ON EUCLIDEAN CONNECTIONS IN A FINSLER MANIFOLD

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Introduction. In a general Finsler space S. S. Chern [2] defined an infinite number of Euclidean connections which include the connection defined by E. Cartan and others as a particular cases. The purpose of this note is to discuss in a modern view-point Euclidean connections in a Finsler manifold and to give a geometrical interpretation to the connections defined by S. S. Chern. Throughout the whole discussion the following conventions are adopted: By differentiability we understand always that of class C^∞, and differential forms of degree 1 of class C^∞ are all called as 1-forms. Given a 1-form over a manifold M by ω, we denote the restriction of it at a point x of M by ω_x. We denote ω_x by ω, too for brevity if it is not ambiguous from the context. For vector field and so on this convention is also applied.

Let us assume that Latin indices b, c, d, h, i, j, k, l run from 1 to n and Greek indices α, β, γ, δ from 1 to $n-1$.

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1. Preliminaries. Let M be an n-dimensional connected differentiable manifold. Let $T(M)$ be the tangent bundle over M so that each point of the bundle space T is represented by a pair (x, y) of a point $x \in M$ and a tangent vector y at x. Let ρ be the canonical projection $T \to M$. For a coordinate neighbourhood V of M we can endow to each point (x, y) of $\rho^{-1}(V)$ with coordinates x^i, y^i where x^i are the coordinates of x and y^i are the components of y there. Such coordinates are called *canonical local coordinates*, briefly *C-coordinates*, in T (induced from V). In T we denote by T^0 the open submanifold $T - N$ where N is the set of all zero-vectors of M. If (x_1, y_1) and (x_2, y_2) are two points of T^0 such that $x_1 = x_2$ and $y_1 = \lambda y_2$ for $\lambda > 0$, we shall say that these two points are equivalent. We denote the quotient space of T^0 by the equivalence relation by Q and its point, say a coset containing (x, y), by $(x, \lambda y)$. Q is also regarded as the bundle space of the tangent sphere bundle $Q(M)$ over M and let q be the canonical projection $Q \to M$ of the bundle. Given a coordinate neighbourhood V of M, each point $(x, \lambda y)$ of $q^{-1}(V)$ can be endowed as its coodinates with x^i, λy^i, where x^i, y^i are the C-coordinates of (x, y) induced from V and λy^i mean homogeneous coordinates up to positive number. Such coordinates are also called *C-coordinates* in Q. Next, let us attach at every point $(x, \lambda y)$ of Q the tangent vector space M_x of M at $x \in M$ and denote the attached space by $M(x, \lambda y)$. (Note that the space $M(x, \lambda y)$ is not related to the tangent space of Q at $(x. \lambda y)$.) Let

$P(x, \lambda y)$ be the set of all n-frames in $M(x, \lambda y)$ and put $P = \bigcup_{(x,\lambda y)\in Q} P(x, \lambda y)$. Let p be the mapping $P \rightarrow Q$ which maps all elements of $P(x, \lambda y)$ to $(x, \lambda y)$. A point $z \in P$ can be thereby represented by $(x, \lambda y, X_1, \ldots\ldots, X_n)$ where $(x, \lambda y) = p(z)$ and $(X_1, \ldots, X_n)$ is an n-frame in $M(x, \lambda y)$. Moreover, given a coordinate neighborhood V of M, we can endow each point $(x, \lambda y, X_1, \ldots,$ $X_n)$ of $(qp)^{-1}(V)$ with coordinates x^i, λy^i, $X_1^i, \ldots, X_n^i$ where x^i, λy^i are the C-coordinates of $(x, \lambda y)$ induced from V and X_j are the components of a vactor X_j in V. Such coordinates are also called *C-coordinates* in P. P is also regarded as the bundle space of a principal fibre bundle with base space Q, standard fibre $GL(n)$ and canonical projection p. We denote this bundle by $P(Q)$. In the spaces T, Q and P we admit their C-coordinate systems as allowable ones and we shall treat them as differentiable manifolds.

Next, let us suppose that M has a *Finsler metric*, that is a continuous function $L(x, y)$ defined on T, which satisfies the following conditions:

1) $L(x, ty) = |t| L(x,y)$ for any real number t.

2) $L(x, y)$ is differentiable and positive on T^0.

Let V be any coordinate neighbourhood of M. In $\rho^{-1}(V) \cap T^0$ with the C-coordinate system (x^i, y^i) we put

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2(x, y)}{\partial y^i \partial y^j}$$

Then $g_{ij} = g_{ij}(x, y)$ are homogeneous functions of degree 0 with respect to y^i and hence are also regarded as functions in $q^{-1}(V)$. It is well known that all of them determine on M a covariant tensor field of degree 2 which is called the fundamental tensor field of M. If the matrix (g_{ij}) is positive definite everywhere, we say that the Finsler metric is *positive definite*. We shall impose this condition upon our Finsler metric from now on. On the other hand, since we treat $L(x, y)$ as the length of the vector y the lengths of curves (of class D^1) are naturally defined, and by geodesics we mean the curves which satisfy Euler's differential equations, as is well-known.

REMARK 1. *In a connected differentiable manifold, if the second countability axiom holds it always admits a positive definite non-Riemannian Finsler metric, and the converse is also true* (see Appendix).

As M has a positive definite Finsler metric we can moreover construct a subbundle of $P(Q)$ as follows: Let V be any coordinate neighbourhood of M with coordinate system (x^i) and $(x, \lambda y)$ be any point of $q^{-1}(V)$. In the tangent vector space M_x at $x \in V$ we define a scalar product of every two vectors $\partial/\partial x^j$, $\partial/\partial x^k$ by $g_{jk}(x, \lambda y)$, and suppose that such M_x is attached at $(x, \lambda y)$. Hence it results that the vector spaces attached on Q are Euclidean. At any point $(x, \lambda y) \in Q$, the unit vector $y/L(x, y)$ considered as one of $M(x, \lambda y)$ is called the *supporting vector* of $M(x, \lambda y)$ and let P^0 be the set of all orthonormal frames $(X_1, \ldots, X_n)$ in all $M(x, \lambda y)$ such that their n-th vectors X_n consist of supporting vectors. Then it follows that P^0 has subbundle structure of $P(Q)$. We denote the bundle by $P^0(Q)$ and the canonical projection $P^0 \to Q$ by $p^0(=p|P^0)$. We denote, in general, any point $(x, \lambda y, X_1, \ldots, X_n)$ of P^0 and its C-coordinates by $(x, \lambda y, u_1, \ldots, u_n)$ and $x^i, \lambda y^i, u_1^i, \ldots, u_n^i$, using u instead of X.

Now take a point $z=(x, \lambda y, u_1, \ldots, u_n)$ of P^0 and let $x^i, \lambda y^i, u_1^i, \ldots, u_n^i$, be its C-coordinates. Then it is clear that

$$u_n^i = y^i/L(x, y), \quad g_{ij} u_\alpha^i u_\beta^j = \delta_{\alpha\beta}, \quad g_{ij} u_\alpha^i y^j = 0.$$

Moreover let Z be any tangent vector of P^0 at z. If we project Z by qp^0, we get a tangent vector $qp^0(Z)$ of M at x and we denote by $\omega^i(Z)$ its components with respect to the frame (u_i) of M_x, i.e. the frame (u_i) of $M(x, \lambda y)$ regarded as that of M_x (such a note is omitted from now on). Then it is easily verified that ω^i are expressed by 1-forms $\omega^i = v_j^i\, dx^j$ on P^0 in terms of C-coordinates where we have put $v_i^j u_k^i = \delta_k^j$. Note that $v_i^n = \partial L/\partial y^i$.

LEMMA 1. 1) *There exists only one set of 1-forms* ω_i^j *and differentiable functions* H_{ij}^k $(H_{nj}^k = H_{in}^k = H_{ij}^n = 0)$ *on* P^0 *which satisfy* $d\omega^i = \omega^j \wedge \omega_j^i$ *and* $\omega_i^j + \omega_j^i = H_{ij}^k \omega_k^n$. 2) *1-forms* ω^i, $\omega_\alpha^\beta (\alpha < \beta)$, ω_α^n *are linearly independent*. (See[1])

PROOF. Let $\lambda_{\alpha\beta}$, $\mu_{\alpha\gamma}^\beta$ be unknown functions on P^0 such that $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$, $\mu_{\alpha\gamma}^\beta = \mu_{\gamma\alpha}^\beta$. Under a C-coordinate system $(x^i, \lambda y^i, u_1^i, \ldots, u_n^i)$ if we put

$$\omega_\alpha^\beta = v_k^\beta\, du_\alpha^k - \left(u_\beta^j u_\alpha^k \frac{\partial^2 L}{\partial x^j \partial y^k} + \lambda_{\alpha\beta}\right)\omega^n + \mu_{\alpha\gamma}^\beta \omega^\gamma,$$

$$\omega_\alpha^n = -u_\alpha^k \frac{\partial^2 L}{\partial y^j \partial y^k}\, dy^j + \frac{1}{L} u_\alpha^j \left(\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j} y^k\right)\omega^n$$

$$+ u_\alpha^j u_\beta^k \frac{\partial^2 L}{\partial x^j \partial y^k}\omega^\beta + \lambda_{\alpha\beta}\omega^\beta, \tag{1.1}$$

$$\omega_n^\alpha = -\omega_\alpha^n, \quad \omega_n^n = 0,$$

$$H_{ij}^k = L \frac{\partial g_{bc}}{\partial y^a} u_i^b u_j^c u_k^a,$$

where $L = L(x, y)$, then we can find only one set of $\lambda_{\alpha\beta}$, $\mu_{\alpha\gamma}^\beta$ which satisfy

$$d\omega^i = \omega^j \wedge \omega_j^i. \quad \omega_i^j + \omega_j^i = H_{ij}^k \omega_k^n. \tag{1.2}$$

Moreover it is easily seen that ω_i^j and H_{ij}^k which satify the relation (1.2) do not exist except those of (1.1). Next 2) is proved from the facts that the matrix $(\partial^2 L/\partial y^j \partial y^k)$ is of rank $n-1$ and 1-forms $v_k^\beta du_\alpha^k$ $(\alpha < \beta)$ are linearly independent.

Let $a = (a_i^j)$ be any element of $O(n)$ satisfying $a_\alpha^n = a_n^\alpha = 0$ and put $b_i^j a_k^i = \delta_k^j$. Let R_a be a right translation on P^0 by a. Then we can easily verify

$$R_a^* \cdot \omega^i = b_k^i \omega^k, \quad \omega^i = R_a^* \cdot a_k^i \omega^k. \tag{1.3}$$

We shall now denote the matrix (ω_i^j) by ω, and by $R_a^* \cdot \omega$ we mean the matrix $(R_a^* \cdot \omega_i^j)$.

LEMMA 2. $R_a^* \cdot \omega = a^{-1}\omega a$.

PROOF. From (1.3) and Lemma 1,

(1.4) $d(R_a^*\omega^i) = b_k^i d\omega^k = b_k^i \omega^j \wedge \omega_j^k = b_k^i (R_a^* \cdot a_h^j \omega^h) \wedge \omega_j^k = (R_a^* \cdot \omega^h) \wedge b_k^i \omega_j^k a_h^j,$

(1.5) $d(R_a^* \cdot \omega^i) = R_a^* \cdot d\omega^i = R_a^* \cdot (\omega^h \wedge \omega_h^i) = (R_a^* \cdot \omega^h) \wedge (R_a^* \cdot \omega_h^i).$

If we compare (1.4) with (1.5), for certain functions ρ_{ik}^j ($\rho_{ik}^j = \rho_{ki}^j$) on P^0 we obtain

$$R_a^* \cdot \omega_i^j = b_l^j \omega_h^l a_i^h + \rho_{il}^j (R_a^* \cdot \omega^l).$$

Using (1.3), the last equation can be written as

(1.6) $R_a^* \cdot \omega_i^j = b_l^j \omega_h^l a_i^h + \rho_{il}^j b_h^l \omega^h.$

On the other hand, from Lemma 1,

$$R_a^* \cdot \omega_i^j + R_a^* \cdot \omega_j^i = R_a^* \cdot H_{ij}^k \omega_k^n.$$

Substituting (1.6) into this,

(1.7) $$b_l^j \omega_h^l a_i^h + \rho_{il}^j b_h^l \omega^h + b_l^i \omega_h^l a_j^h + \rho_{jl}^i b_h^l \omega^h = \sum_{d,k} H_{bc}^d a_i^b a_j^c a_k^d (b_l^n \omega_h^l a_k^h + \rho_{kl}^n b_h^l \omega^h).$$

Now, when we put

(1.8) $A_{ijh} = \sum_{d,k} H_{bc}^d a_i^b a_j^c a_k^d \rho_{kl}^n b_h^l - \rho_{il}^j b_h^l - \rho_j^i b_h^l,$

(1.7) is rewritten as

$$b_l^j \omega_h^l a_i^h + b_l^i \omega_h^l a_j^h = \sum_{d,k} H_{bc}^d a_i^b a_j^c a_k^d b_l^n \omega_h^l a_k^h + A_{ijh}\omega^h.$$

This implies

$$\omega_i^j + \omega_j^i = H_{ij}^k \omega_k^n + b_i^h A_{hkl} \omega^l b_j^k.$$

By applying Lemma 1,

$$A_{hkl} = 0.$$

Therefore, if we put

$$B_{ij}^k = \sum_d H_{bc}^d a_i^b a_j^c a_k^d,$$

(1.8) is rewritten as

(1.9) $B_{ij}^l \rho_{lk}^n = \rho_{ik}^j + \rho_{jk}^i.$

Now, using $B_{nj}^k = 0$, we get $\rho_{nk}^j = -\rho_{jk}^n$ from (1.9), and hence $\rho_{nk}^n = 0$. So, putting $k = n$ in (1.9) we get $\rho_{ij}^n = 0$, and from (1.9), $\rho_{ik}^j = 0$. By applying this in (1.6) our assertion is proved.

From Lemma 1, the 1-forms ω^i, $\omega_\alpha^\beta (\alpha < \beta)$, ω_α^n are linearly independent and the number is equal to the dimension of P^0. Therefore, corresponding to the set of the 1-forms, the dual base i.e. a set of vector fields which we call their *dual vector fields*, does exist and we denote it by $(E^i, E_\alpha^\beta, E_\alpha^n)$. The vectors $E^i, E_\alpha^\beta, E_\alpha^n$ at a point span evidently the tangent vector space of P^0 and an n-dimensional tangent vector subspace spanned by E^i only is said to be a *natural subspace*.

LEMMA 3. *For the vectors* $E^i, E_\alpha^\beta (\alpha < \beta), E_\alpha^n$ *at a point* $z = (x, \lambda y, u_1, \ldots, u_n)$

of P^0, 1) $qp^0(E^i) = u^i$, 2) *vectors* $p^0(E^n_\alpha)$ *at a point* $(x, \lambda y) \in Q$ *are linearly independent and are tangent to the fibre of* $Q(M)$ *over* x, 3) *vectors* E^β_α *are tangent to the fibre of* $P^0(Q)$ *over* $(x, \lambda y)$.

PROOF. As the j-th component of a vector $qp^0(E^i)$ with respect to a frame (u_i) of M_x is $\omega^j(E^i) = \delta^{ji}$, 1) holds good. Next, from $\omega^i(E^n_\alpha) = 0$, E^n_α is expressed by the following type:

$$(1.10) \qquad \sum_{s=1}^{r} a_s \frac{\partial}{\partial\sigma^s} + \sum_{t=1}^{n-1} b_t \frac{\partial}{\partial\tau^t} \qquad \left(r = \frac{(n-1)(n-2)}{2}\right)$$

where σ^s and τ^t mean essential parameters of the standard fibres $O(n-1)$ and S^{n-1} of $P^0(Q)$ and $Q(M)$ respectively. So, using $\omega^n_\beta(E^n_\alpha) = \delta_{\beta\alpha}$ we see easily that 2) is true. Moreover, as it follows that E^β_α are of the type $\sum a_s \cdot \partial/\partial\sigma^s$, 3) is also clear.

LEMMA 4. *Given a vertical vector* Y *of* $P^0(Q)$ *at* $z \in P^0$, *an element of the Lie algebra of the standard fibre* $GL(n)$ *of* $P(Q)$ *generated by* Y *has a matrix* $\omega(Y)$, *as its components with respect to the natural frame of the Lie algebra.*

PROOF. Let $(x^i, \lambda y^i, u^i_1, \ldots, u^i_n)$ be a C-coordinate system at z. It is clear that $\omega^\beta_\alpha(Y) = v^\beta_k du^k_\alpha(Y)$ and $\omega^n_\alpha(Y) = 0$. Now, since $du^i_n(Y) = d(y^i/L)(Y) = 0$, we have $v^k_i du^i_n(Y) = 0$. And $v^n_i du^i_k(Y) = -u^i_k dv^n_i(Y) = -u^i_k d(\partial L/\partial y^i)(Y) = 0$. So $\omega(Y) = (\omega^j_k(Y)) = (v^j_i du^i_k(Y))$. This fact suffices to show our assertion.

From now on, we shall treat the Euclidean vector spaces attached on Q as Euclidean spaces. Let I be a differentiable transformation of Q onto itself defined by the mapping $((x, \lambda y) \to (x, -\lambda y))$ where $-\lambda y$ means $\lambda(-y)$. Then, I induces for every point $(x, \lambda y) \in Q$ the mapping $M(x, \lambda y) \to M(x, -\lambda y)$ whose mutually corresponding points are of the same point of M_x. We denote the mapping by I too. I induces moreover a differentiable transformation of P^0, that is a mapping $((x, \lambda y, u_1, \ldots, u_{n-1}, u_n) \to (x, -\lambda y, u_1, \ldots, u_{n-1}, -u_n))$of $R_a P^0$ onto itself. It is also denoted by I.

LEMMA 5. *The natural subspace field is invariant by the right translation and the differentiable transformation* I *on* P^0.

PROOF. We denote the natural subspace at $z \in P^0$ by N_z and put $\bar{z} = R_a \cdot z$. If $X \in N_z$, $\omega(X) = 0$. Therefore, $(R^*_a \cdot \omega)(X) = 0$ by virtue of Lemma 2. Hence, $\omega(R_a \cdot X) = 0$. So we have $R_a \cdot N_z = N_{\bar{z}}$. Next, from (1.1) we can easily show

$$(1.11) \qquad I^* \cdot \omega^\beta_\alpha = \omega^\beta_\alpha \qquad \text{and} \qquad I^* \cdot \omega^n_\alpha = -\omega^n_\alpha.$$

We put $z' = Iz$. Then $(I^* \cdot \omega)(X) = 0$ by $\omega(X) = 0$. Hence, $\omega(I \cdot X) = 0$. This shows $I \cdot N_z = N_{z'}$.

2. Euclidean connections. Let C be an isometric mapping between two infinitesimally neighbouring Euclidean spaces attached on Q, for which the following conditons are satisfied:

1) For a coordinate neighbourhood V of M and for each point $u=(x, \lambda y) \in q^{-1}(V)$, if a frame (e_i) in $M(u)$ ($=M(x, \lambda y)$ and u also denotes the origin of (e_i)) is the natural frame at x (its coordinates are x^i) $\in V$, we can find 1-forms $\overline{\omega}_i^j$ in $q^{-1}(V)$ such that C is determined by $du = dx^i e_i$ and $de_i = \overline{\omega}_i^j e_j$.

2) C is invariant by I, that is $C = ICI$.

Such a mapping C is called an *Euclidean connection* in M. Moreover, an Euclidean connection in M, by which supporting vectors in Euclidean spaces attached at points on the same fibre of $Q(M)$ are never mutually parallelly mapped, is said to be *regular*. When an Euclidean connection is given in M and we express it in terms of orthonormal frames (u_i) in $M(u)$, $u=(x, \lambda y) \in Q$, we can find 1- forms π_i^j on P^0 such that the connection is determined by

$$\text{(A)} \qquad du = \pi^i u_i, \quad du_i = \pi_i^j u_j \quad (\pi^i = \omega^i).$$

A matrix $\pi = (\pi_i^j)$ is called its *connection form* on P^0, and by a *horizontal subspace* (in P^0) at $z \in P^0$ we mean a tangent vector subspace consisting of all vectors X at z which satisfy $\pi(X)=0$. The 1-forms π_i^j are characterized by the following conditions:

a) For any vertical vector Y of $P^0(Q)$, $\pi(Y) = \omega(Y)$ (cf. Lemma 4), b) $\pi_i^j + \pi_j^i = 0$, c) $R_a^* \cdot \pi = a^{-1} \pi a$, d) $I^* \cdot \pi_\alpha^\beta = \pi_\alpha^\beta$, $I^* \cdot \pi_\alpha^n = -\pi_\alpha^n$.

THEOREM 1. *A necessary and sufficient conditions that a mapping* (A) *determines an Euclidean connection is that π_i^j be expressed by the type*

$$\text{(B)} \qquad \pi_i^j = \omega_i^j + \gamma_i^{jk} \omega_k^n + \gamma_{ik}^j \omega_k.$$

where γ_i^{jk}, γ_{ik}^j are differentiable functions on P^0 which satisfy the following conditions:

1) $\gamma_i^{jn} = 0$, $\gamma_i^{jk} + \gamma_j^{ik} = -H_{ij}^k$ *and* $\gamma_{ik}^j + \gamma_{jk}^i = 0$.

2) *In a neighbourhood with C-coordinate system $(x^i, \lambda y^i, u_1^i, \ldots, u_n^i)$ of P^0 there exist functions G_{bcd} and H_{bcd} of $x^i, \lambda y^i$ only, which satisfy*

$$\gamma_i^{jk} = G_{bcd} u_i^b u_j^c u_k^d \quad \text{and} \quad \gamma_{ik}^j = H_{bcd} u_i^b u_j^c u_k^d.$$

3) *For a point $z \in P^0$: $\gamma_\alpha^{\beta\gamma}(I \cdot z) = -\gamma_\alpha^{\beta\gamma}(z)$, $\gamma_{\alpha\gamma}^\beta(I \cdot z) = \gamma_{\alpha\gamma}^\beta(z)$;*
if any one of i, j, k, is n, $\gamma_i^{jk}(I \cdot z) = \gamma_i^{jk}(z)$, $\gamma_{ik}^j(I \cdot z) = -\gamma_{ik}^j(z)$;
if any two of i, j, k are n, $\gamma_{ik}^j(I \cdot z) = \gamma_{ik}^j(z)$.

Further, if and only if the Euclidean connection is regular,

$$\det. \mid \delta_\alpha^\beta - \gamma_n^{\alpha\beta} \mid (= \det. \mid \delta_\alpha^\beta + \gamma_\alpha^{n\beta} \mid) \neq 0.$$

PROOF. We shall first prove the necessity. By Lemma 1, π_i^j $(i<j)$ can be put

$$\pi_i^j = \sum_{\alpha<\beta} \gamma_{i\alpha}^{j\beta} \omega_\alpha^\beta + \gamma_i^{j\alpha} \omega_\alpha^n + \gamma_{ik}^j \omega^k.$$

As the condition a) must hold good, $\pi_i^j(Y) = \omega_i^j(Y)$ for any vertical vector Y of $P^0(Q)$. If we substitute a vector $E_\gamma^\delta(\gamma<\delta)$ for Y we get

$$\sum_{\alpha<\beta} \gamma_{i\alpha}^{j\beta} \delta_{\alpha\gamma} \delta^{\beta\delta} = \delta_{i\gamma} \delta^{j\delta}, \text{ i. e. } \gamma_{i\gamma}^{j\delta} = \delta_{i\gamma} \delta^{j\delta}.$$

Therefore, by using the condition b) we see that π_i^j are expressed by the type (B), i.e.

$$\pi_i^j = \omega_i^j + \gamma_i^{jk}\omega_k^n + \gamma_{ik}^j\omega^k, \tag{2.1}$$

together with the relations

$$\gamma_i^{jn} = 0,\ \ \gamma_i^{jk} + \gamma_j^{ik} = -H_{ij}^k,\ \ \gamma_{ik}^j + \gamma_{jk}^i = 0,$$

which are our assertion 1). On the other hand, for $z \in P^0$ let $x^i, \lambda y^i, u_1^i, \ldots, u_n^i$ and $x^i, \lambda y^i, \bar{u}_1^i, \ldots, \bar{u}_n^i$ be C-coordinates of z and $\bar{z}(=R_a\cdot z)$ respectively. Then $\bar{u}_i^j = u_k^j a_i^k$. Let us put

$$\gamma_i^{jk}(z) = G_{bcd}(z)u_i^b u_c^j u_k^d,\ \ \gamma_{ik}^j(z) = H_{bcd}(z)u_i^b u_j^c u_k^d,$$

$$\gamma_i^{jk}(\bar{z}) = G_{bcd}(\bar{z})\bar{u}_i^b\bar{u}_j^c\bar{u}_k^d,\ \ \gamma_{ik}^j(\bar{z}) = H_{bcd}(\bar{z})\bar{u}_i^b\bar{u}_j^c\bar{u}_k^d,$$

$$\phi = (\phi_i^j) = (\gamma_i^{j\alpha}\omega_\alpha^n),\qquad \psi = (\psi_i^j) = (\gamma_{ik}^j\omega^k).$$

By using properties of a matrix a and R_a^* (see Lemma 2), we get

$$
\begin{aligned}
&R_a^*\cdot(\phi_i^j)_{\bar{z}} = \Sigma_k G_{bcd}(\bar{z})\,u_h^b u_l^c u_k^d a_i^h a_j^l(\omega_k^n)_z,\\
&b_l^j(\phi_h^l)_z a_i^h = \Sigma_k G_{bcd}(z)\,u_h^b u_l^c u_k^d a_i^h a_j^l(\omega_k^n)_z,\\
&R_a^*\cdot(\psi_i^j)_{\bar{z}} = H_{bcd}(\bar{z})\,u_h^b u_l^c u_k^d a_i^h a_j^l(\omega^k)_z,\\
&b_l^j(\psi_h^l)_z a_i^h = H_{bcd}(z)\,u_h^b u_l^c u_k^d a_i^h a_j^l(\omega^k)_z.
\end{aligned}
\tag{2.2}
$$

Now, from Lemma 2 the condition c) is equivalent to

$$R_a^*\cdot(\phi + \psi) = a^{-1}(\phi + \psi)a.$$

If we substitute (2.2) into this, from Lemma 1 we get easily

$$G_{bcd}(\bar{z}) = G_{bcd}(z),\quad H_{bcd}(\bar{z}) = H_{bcd}(z).$$

This means that 2) holds true. Next we shall apply the condition d) to (2.1), then

$$I^*\cdot(\omega_\alpha^\beta + \gamma_\alpha^{\beta\gamma}\omega_\gamma^n + \gamma_{\alpha k}^\beta\omega^k) = \omega_\alpha^\beta + \gamma_\alpha^{\beta\gamma}\omega_\gamma^n + \gamma_{\alpha k}^\beta\omega^k,$$

$$I^*\cdot(\omega_\alpha^n + \gamma_\alpha^{n\gamma}\omega_\gamma^n + \gamma_{\alpha k}^n\omega^k) = -\omega_\alpha^n - \gamma_\alpha^{n\gamma}\omega_\gamma^n - \gamma_{\alpha k}^n\omega^k.$$

However, $I^*\cdot\omega^\alpha = \omega^\alpha$ and $I^*\cdot\omega^n = -\omega^n$. Using these together with (1.11) if we simplify the above relations, we get

$$I^*\cdot\gamma_\alpha^{\beta\gamma} = -\gamma_\alpha^{\beta\gamma},\quad I^*\cdot\gamma_{\alpha\gamma}^\beta = \gamma_{\alpha\gamma}^\beta,\quad I^*\cdot\gamma_\alpha^{n\gamma} = \gamma_\alpha^{n\gamma},$$

$$I^*\cdot\gamma_{\alpha n}^\beta = -\gamma_{\alpha n}^\beta,\quad I^*\cdot\gamma_{\alpha\gamma}^n = -\gamma_{\alpha\gamma}^n,\quad I^*\cdot\gamma_{\alpha n}^n = \gamma_{\alpha n}^n.$$

These relations are expressed as 3). So, the necessity has been proved, and the sufficiency is now clear from the above proof.

Finally if det. $|\delta_\alpha^\beta - \gamma_n^{\alpha\beta}| = 0$ at a point $z_0 \in P^0$, we can find a vector Z at z_0 which is spanned by $n-1$ vectors E_α^n and satisfies $\pi_n^\alpha(Z) = 0$. This contradicts with the regularity. So, det. $|\delta_\alpha^\beta - \gamma_n^{\alpha\beta}| \neq 0$ on P^0. The converse is clear.

By Theorem 1 it is evident that a regular Euclidean connection (A) may exist satisfying $\gamma_{ik}^j = 0$, i.e. the connection such that π_i^j are expressed as

(C) $$\pi_i^j = \omega_i^j + \gamma_i^{jk}\,\omega_k^n,$$

where γ_i^{jk} are differentiable functions on P^0 satisfying the conditions of Theorem 1. Such a regular Euclidean connection is called a *Chern's connection* [2].

REMARK 2. In an Euclidean connection (A) associated with (C), when we put $\gamma_i^{jk} = -(1/2)\cdot H_{ij}^k$ we get a Chern's connection defined by E. Cartan [1] [2]. We shall call this connection *Cartan's connection.*

REMARK 3. *There exists an infinite number of regular Euclidean connections, especially Chern's connections* (see Appendix).

THEOREM 2. *If a Chern's connection is given in M, its horizontal subspace field in P^0 coincides with the natural subspace field. Conversely, an Euclidean connection of M whose horizontal subspace field in P^0 coincides with the natural subspace field is a Chern's connection.*

PROOF. We denote the given Chern's connection by (A) associated with (C). As the 1-forms $\pi^i(=\omega^i)$, $\pi_\alpha^\beta(\alpha<\beta)$, π_α^n on P^0 are linearly independent from the regularty, their dual vector fields do exist, and we denote the ones corresponding to π^i by V^i. Then we see easily $V^i = E^i$. Accordingly the former part is true.

In order to prove the latter part, we denote an Euclidean connection satisfying the given condition by (A) associated with (B). However this condition shows that $\pi_i^j(E^k) = 0$. Hence $\gamma_{ik}^j = 0$. Therefore, π_i^j are expressed by the type (C). On the other hand, if det. $|\delta_\alpha^\beta + \gamma_\alpha^{n\beta}| = 0$ at $z_0 \in P^0$, we may find a vector Z of the type $\Sigma_{\alpha<\beta}\, a_\alpha^\beta E_\alpha^\beta + \Sigma_\alpha b_\alpha E_\alpha^n$ at z_0 satisfying $\pi(Z) = 0$. This contradicts evidently with the given condition. Hence det. $|\delta_\alpha^\beta + \gamma_\alpha^{n\beta}| \neq 0$. So the Euclidean connection in consideration is a Chern's con- nection. This is our assertion.

THEOREM 3. *In a regular Euclidean connection of M, a necessary and sufficient condition for it to be a Chern's connection is that it gives rise to no torsion when the supporting vector is parallelly displaced.*

PROOF. We shall denote a Chern's connection by (A) associated with (C). Then, for the torsion (Π^i) $(\Pi^i = d\pi^i - \pi^j \wedge \pi_j^i)$ we have $\Pi^i = -\gamma_j^{i\alpha}\,\omega^j \wedge \omega_\alpha^n$. Hence, $\Pi^i(Z_1, Z_2) = 0$ for any vector fields Z_1, Z_2 which satisfy $\pi_n^\alpha(Z_1) = \pi_n^\alpha(Z_2) = 0$, because $\omega_\alpha^n(Z_1) = \omega_\alpha^n(Z_2) = 0$. This means that the condition of Theorem 3 for the torsion holds true.

Next, we denote a regular Euclidean connection which satisfies the condition for the torsion by (A) associated with (B). Now, this condition is equivalent to $\Pi^i(Z_1, Z_2) = 0$ for any two vector fields Z_1, Z_2 which satisfy $\pi_n^\alpha(Z_1) = \pi_n^\alpha(Z_2) = 0$. However $\Pi^i = -\gamma_j^{i\alpha}\,\omega^j \wedge \omega_\alpha^n - \gamma_{jk}^i\,\omega^j \wedge \omega^k$. By using this the given condition implies

(2.3) $$\gamma_i^{j\alpha}\, C_\alpha^\beta\, \gamma_{\beta k}^n - \gamma_k^{j\alpha}\, C_\alpha^\beta\, \gamma_{\beta i}^n - \gamma_{ik}^j + \gamma_{ki}^j = 0,$$

where (C^{β}_{α}) is the inverse matrix of $(\delta^{\beta}_{\alpha} + \gamma^{n\beta}_{\alpha})$. From (2.3) the following relations follow:

$$\begin{aligned} \gamma^{n\alpha}_{\gamma} C^{\beta}_{\alpha} \gamma^{n}_{\beta n} - \gamma^{n\alpha}_{n} C^{\beta}_{\alpha} \gamma^{n}_{\beta\gamma} - \gamma^{n}_{\alpha n} + \gamma^{n}_{n\gamma} &= 0, \\ \gamma^{n\alpha}_{\gamma} C^{\beta}_{\alpha} \gamma^{n}_{\beta\delta} - \gamma^{n\alpha}_{\delta} C^{\beta}_{\alpha} \gamma^{n}_{\beta\gamma} - \gamma^{n}_{\gamma\delta} + \gamma^{n}_{\delta\gamma} &= 0, \\ \gamma^{\gamma\alpha}_{n} C^{\beta}_{\alpha} \gamma^{n}_{\beta\delta} - \gamma^{\gamma\alpha}_{\delta} C^{\beta}_{\alpha} \gamma^{n}_{\beta n} - \gamma^{\gamma}_{n\delta} + \gamma^{\gamma}_{\delta n} &= 0. \end{aligned} \tag{2.4}$$

On the other hand, $C^{\beta}_{\alpha}(\delta^{\alpha}_{\gamma} + \gamma^{n\alpha}_{\gamma}) = \delta^{\beta}_{\gamma}$ and hence

$$C^{\beta}_{\alpha} \gamma^{n\alpha}_{\gamma} = \delta^{\beta}_{\gamma} - C^{\beta}_{\gamma}. \tag{2.5}$$

By substituting (2.5) into $(2.4)_1$ and using 1) (of Theorem 1), we get

$$\gamma^{n}_{\beta n} = 0. \tag{2.6}$$

Next, substituting (2.5) into $(2.4)_2$,

$$C^{\beta}_{\gamma} \gamma^{n}_{\beta\delta} = C^{\beta}_{\delta} \gamma^{n}_{\beta\gamma}. \tag{2.7}$$

If we simplify $(2.4)_3$ by 1), (2.5) and (2.6),

$$- C^{\beta}_{\gamma} \gamma^{n}_{\beta\delta} = \gamma^{\gamma}_{\delta n}. \tag{2.8}$$

By (2.7) and 1), from (2.8) $\gamma^{\gamma}_{\delta n} = 0$. Again from (2.8), $\gamma^{n}_{\beta\delta} = 0$. By this and (2.6), from (2.3) $\gamma^{j}_{ik} = \gamma^{j}_{ki}$. Moreover, by 1) we get $\gamma^{j}_{ik} = 0$. Therefore (B) is rewritten as

$$\pi^{j}_{i} = \omega^{j}_{i} + \gamma^{jk}_{i} \omega^{n}_{k}.$$

That is, the Euclidean connection in consideration becomes a Chern's connection. Our assertion has thereby been proved.

In M let $l: x(t)$ $(0 \leqq t \leqq 1)$ be a curve of class C^2 whose tangent vector $x'(t)$ is not zero. Then a curve $l': (x(t), \lambda x'(t))$ on Q is called the ***tangent curve*** of l. If an Euclidean connection is given in M, a curve on the Euclidean space $M(x(0), \lambda x'(0))$ obtained by developing Euclidean spaces $M(x(t), \lambda x'(t))$ successively along l'^{-1} is called the ***naturally developed curve*** of l.

LEMMA 6. *For any integral curve $g^0(s)$ of a vector field E^n the curve $g(s) = qp^0(g^0(s))$ in M is a geodesic whose parameter s is curve-length. Conversely, for any geodesic $g(s)$ (s: curve-length) there exists an integral curve $g^0(s)$ of E^n such that $qp^0(g^0(s)) = g(s)$.*

PROOF. We shall first prove the former part. By Lemma 3 it follows directly that tangent vectors of $g(s)$ are unit vectors. So, s is regarded as the curve-length of the curve $g(s)$. If we put $g'(s) = p^0(g^0(s))$, then $g'(s)$ is the tangent curve of $g(s)$. By this and Theorem 2, when Cartan's connection is given we see easily that the naturally developed curve of $g(s)$ is a straight line. Therefore $g(s)$ is a geodesic of M, i. e. the former part is true. Next, if we use the fact that, when a point of M and there a direction are given, a geodesic passing through the point and having the direction is completely determined, the latter part is easily verified.

THEOREM 4. *In a regular Euclidean connection* (A) *associated with* (B), *a necessary and sufficient condition that the naturally developed curves of* *all geodesics be straight lines is that* $\gamma^n_{\alpha n} = 0$ (cf. Proof of Remark 3, Appendix).

PROOF. At first, by the regularity the 1-forms $\pi^i(=\omega^i)$, $\pi^\beta_\alpha(\alpha<\beta)$, π^n_α are linearly independent on P^0. Hence their dual vector fields do exist, and we denote the one corresponding to π^n by V^n. In the vector field V^n, let h^0 be any of its integral curves, and put $h'=p^0(h^0)$, $h=q(h')$. From $\omega^i(V^n)=\delta^{in}$ we see that the curve h' is the tangent curve of h, and the naturally developed curve of h is a straight line.

Assume now that the naturally developed curves of geodesics are of straight lines. From this and the regularity, we can easily see that the curve h is a geodesic. So, by Lemma 6 there exists an integral curve of the field E^n, which is mapped onto h' by p^0. Hence we can find a vertical vector field Y of $P^0(Q)$ which satisfies $V^n=Y+E^n$ at each point of P^0. However $\pi^n_\alpha(V^n)=0$. So, from Lemma 3 we get $\gamma^n_{\alpha n}=0$. The necessity has thereby been proved, and the sufficiency is now clear.

COROLLARY. *In any Chern's connection, the naturally developed curves of all geodesics are straight lines.*

(This is also verified from Lemma 6 and Theorem 2.)

In $P(Q)$, let z be a point of P^0 and let x^i, λy^i, X_1, ..., X_n be C-coordinates of z. In such a C-coordinate system, if we vary the coordinates λy^i only, leaving x^i, X_1, ..., X^i_n fixed, a submanifold of P is obtained. We denote it by $S(z)$. If we project a vector E^n_α at z on the tangent space of $S(z)$ (with respect to the fibre), we obtain a vector of the type $\sum^{n-1}_{t=1} b_t \partial/\partial\tau^t$ (put $a_s=0$ in (1.10)). We denote this by F^n_α. Moreover we denote by L^{j*}_i the vertical vector fields of $P(Q)$ generated from the vectors L^j_i which form the natural frame in the Lie algebra of the standard fibre $GL(n)$. Next, given a Chern's connection in M, the horizontal subspaces on P and so on are defined as defined in P^0.

THEOREM 5. *Given in M a Chern's connection* (A) *associated with* (C), *then its horizontal subspace in P at a point $z\in P^0$ is spanned by the natural subspace and $n-1$ linearly independent vectors $F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i$ at z.*

Proof. When we consider on P the given Chern's connection, we assume that it is expressed by

$$du=\theta^i X_i,\quad dX_i=\theta^j_i X_j.$$

Now let $(x^i, \lambda y^i, X^i_1, \ldots, X^i_n)$ and $(x^i, \lambda y^i, u^i_1, \ldots, u^i_n)$ be the C-coordinate systems in P and P^0 respectively induced from a coordinate neighbourhood of M. Then we have following relations:

$$\theta^i = Y^i_j dx^j,$$

$$\theta^j_i = Y^j_k dX^k_i - Y^j_k du^k_h v^h_l X_i + Y^j_k u^k_h \pi^h_l v^l_b X^b_i,$$

where we have put $Y^i_k X^k_j=\delta^i_j$. When we consider the restriction of the 1-forms

θ^i, θ^j_i, on P^0 we shall denote them by θ^{i0}, θ^{j0}_i. Then,

$$\theta^{i0} = \pi^i = \omega^i,$$
$$\theta^{j0}_i = v^j_k dX^k_i - v^j_k du^k_i + \pi^j_i \equiv v^j_k dX^k_i + \gamma^{j\alpha}_i \omega^n_\alpha (\text{mod } \omega^i).$$

However, as the 1-forms ω^i, θ^{j0}_i, ω^n_α are linearly independent, their dual vector fields in P defined on P^0 do exist We denote them by K^i, K^{j0}_i, K^n_α respectively. These vectors at $z \in P^0$ span the tangent space of P and at the point we have the following relations:

$$K^i = E^i,\ K^{j0}_i = L^{j*}_i,\ K^n_\alpha = F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i. \tag{2.9}$$

This is easily verified by

$$\omega^k(E^i) = \delta^{ki},\ \theta^{l0}_k(E^i) = (v^l_h dX^h_k - v^l_h du^h_k + \pi^l_k)(E^i) = 0,\ \omega^n_\beta(E^i) = 0,$$
$$\omega^k(L^{j*}_i) = 0,\ \theta^{l0}_k(L^{j*}_i) = v^l_h dX^h_k(L^{j*}_i) = \delta_{ki}\delta^{lj},\ \omega^n_\beta(L^{j*}_i) = 0,$$
$$\omega^k(F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i) = 0,\ \theta^{l0}_k(F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i)$$
$$= (v^l_h dX^h_k + \gamma^{l\beta}_k \omega^n_\beta)(F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i) = \gamma^{l\alpha}_k - \gamma^{l\alpha}_k = 0,$$
$$\omega^n_\beta(F^n_\alpha - \Sigma_{i,j}\gamma^{j\alpha}_i L^{j*}_i) = \delta^{\beta\alpha}.$$

The relation (2.9) shows us that Theorem 5 is true.

REMARK 4. Given in M a Chern's connection, the horizontal subspace field on P is invariant by any right translation $R_g(g \in GL(n))$ and by a differentiable transformation $J((x, \lambda y, X_1, \ldots, X_n) \to (x, -\lambda y, X_1, \ldots, X_n))$, and the canonical projection p maps the horizontal subspace at a point $z \in P$ isomorphically onto the tangent vector space of Q at $p(z)$.

APPENDIX

We shall here attempt to prove Remarks 1 and 3. Unless defined otherwise, the previous notations will be adopted, as it is.

PROOF OF REMARK 1. Let M be an n-dimensional connected differentiable manifold. Assume that the second contability axiom holds in M. Then M has a positive definite diferentiable Riemannian metric. Under this metric we denote by $L_1(x, y)$ the length of a vector y at $x \in M$. We can moreover find a continuous function $L_2(x, y)$ on T which satisfies the following conditions: 1) $L_2(x, ty) = |t|\, L_2(x, y)$ for any real number t, 2) $L_2(x, y)$ is differentiable and non-negative on T^0, 3) the carrier is mapped by ρ onto a compact subset in M. When we put

$$L(x, y) = \sqrt{L_1^2(x, y) + \varepsilon L_2^2(x, y)}$$

for a positive constant ε, $L(x, y)$ is a continuous function on T which satisfies the conditions for Finsler metric on M, that is, M has a Finsler metric. We shall determine $\varepsilon\,(>0)$ such that this Finsler metric becomes positive definite. This is possible by the above 3). Accordingly there exists a positive definite Finsler metric in M, and for a suitable $L_2(x, y)$ it is evident that it becomes non-Riemannian.

Next, assume that M has a positive definite (non-Riemannian) Finsler metric. For each point $x \in M$, we denote by $Q(x)$ a countable dense subset in the fibre of $Q(M)$ over x and we regard $Q(x)$ as the set of unit vectors at x. Let o be a fixed point of $M(x)$ and we describe as a broken geodesic a curve obtained by joining a finite number of geodesic-arcs (which we call simply its arcs). Let Φ be the set of all broken geodesics starting from the point o such that, any of their arcs has a vector of $Q(x')$ as the tangent vector at the initial point x' and a rational number as the length. Let Ψ be the set of terminal points of all broken geodesics of Φ, then Ψ is evidently countable. We shall prove that Ψ is dense in M. For $x \in M - \Psi$, join o and x with a broken geodesic l, and cover l by a finite number of simple convex neighbourhoods. (By a simple convex neighbourhood we mean a neighbourhood V such that any two points in V are joined by only one geodesic-arc wholly contained in V.) In these neighbourhoods we denote the union set by W. If we consider broken-geodesics of Φ contained in W, we can find a subset of Ψ which has x as a accumlating point. Hence, Ψ is a countable dense subset of M. However, since M is a metric space similarly as in Riemannian case, the second countability axiom holds in M.

PROOF OF REMARK 3. Let M be an n-dimensional connected differentiable manifold with a positive definite Finsler metric. Let V be any coordinate neighbourhood of M. and let $(x^i, \lambda y^i, u_1^i, \dots, u_n^i)$ be the C-coordinate system of $(qp^0)^{-1}(V)$ induced from V. We can find a differentiable skew-symmetric covariant tensor field S of degree 2 on M whose carrier is compact and whose components S_{bc} in V are expressed as functions in $q^{-1}(V)$ satisfying $S_{bc}(x, \lambda y) = S_{bc}(x, -\lambda y)$. If we put there

$$\Gamma^{\alpha\beta} = L \frac{\partial S_{bc}}{\partial y^a} u_n^b u_\alpha^c u_\beta^a,$$

then we obtain functions $\Gamma^{\alpha\beta}$ defined on P^0 and their carriers are compact. Hence there exists a constant $\varepsilon(\neq 0)$ such that det. $|\delta_\alpha^\beta - \varepsilon\Gamma^{\alpha\beta}| \neq 0$ on P^0. Further we can find a differentiable covariant tensor field H of degree 3 on M whose components H_{bca} in V are expressed as functions in $q^{-1}(V)$ satisfying $H_{bca} = -H_{cba}$ and $H_{bca}(x, \lambda y) = H_{bca}(x, -\lambda y)$. Using the above ε we shall put

$$\gamma_i^{jk} = \left(-\frac{1}{2} L \frac{\partial g_{bc}}{\partial y^a} + \varepsilon L \frac{\partial S_{bc}}{\partial y^a}\right) u_i^b u_j^c u_k^a, \quad \gamma_{ik}^j = H_{bca} u_i^b u_j^c u_k^a.$$

Then we obtain also functions γ_i^{jk} and γ_{ik}^j defined on P^0. They satisfy all the conditions of Theorem 1. Accordingly a regular Euclidean connection is obtained. By the above statements we can easily understand that our assertion is true. (Note that $\gamma_{\alpha n}^n = 0$ if H is skew-symmetric with respect to all indices.)

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