression are satisfied. Namely, f_n is compressed into $A \cup M_{n-1}$ for $n \geqq 2$.

Proof of our theorem is given except for $n = 1$. In this case, however, any closed path is deformed into a point outside of F_r. Thus the the rem is completely proved.

THEOREM 4.2. *Let f be a differentiable mapping of V of dimension $\leqq n$ into a simply connected manifold M_{n+r}, and let F_s be a subvariety in M_{n+r} which has the singular locus F_r. If $f^*DF_r = 0$, then f^*DF_s is realized by a non-singular submanifold induced from F_s by f.*

PROOF. If the dimension of V is less than n, our theorem is obvious. It is sufficient to consider the case where the dimension of V is exactly equal to n. From theorem 4.1 and the property (1) in the beginning of this section, there is a differentiable mapping which is homotopic to f, transversally regular with respect to F_s and has no image points in F_r. The inverse image by this mapping is the required manifold.

COROLLARY 4.1. *If $n < 2(i+1)$ then $W_i(\mathfrak{E}^m)$ can be realized by an induced Schubert submanifold.*[5]

PROOF. Put $M_{mN} = G_{m,N}$, $DF_s = (0, \dots, 0, \underbrace{1, \dots, 1}_{i})$ and $DF_r = (0, \dots, 0, \underbrace{2, \dots, 2}_{i+1})$, where ( ) means a Schubert cochain. f^*DF_r is of dimension $2(i+1)$. Hence it is obviously 0. From theorem 4.2, we get the result.

COROLLARY 4.2. *$W_i(\mathfrak{E}^m(V_{2(i+1)}))$ is realized by an induced Schubert submanifold, if $f^*(0, \dots 0, 2, \dots, 2) = 0$ in integer coefficients.*

COROLLARY 4.3. *$W_{2k+1}(\mathfrak{E}^m(V_{4(k+1)}))$ is realized by an induced Schubert submaifold, if $P_{4(k+1)}(\mathfrak{E}^m(V_{4(k+1)})) = 0$ in integer coefficients.*

CHAPTER II

REALIZATION OF THE STIEFEL-WHITNEY CLASSES BY THE CONSTRUCTION OF MAPPINGS

5. A Necessary Condition.

Let $\mathfrak{E}$ be an m-vector bundle over a compact differentiable manifold V_n. We denote by $W_i(\mathfrak{E})$ the i-th Stiefel-Whitney characteristic class.

THEOREM 5.1. *If $W_i(\mathfrak{E})$ is realizable by a submanifold in V_n, then the class $(W_{2i+1}(\mathfrak{E}))^2$ belongs to the ideal generated by $W_{2i}(\mathfrak{E})$.*

PROOF. From the result stated in section 1, we can see that there is a mapping f of V_n into $M(O(2i))$ such that

$$f^*U_{2i} = W_{2i}(\mathfrak{E}), \tag{5.1}$$

where U_{2i} denotes the fundamental class of $M(O(2i))$. It also follows that

$$Sq^1U_{2i} = \varphi^*_{G,2i}\, W_1. \tag{5.2}$$

Using (5.1), we get

$$\begin{aligned}\varphi^*_{\mathfrak{E}} W_1(\mathfrak{E}) &= f^*\varphi^*_{G,2i} W_1 \\ &= f^*\, Sq^1\, U_{2i} \\ &= Sq^1 f^* U_{2i} \\ &= Sq^1 W_{2i}(\mathfrak{E}).\end{aligned}$$

Taking square in the sense of cup-product, we obtain

$$\begin{aligned}(Sq^1 W_{2i}(\mathfrak{E}))^2 &= f^*(\varphi^*_{G,2i} W_1)^2 \\ &= f^*(U^2_{2i}\, W^2_1) \\ &= f^*(U_{2i})\, f^*(U_{2i}(W_1))^2\end{aligned}$$

5) If singularities of an induced Schubert variety vanish, we call it an *induced Schubert manifold*.

$$= W_{2i}(\mathfrak{S}) f^*(\varphi^*_{G,2i} W_1)^2$$
$$= W_{2i}(\mathfrak{S}) \varphi^*_{\mathfrak{S}}(W_1(\mathfrak{S}))^2. \tag{5.3}$$

Using Wu's formula [9], we get

$$(Sq^1 W_{2i}(\mathfrak{S}))^2 = (W_1(\mathfrak{S}) W_{2i}(\mathfrak{S}))^2 + (W_{2i+1}(\mathfrak{S}))^2 \tag{5.4}$$

(5.3) and (5.4) prove our theorem.

The condition of theorem 5.1 is necessarily satisfied if $n \leqq 2(i+1)$. For $n < 2(i+1)$, it is obvious that $W_{2(i+1)}(\mathfrak{S}) = 0$. For $n = 2(i+1)$, the Poincaré-Veblen's duality shows the decomposition. The simplest example is the case of $W_3(\mathfrak{S})$. If $W_2(\mathfrak{S})$ is realizable by a submanifold in V_n, then we have $(W_3(\mathfrak{S}))^2 = W_2(\mathfrak{S})\{X\}$, which holds necessarily if $n \leqq 6$.

THEOREM 5.2. *If $W_{2i+1}(\mathfrak{S})$ is realizable by a submanifold, then $(W_1(\mathfrak{S}) W_{2i+2}(\mathfrak{S}))^2$ belongs to the ideal generated by $W_{2i+1}(\mathfrak{S})$ if i is even and $(Sq^1 W_{2i+1}(\mathfrak{S}))^2$ belongs to the ideal if i is odd.*

PROOF. From the same argument as (5.3), it follows that

$$(Sq^2(W_{2i+1}(\mathfrak{S}))^2 = W_{2i+1}(\mathfrak{S}) \varphi^*_{\mathfrak{S}}(W_2(\mathfrak{S}))^2. \tag{5.5}$$

Using Wu's formula, we obtain

$$Sq^2 W_{2i+1}(\mathfrak{S}) = W_2(\mathfrak{S}) W_{2i+1}(\mathfrak{S}) + \begin{Bmatrix} 2i-1 \\ 1 \end{Bmatrix} W_1(\mathfrak{S}) W_{2i+2}(\mathfrak{S})$$
$$+ \begin{Bmatrix} 2i \\ 2 \end{Bmatrix} W_{2i+3}(\mathfrak{S}). \tag{5.6}$$

If $i = 2k$, then we have

$$\begin{Bmatrix} 2i-1 \\ 1 \end{Bmatrix} = 1 \mod 2,$$
$$\begin{Bmatrix} 2i \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4k \\ 2 \end{Bmatrix} = 0 \mod 2.$$

Substituting these values in (5.6), we get

$$Sq^2 W_{2i+1}(\mathfrak{S}) = W_2(\mathfrak{S}) W_{2i+1}(\mathfrak{S}) + W_1(\mathfrak{S}) W_{2i+2}(\mathfrak{S}).$$

From (5.5) it follows that

$$(W_1(\mathfrak{S}) W_{2i+2}(\mathfrak{S}))^2 = 0 \quad (W_{2i+1}(\mathfrak{S})).$$

If $i = 2k+1$, then we have

$$\begin{Bmatrix} 2i \\ 2 \end{Bmatrix} = \begin{Bmatrix} 2(2k+1) \\ 2 \end{Bmatrix} = 1 \quad \mod 2,$$

hence, we obtain

$$Sq^2 W_{2i+1}(\mathfrak{S}) = W_2(\mathfrak{S}) W_{2i+1}(\mathfrak{S}) + W_1(\mathfrak{S}) W_{2i+2}(\mathfrak{S}) + W_{2i+3}(\mathfrak{S})$$
$$= W_2(\mathfrak{S}) W_{2i+1}(\mathfrak{S}) + Sq^1 W_{2i+2}(\mathfrak{S}).$$

From (5.5) it follows that

$$(Sq^1 W_{2i+2}(\mathfrak{S}))^2 = 0 \quad (W_{2i+1}(\mathfrak{S})).$$

REMARK. (1) Theoremes 5.1 and 5.2 hold not only for the Stiefel-Whitney classes but also for any classes which satisfy the squaring formula by Wu:

$$Sq^r W_i = \sum_t \left\{ \begin{matrix} i-r+t-1 \\ t \end{matrix} \right\} W_{r-t} W_{i+t}.$$

(2) For large i, Theorem 5.1 takes a little more detailed form; if $W_{2i}(\mathfrak{E})$ is realizable, then it follows that

$$(W_{2i+k}(\mathfrak{E}))^2 = 0 \qquad (W_{2i+1}(\mathfrak{E})),$$

for $1 \leqq k \leqq 2i-1$.

We will give an example of a tangent bundle with a non-realizable class W_2. This is the tangent bundle of the manifold $P = P(2) \times P(4) \times P(5)$. It is easily seen that

$$W(P(2)) = 1 + h_1 + h_1^2,$$
$$W(P(4)) = 1 + h_2 + h_2^4,$$
$$W(P(5)) = 1 + h_3^2 + h_3^4.$$

Then the Stiefel-Whitney classes of the above product manifold are given by

$$W_2(P) = h_1^2 + h_3^2 + h_1h_2,$$
$$W_3(P) = h_3^2h_1 + h_1^2h_2 + h_3^2h_2,$$

consequently

$$(W_3(P))^2 = h_3^4h_1^2 + h_1^4h_2^2 + h_3^4h_2^2$$
$$= h_3^4h_1^2 + h_3^4h_2^2.$$

On the other side, any classes of $H^4(P; Z_2)$ are sums of the following elements;

$$h_1^2h_2^2,\ h_1^2h_2h_3,\ h_1^2h_3^2,$$
$$h_1h_2^3,\ h_1h_2^2h_3,\ h_1h_2h_3^2,\ h_1h_3^3,$$
$$h_2^4,\ h_2^3h_3,\ h_2^2h_3^2,\ h_2h_3^3,\ h_3^4.$$

$H^*(P; Z_2)$ is a free commutative ring over Z_2 generated by h_1, h_2 and h_3 with relations $h_1^3 = h_2^5 = h_3^6 = 0$. Possible forms of the right side in the equation,

$$(W_3(P))^2 = W_2(P)\ \{X\} \quad \text{mod } 2 \tag{5.7}$$

are sums of the following elements;

$$u_1 = h_1^2h_2^2h_3^2,$$
$$u_2 = h_1^2h_2h_3^3,$$
$$u_3 = h_1^2h_3^4,$$
$$v_1 = h_1^2h_2^4 + h_1h_2^3h_3^2,$$
$$v_2 = h_1^2h_2^3h_3 + h_1h_2^2h_3^3,$$
$$v_3 = h_1^2h_2^2h_3^2 + h_1h_2^2h_3^4,$$
$$v_4 = h_1^2h_2h_3^3 + h_1h_3^5,$$
$$w_1 = h_1^2h_2^4 + h_2^4h_3^2,$$
$$w_2 = h_1^2h_2^3h_3 + h_1h_2^4h_3 + h_2^3h_3^3,$$

$$w_3 = h_1^2h_2^2h_3^2 + h_1h_2^3h_3^2 + h_2^2h_3^4,$$
$$w_4 = h_1^2h_2h_3^3 + h_1h_2^2h_3^3 + h_2h_3^5,$$
$$w_5 = h_1^2h_3^4 + h_1h_2h_3^4.$$

It is easily shown that

$$h_1^2h_3^4 = w_5 + v_3 + u_1,$$

namely

$$h_1^2h_3^4 \equiv 0 \quad \mathrm{mod}\ (u_1,\ u_2,\ u_3,\ v_1,\ v_2,\ v_3,\ v_4,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5) = M$$

and

$$h_2^2h_3^4 = w_3 + u_1 + v_1 + h_1^2h_2^4$$
$$= w_3 + u_1 + v_1 + w_1 + h_2^4h_3^2,$$

namely

$$h_2^2h_3^4 \equiv h_1^2h_2^4 \equiv h_2^4h_3^2.$$

Since $h_2^4h_3^2$ appears only in w_1, it does not belong to M. Consequently, (5.7) has no solution, that is to say, $(W_3(P))^2 \neq 0$ $(W_2(P))$, which implies that $W_2(P)$ can not be realized by a submanifold.

6. On the Spaces $K(Z_2, 2;\ Z,\ 4;\ \mathbf{k}^5)$ and $M(O(2))$.

Now we shall consider the condition of Theorem 5.1 for W_2. Our theorem can be stated as follows; *if* $W_2(\mathfrak{S})$ *is realizable then*

$$(W_3(\mathfrak{S}))^2 = 0 \qquad (W_2(\mathfrak{S})),$$

that is to say, we can find a cohomology class $\{X\}$ mod 2 of dimension 4, satisfying

$$(W_3(\mathfrak{S}))^2 = W_2(\mathfrak{S})\ \{X\} \quad \mathrm{mod}\ 2. \tag{6.1}$$

Suppose the base space of $\mathfrak{S}$ is a manifold of dimension 6. It is known that $K(Z_2, 2)$ and $M(O(2))$ are of same 4 type. Let f be the canonical mapping of $M(O(2))$ to $K(Z_2, 2)$. When we extend the homotopy inverse $\bar{f}$ of f from 4-skeleton to 5-skeleton, obstruction is given by the Eilenberg-MacLane invariant which is an element of $H^5(Z_2, 2;\ \pi_4(M(O(2))))$. Let ι be the fundamental cocycle of $K(Z_2, 2)$. The invariant generates the kernel of the homomorphism f^* of $H^5(Z_2, 2; Z)$ to $H^5(M(O(2)); Z)$. Here we notice that $\pi_4(M(O(2))) = Z$. $H^5(Z_2, 2; Z)$ is a cyclic group of order 4 generated by $(1/4)\delta\mathbf{p}(\iota)$ where $\mathbf{p}$ is the Pontrjagin square and $H^5(M(O(2)); Z)$ is a cyclic group of order 2. The kernel of f^* is generated by $(1/2)\delta\mathbf{p}(\iota)$ which is exactly the invariant (see Thom [2] and Eilenberg-Mac Lane [10]).

We construct a mapping h of V_6 to $K(Z_2, 2)$ such that

$$h^*\iota = W_2(\mathfrak{S}).\ ^{6)}$$

---

6) Let u be a 2-cell of $K(Z_2, 2)$ which gives the fundamental cycle. Define $h\,|\,(V_6)_1$ as a constant mapping. Extend h over $(V_6)_2$ in such a way that each 2-simplex σ_2 of V_6 goes to u in a degree which is equal to $W_2(\mathfrak{S})(\sigma_2)$ mod 2. Since we have $\delta W_2(\mathfrak{S}) = 0$, h can be extended over $(V_6)_3$, namely over V_6.

Since $K(Z_2, 2) = PC(\infty)$ has a simplicial subdivision, we can assume that h is simplicial. According to Eilenberg-Mac Lane's paper [10], the obstruction to extend the mapping $\bar{f}h$ of the 4-skeleton $(V_6)_4$ of V_6 to $M(O(2))$ over the 5-skeleton $(V_6)_5$ is given by

$$f^*((1/2)\delta\mathbf{p}(\iota)) = (1/2)\delta\mathbf{p}(f^*\iota)$$
$$= (1/2)\delta\mathbf{p}(W_2). \tag{6.2}$$

From Wu's paper [11], we have

$$\mathbf{p}(W_2) = (P_4)_4 + \theta_2(W_1{}^2W_2) \qquad \text{mod } 4,$$

where $(P_4)_4$ is the Pontrjagin class P_4 reduced mod 4 and θ_2 is a natural homomorphism of Z_2 to Z_4 defined by the exact sequence,

$$0 \to Z_2 \xrightarrow{\theta_2} Z_4 \to Z_2 \to 0.$$

Let p_4, w_1 and w_2 be representative cocycles of P_4, W_1 and W_2 respectively. It follows that

$$(1/2)\delta\mathbf{p}(W_2) = (1/2(\delta(p_4 + 2(w_1{}^2w_2))$$
$$= \{\delta(w_1{}^2w_2)\}$$
$$= 0. \tag{6.3}$$

We denote by $K = K(Z_2, 2\,;\, Z, 4\,;\, \mathbf{k}^5)$ a space with $\pi_2(K) \simeq Z_2$, $\pi_4(K) \simeq Z$, $\pi_i(K) = 0$ for $i \neq 2$, 4 and with the Eilenberg-Mac Lane invariant $\mathbf{k}^5$. In particular, killing homotopy groups of dimension $i \geqq 5$, we can get the cell complex $K(Z_2, 2\,;\, Z, 4\,;\, \mathbf{k}^5(M(O(2))))$, which is regarded as the second step of the Postnikov system of $M(O(2))$.

Now suppose $\mathbf{k}^5 = \mathbf{k}^5(M(O(2))) = (1/2)\delta\mathbf{p}(\iota)$. Because of (6.3), we obtain. the following diagram of mappings;

$$(V_6)_5 \xrightarrow{g_1} K \underset{5}{\cong} M(O(2)), \quad \text{with } \bar{g}: K \to M(O(2)),\ g: M(O(2)) \to K \tag{6.4}$$

where the notation $\underset{5}{\cong}$ means that spaces of both sides are of same 5 type.

LEMMA 6.1 *The following relation holds*:

$$H^*(K\,;\, Z_2) \approx H^*(Z_2, 2\,;\, Z_2) \otimes H^*(Z, 4\,;\, Z_2). \tag{6.5}$$

PROOF. According to the theory of Eilenberg-Mac Lane complexes, $H^*(Z_2, 2\,;Z_2)$ and $H^*(Z, 5\,;\, Z_2)$ are generated by cohomology operations of their fundamental cocycles ι and ν respectively (see [13, Exp. 16]). We have the exact cohomology sequence of $(K, K(Z, 4))$ with coefficient group Z_2:

$$\to H^i(K, K(Z, 4)\,;\, Z_2) \xrightarrow{j^*} H^i(K\,;\, Z_2) \xrightarrow{i^*} H^i(Z, 4\,;\, Z_2) \xrightarrow{\delta^*} H^{i+1}(K, K(Z, 4)\,;\, Z_2) \to. \tag{6,6}$$

Let p^* be the homomorphism of $H^*(Z_2, 2\,;\, Z_2)$ to $H^*(K, K(Z, 4)\,;\, Z_2)$ induced by the projection $p : (K, K(Z, 4)) \to (K(Z_2, 2), 0)$. From the definition of $\mathbf{k}$-invariant, we have $p^*\mathbf{k}^5 = -\delta^*\nu$. By our assumption, we get $\mathbf{k}^5 = (1/2)\delta\mathbf{p}(\iota) = 2(1/4)\delta\mathbf{p}(\iota) = 0 \bmod 2$. Hence we have $\delta^*\nu = 0$. Because of the exactness of (6.6), there is a class $\bar{\nu} \in H^4(K; Z_2)$ such that $i^*\bar{\nu} = \nu$. Since the inclusion map i commutes with any cohomology operations, i^* is a homomorphism onto. (6.6) causes the following exact sequence;

$$0 \to H^i(K, K(Z,4); Z_2) \xrightarrow{j^*} H^i(K; Z_2) \xrightarrow{i^*} H^i(Z,4; Z_2) \to 0. \qquad (6.7)$$

Consequently, i^* is onto, that is to say, the fiber $K(Z,4)$ is *totally non-homologous to zero with respect to* Z_2. It is obvious that $H^i(Z_2, 2; Z_2)$ is of finite dimension over Z_2 for all $i \geqq 0$.

Define a homomorphism q^* of $H^*(Z,4; Z_2)$ into $H^*(K; Z_2)$ in such a way that $q^*\nu = \bar{\nu}$ which induces the homomorphism whole over $H^*(Z,4; Z_2)$, taking corresponding cohomology operations of ν and $\bar{\nu}$ respectively. Obviously we have $i^*q^* = 1$. Hence our lemma is a direct consequence of Chap. III, Prop. 8 by Serre [13].

LEMMA 6.2. *There is* $\bar{\nu}$ *of Lemma* 6.1 *such that the second* **k**-*invariant of* $M(O(2))$ *is given by*

$$\mathbf{k}^6(M(O(2))) = (Sq^1\bar{\iota})^2 + \bar{\iota}\,\bar{\nu} \quad mod\ 2, \qquad (6.8)$$

where $\bar{\iota} = p^*\iota$.

PROOF. $\bar{\iota}$ is a generator of $H^2(K; Z_2)$. In the diagram (6.4), we have

$$g^*\bar{\iota} = U_{O(2)}.$$

It is known that $H^4(M(O(2)); Z_2) = Z_2(U^2_{O(2)}, U_{O(2)}(W_1)^2)$. From Lemma 6.1, we have $H^4(K; Z_2) = Z_2((\bar{\iota})^2, \bar{\nu})$. Since g^* is an isomorphism of $H^i(K; Z_2)$ to $H^i(M(O(2)); Z_2)$ for all $i \leqq 4$, including cup-products, there is a unique $\bar{\nu}$ satisfying the relation,

$$g^*\bar{\nu} = \varphi^*_{O(2)}(W_1)^2 = U_{O(2)}(W_1)^2.$$

Obviously we have

$$g^*(Sq^1\bar{\iota})^2 = (Sq^1U_{O(2)})^2.$$

It is known that $\mathbf{k}^6$ generates the kernel of g^*, i.e., $\{\mathbf{k}^6\} = g^{*-1}(0)$. We can see from Lemma 6.1 that $H^6(K; Z_2) = Z_2((\bar{\iota})^3, (Sq^1\bar{\iota})^2, \bar{\iota}\,\bar{\nu}, Sq^2\bar{\nu})$. It follows that

$$g^*(\bar{\iota})^3 = (W_2)^3 \neq 0,$$
$$g^*(Sq^1\bar{\iota})^2 = U^2_{O(2)}W_1^2 \neq 0,$$
$$g^*(\bar{\iota}\,\bar{\nu}) = U^2_{O(2)}W_1^2,$$
$$g^*(Sq^2\bar{\nu}) = Sq^2(U_{O(2)}W_1^2) = U^2_{O(2)}W_1^2 + U_{O(2)}W_1^4. \neq 0.$$

Therefore only possible generator of $g^{*-1}(0)$ is given by

$$(Sq^1\bar{\iota})^2 + \bar{\iota}\bar{v}.$$

We denote again by K the mapping cylinder of g in (6.4). K contains $M(O(2))$ which is denoted simply by M, here. M is closed in K. From the exact cohomology sequence, it follows that

$$H^r(K, M; Z_2) = 0 \text{ for } r < 6, \quad H^6(K, M; Z_2) = Z_2,$$

$$H^r(K, M; Z_p) = 0 \text{ for } r \leqq 6 \text{ and any odd prime } p.$$

Using the duality with respect to Z_2 and Z_p, we have

$$H_r(K, M; Z_2) = 0 \text{ for } r < 6, \quad H_6(K, M; Z_2) = Z_2,$$

$$H_r(K, M; Z_p) = 0 \text{ for } r \leqq 6 \text{ and any odd prime } p.$$

From the universal coefficient formula, we get

$$H_r(K, M; Z) = 0 \text{ for } r < 6, \quad H_6(K, M; Z) = Z_2.$$

Because of the relative Hurewicz theorem, we obtain

$$\pi_6(K, M) = Z_2,$$

hence

$$\pi_5(M) = Z_2.$$

Thus (6.8) is exactly the second invariant of $M(O(2))$.

Define in the singular complex $S(K)$ the following relation of equivalence (ρ): Two simplices $\sigma^q_{f_1}$ and $\sigma^q_{f_2}$ $(q \geqq 4)$ in $S(K)$ are equivalent by ρ if and only if the mappings f_1 and f_2 of the standard q-simplex Δ^q in K coincide up to the 3-skeleton of Δ^q. It is then well known (definition of the Postnikov system) that the quotient complex $S(K)/\rho$ can be identified (up to homotopy) to $S(K(Z_2, 2))$.

Let w be a 4-dimensional cochain in $S(K)$ with values in Z, with the following property: If two 4-simplices $\sigma^4_{f_1}$ and $\sigma^4_{f_2}$ are ρ-equivalent, then

$$w(\sigma^4_{f_1}) - w(\sigma^4_{f_2}) = d(f_1, f_2) \in \pi_4(K) \tag{6.9}$$

It is obvious that such cochains do exist. Take arbitrarily the value of w on some representative of any ρ-class, and compute the value of w on any other simplex of the class according to (6.9). Consider now the 5-skeleton P^5 of the "base" $K(Z_2, 2)$. We can extend the map of P^5 to K already given on the 3-skeleton to the 4-skeleton (because of $\pi_3(K) = 0$). Let f_1 be such an extension. The extension of f_1 to the 5-skeleton gives rise to an obstruction cocycle $\alpha_{f_1} \in C^5(K(Z_2, 2): Z)$. This cocycle depends on the extension f_1. But I claim that the cocycle,

$$\alpha_{f_1} - \delta[f_1^*(w)]$$

does not depend on the particular extension f_1. In fact, if we replace f_1 by an other extension f_2, we have

$$\alpha_{f_1} - \alpha_{f_2} = \delta d(f_1, f_2).$$

hence, according to (6.9)

$$\alpha_{f_1} - \delta[f_1^*(w)] = \alpha_{f_2} - \delta[f_2^*(w)], \tag{6.10}$$

ehT cohomology class of this cocycle does not depend on the particular choice of the cochain w: Suppose we replace the cochain w by another cochain $\bar{w}$ satisfying also

$$\bar{w}(\sigma^4_{f_1}) - \bar{w}(\sigma^4_{f_2}) = d(f_1, f_2).$$

Then the difference $\bar{w} - w$ takes the same value of any couple of 4-simplexes $\sigma^4_{f_1}$ and $\sigma^4_{f_2}$ which are ρ-equivalent, hence $\bar{w} - w$ is a 4-dimensional cochain u in the base space $K(Z_2, 2)$. And we get

$$(\alpha_{f_1} - f_1{}^*\delta w) - (\alpha_{f_1} - f_1{}^*\delta \bar{w}) = \delta u.$$

It is readily seen that the cohomology class of $\alpha_{f_1} - f_1{}^*\delta w$ is nothing but the Eilenberg-Mac Lane invariant, as defined in [14], with the use of a minimal complex. Using such a minimal complex, and taking the lifting f_1 for which $w = 0$, we get into the original definition of the Eilenberg-Mac Lane invariant $\mathbf{k}^5$ as an obstruction.

$\alpha_{f_1} - f_1{}^*(\delta w) = k$ is a representative cocycle of the invariant $\mathbf{k}^5$. Let us compute its image p^*k in $S(K)$. On any 5-dimensional simplex σ^5 in $S(K)$, we have $< p^*k, \sigma^5 > = < k, p_*(\sigma^5) > = < \alpha_f - f^*(\delta w), p_*(\sigma^5) >$ for any lifting f. But if we take the lifting f already given by σ^5 in $S(K)$, we have $\alpha_f(\sigma^5) = 0$, hence $p^*k = -\delta w$.

The cochain w plays the role of a *transgression cochain* (it is obvious that w restricted to the fiber of P gives the fundamental cocycle). It should be observed that the Eilenberg-Mac Lane invariant $\mathbf{k}$ and the image by transgression of the fundamental cocycle are of opposite signs.

The relation $p^*k = -\delta w$ leads

$$w = (1/2)\delta \mathfrak{p}(\bar{\iota}) = 0 \mod 2. \tag{6.11}$$

Obviously, w is a cocycle mod 2. We denote by $\{w\}$ the cohomology class mod 2 determined by w. There are two possibilities in the value of $g^*\{w\}$: (A) $U_{O(2)}W_1{}^2$ or (B) $U_{O(2)}W_1{}^2 + U_{O(2)}{}^2$. In the case (A), we have $\bar{\nu} = \{w\}$. In the case (B), we replace $\{w\}$ by $\{\bar{w}\} = \{w\} + (\bar{\iota})^2$, namely $\bar{\nu} = \{w\}$.[7] In both cases, therefore, we can find a cocyle w mod 2 satisfying (6.9) and $\bar{\nu} = \{w\}$.

LEMMA 6.3. *One can choose a mapping g_2 of $(V_6)_5$ to K instead of g_1 in (6.4), satisfying the following conditions;*

$$g_2{}^*(\bar{\iota}) = W_2(\mathfrak{S}), \tag{6.12}$$

$$g_2{}^*(w) = X \text{ for given } X \in Z^4(V_6; Z). \tag{6.13}$$

PROOF. Suppose $g_1{}^*w = X_0 \neq X$. $g_1|(V_6)_4$ is a mapping induced by a mapping f_1, namely $g_1 = f_1 h$. $g_1{}^*w$ is a cocycle, since g_1 is defined over $(V_6)_5$. we can find a mapping f_2 of the 4-skeleton of $K(Z_2, 2)$ to K in such a way that its induced mapping $g_2 = f_2 h$ of $(V_6)_4$ to K satisfies the following condi-

7) Since f_1 and f_2 coincide on the 3-skeleton of $K(Z_2, 2)$, the added term $(\bar{\iota})^2$ does not affect at all the formulae (6.9), (6.10) and (6.11). The cocycle $(\bar{\iota})^2$ is zero on any 4-dimensional spherical cycle.

tions,

$$g_1|(V_6)_3 = g_2|(V_6)_3,$$
$$d(g_1, g_2) = X_0 - X.$$

(6.9) leads the relation,

$$\begin{aligned} g_1{}^*w - g_2{}^*w &= h^*(f_1{}^*w - f_2{}^*w) \\ &= h^*d(f_1, f_2) \\ &= d(f_1h, f_2h) \\ &= d(g_1, g_2) \\ &= X_0 - X, \end{aligned}$$

namely

$$\begin{aligned} g_2{}^*w &= X_0 + (X - X_0) \\ &= X. \end{aligned}$$

Taking coboundary, the obstruction cocycle to extend g_2 over $(V_6)_5$ is given by

$$\begin{aligned} Z(g_2) &= \delta(g_2{}^*w) \\ &= \delta X \\ &= 0, \end{aligned}$$

with respect to integer coefficients. Hence g_2 can be extented over $(V_6)_5$. Since g_2 is not changed on $(V_6)_3$, the relation (6.12) holds for g_2 as for g_1. From the construction of g_2, we have $g_2{}^*w = X$ which is the second relation.

7. W_2 in V_6.

THEOREM 7.1. *If there is an integral class $\{X\} \in H^4(V_6; Z)$ such that*

$$(Sq^1W_2(\mathfrak{S}))^2 = W_2(\mathfrak{S})\ \{X\} \quad mod\ 2 \tag{7.1}$$

where the right side of (7.1) *is the cup-product by the natural pairing*

$$Z_2 \otimes Z \to Z_2,$$

then $W_2(\mathfrak{S})$ is realizable by a submanifold in V_6.

PROOF. Using Lemmas 6.2 and 6.3, the obstruction class to extend the mapping $\bar{g}g_2$ of $(V_6)_5$ to $M(O(2))$ over the whole manifold V_6 is given by the following formula,

$$\begin{aligned} g_2{}^*\mathbf{k}^6(M(O(2))) &= g_2{}^*(Sq^1\bar{\iota})^2 + g_2{}^*(\bar{\iota}\bar{\nu}) \\ &= (Sq^1g_2{}^*\bar{\iota})^2 + g_2{}^*\bar{\iota}\, g_2{}^*\bar{\nu} \\ &= (Sq^1W_2(\mathfrak{S}))^2 + W_2(\mathfrak{S})\{X\} \\ &= 0. \end{aligned}$$

Therefore we can construct an extension of $\bar{g}g_2$ over V_6 which is denoted by H: $V_6 \to M(O(2))$. It follows from the Lemma 6.3 that

$$\begin{aligned} H^*U_{O(2)} &= g_2{}^*\bar{g}^*U_{O(2)} \\ &= g_2{}^*\bar{\iota} \end{aligned}$$

$$= W_2(\mathfrak{E}).$$

Thus our theorem is a direct consequence of Thom's fundamental theorem which is stated in section 1.

When V_6 is orientable, then the above theorem leads the following result:

THEOREM 7.2. *In an orientable manifold V_6 the W_2 class of any vector bundle is realizable.*

PROOF. In the orientable manifold V_6, using the formula of iterated square operations (see J. Adem [5]), we have the following relation,

$$\begin{aligned}(Sq^1W_2(\mathfrak{E}))^2 &= Sq^3(Sq^1W_2(\mathfrak{E}))\\ &= Sq^1(Sq^2Sq^1W_2(\mathfrak{E}))\\ &= W_1(V_6)(Sq^2Sq^1W_2(\mathfrak{E}))\\ &= 0,\end{aligned}$$

as $W_1(V_6)=0$ in V_6, where $W_1(V_6)$ is the W_1-class of the tangent bundle on V_6(see W. T. Wu[16]). Consequently the equation (7.1)

$$\begin{aligned}(Sq^1W_2(\mathfrak{E}))^2 &= (W_3(\mathfrak{E}))^2\\ &= W_2(\mathfrak{E})\ \{X\}\end{aligned}$$

has always a solution $X=0$.

Examples of non-orientable manifolds of dimension 6. We shall show that (7.1) holds for each generator of cobordism group mod 2 in dimension 6, which are denoted by (i), (ii) and (iii) in section 3. It is easily seen by a direct calculation that

$$(Sq^1W_2(\text{i}))^2 = 0.$$

Hence we can take $X=0$ for manifold (i), $P(6)$.

For the manifold (ii), $P(4)\times P(2)$, we have

$$\begin{aligned}W_1(\text{ii}) &= h_1+h_2,\\ W_2(\text{ii}) &= h_1h_2+h_2^2,\\ W_3(\text{ii}) &= h_1h_2^2,\\ (W_3(\text{ii}))^2 &= h_1^2h_2^4\\ &= 0,\\ (W_1(\text{ii}))^2(W_2(\text{ii}))^2 &= h_2^2(h_1+h_2)^4\\ &= h_1^4h_2^2.\end{aligned}$$

Hence we get

$$\begin{aligned}(Sq^1W_2(\text{ii}))^2 &= (W_1(\text{ii}))^2(W_2(\text{ii}))^2+(W_3(\text{ii}))^2\\ &= h_1^4h_2^2.\end{aligned}$$

Putting $X=h_1^4$ which is an integral class, we obtain (7.1).

For the manifold (iii), $P(2)\times P(2)\times P(2)$, we have

$$W_1(\text{iii}) = \sum_1^3 h_i,$$

$$W_2(\mathrm{iii}) = \sum_1^3 h_i^2 + \sum_{i\neq j} h_i h_j,$$

$$W_3(\mathrm{iii}) = \sum_{i\neq j} h_i^2 h_j + h_1 h_2 h_3,$$

$$(W_3(\mathrm{iii}))^2 = h_1^2 h_2^2 h_3^2,$$

$$W_1(\mathrm{iii})^2 W_2(\mathrm{iii})^2 = \left(\sum_1^3 h_i^2\right)\left(\sum_{i\neq j} h_i^2 h_j^2\right)$$
$$= h_1^2 h_2^2 h_3^2.$$

Consequently we get

$$(Sq^1 W_2(\mathrm{iii}))^2 = (W_1(\mathrm{iii}))^2 (W_2(\mathrm{iii}))^2 + (W_3(\mathrm{iii}))^2$$
$$= 0.$$

Hence we can also take $X = 0$.

No examples are known of a non-orientable V_6 in which W_2 is not realizable.

CHAPTER III

SPECIAL MANIFOLDS

8. Totally Realizable Manifolds.

In the following, $\mathfrak{G}$ denotes any commutative ring. In particular, Z and Z_p denote the ring of integers and the ring of integers modulo p as usual, where p is a prime number. Let M be a real compact differentiable manifold. If every homogeous class of the cohomology ring $H^*(M;\ \mathfrak{G})$ is realized by a compact submanifold which is not necessarily connected, then we say that the manifold M is *totally realizable* for the coefficient ring $\mathfrak{G}$.

For example, an n-sphere S^n and more generally a product space of spheres, $S^{n_1}\times S^{n_2}\times \dots \times S^{n_t}$ is totally realizable for Z_2 but not for $Z_p (p > 2)$ and Z. Now we consider the following manifold. Let S be a sphere of dimension $\geqq 2$ and let q be the symmetric transformation of S with respect to a hyperplane which determines its equater. Then q is a homeomorphism of S onto itself changing its orientation. We denote by x a point of S and denote by I the unit interval. In a product space $S \times I$, we identify a point $(x, 0)$ with $(q(x), 1)$ and denote by L the manifold which is thus obtained. It is easily seen that L is a sphere bundle over a circle S^1 with $G = \{1, q\}$ as its structural group. We denote by s and s^1 the fundamental classes of $H^*(S;\ Z_2)$ and $H^*(S^1;\ Z_2)$ respectively. Let i be an inclusion map of S into L and let π be the projection of L onto S^1. Then we have an isomorphism,

$$H^*(L;\ Z_2) \approx H^*(S;\ Z_2) \otimes H^*(S^1;\ Z_2)$$

by a similar argument to Lemma 6.1, Chap. II. Non-trivial elements of $H^*(L,\ Z_2)$ are;

$$1,\ \pi^*(s^1),\ (i^*)^{-1}s,\ l = (i^*)^{-1}s \cup \pi^*(s^1).$$

where l is the findamental cocyle of L. They are all realizable by compact connected manifolds: L, S_x (a fiber over $x \in S_1$), $y \times S_1$ (y is a point on the equator of S) and a point of L, respectively.

A real projective space $P(n)$ of dimension n is totally realizable for Z_2, because a non-trivial cohomology class of each dimension is realized by a linear subspace of corresponding codimension. In a similar way, we can see that a complex projective space $PC(n)$ of complex dimension n is also totally realizable for Z_2.

9. Complete Intersections of Hypersurfaces.

Let M and V be real compact differentiable manifolds of dimension $n+r$ and n respectively. Suppose V is imbedded in M by a differentiable map. A set of all normal vectors on V in M makes a fiber bundle over V with an r-dimensional vector space as fiber. It is well known that this fiber bundle is isomorphic to an open tubular neighborhood $N(V)$ of V which consists of all normal geodesics with sufficiently small distance from V. We shall call a submanifold of codimension 1 in M a *non-singular hypersurface* in some generalized sense of that of the projective space. We consider r non-singular hypersurfaces $H_1, H_2, \ldots, H_r$ in M which are in *general position*, that is to say, each point x of $H_1 \cap H_2 \cap \ldots \cap H_r$ in M has a neighborhood U in which local coordinates $x_1, \ldots, x_{n+r}$ with x as its center are defined and $U \cap H_i$ is given by $x_{n+i} = 0$. Suppose V_n is such an intersection and we call it a *complete intersection* [8] of non-singular hypersurfaces $H_1, H_2, \ldots, H_r$.

THEOREM 9.1. *The Stiefel-Whitney characteristic classes of normal bundles over a complete intersection V_n of non-singular hypersurfaces in M are all realizable by submanifolds.*

PROOF. The normal bundle decomposes into r real line bundles, each of which is induced by the corresponding hypersurface $H_i (1 \leqq i \leqq r)$. Its characteristic class h_i is the restriction of the cohomology class $\bar{h}_i$ which is dual to the homolcgy class of H_i. The total Stiefel-Whitney class of the normal bundle N is given by the following formula,

$$W(N) = \left[\prod_1^r (1 + \bar{h}_i)\right]_{V_n}$$

$$= \prod_1^r (1 + h_i),$$

$$W_k(N) = \left(\sum_{i_1, \cdots, i_k} \bar{h}_{i_1} \ldots \bar{h}_{i_k}\right)_{V_n}$$

8) This definition is essentially due to F. Hirzebruch, Proc. Int. Congress of Math. (Amsterdam) Vol. III(1954), pp. 457-473.

$$= \sum_{i_1,\dots,i_k} h_{i_1}\dots h_{i_k}. \tag{9.1}$$

Let D be the duality operator of Poincaré-Veblen. We have

$$\begin{aligned} D(h_i) &= \bar{h}_i \cap V_n \\ &= DH_i \cap V_n \\ &= H_i \circ V_n. \end{aligned} \tag{9.2}$$

Therefore each h_i is realizable by submanifold in V_n. We have already known in section 1 that a product of realizable classes is also realizable. On the other side $\sum_{i_1,\dots,i_k} \bar{h}_{i_1}\dots\bar{h}_{i_k}$ is a class in the totally realizable manifold M. Hence, it is realized by a submanifold $\mathbf{H}_k$ in M. By the same way as (9.2), $W_k(N) = \sum_{(i_1,\dots,i_k)} (\bar{h}_{i_1}\dots\bar{h}_{i_k})_{V_n}$ is realizable in V_n.

Consider the restriction of the tangent bundle of M over the submanifold V_n. Since $W_i(M)$ are all realizable in M, it the follows that the Stiefel-Whitney classes $(W_i(M))_{V_n}$ are realizable for $0 \leqq i \leqq n$, The restriction of the tangent bundle of M over V_n is a Whitney sum of the tangent bundle of V_n and the normal bundle over V_n in M. Thus, using the Whitney duality, we get the following relation,

$$\Sigma_{\alpha+\beta=i} W_\alpha(N) W_\beta(V_n) = W_i(M)_{V_n}, \tag{9.3}$$

for $0 \leqq \alpha \leqq r$, $0 \leqq \beta \leqq n$ and $0 \leqq i \leqq n+r$.

LEMMA 9.1. *Let V be a submanifold in a totally realizable manifold M. If one of two fiber bundles, the normal bundle or the tangent bundle over V_n has Stiefel-Whitney classes which are all restrictions of some classes in M, then so does the other.*

PROOF. Characteristic classes of one fiber bundle in (9.3) can be solved with respect to those of the other. Suppose $W_\alpha(N)$ are all restrictions of classes $\bar{W}_\alpha$ in M. Then $W_\beta(V_n)$ are all polynomials in $W_\alpha(N)$ and $W_i(M)_{V_n}$. Since $\bar{W}_\alpha$ and $W_i(M)$ can be realized by submanifolds in the totally realizable manifold M, polynomials in them are also realizable in M. On the other side, polynomials of $W_\beta(V_n)$ in $W_\alpha(N)$ and $W_i(N)_{V_n}$ are restrictions of corresponding polynomials in $\bar{W}_\alpha$ and $W_i(M)$. Hence $W_\beta(V_n)$ are also realizable.

THEOREM 9.2. *A complete intersection V_n of non-singular hypersurfaces in a totally realizable M for Z_2 has the tangent bundle whose Stiefel-Whitney classes are all realizable.*

PROOF. It is seen in the proof of theorem 9.1 that $W_\alpha(N)$ are all restrictions of classes in M. Our theorem is a direct consequence of lemma 9.1.

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