ABSOLUTE SUMMABILITY OF RADEMACHER SERIES

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1. Introduction. A series $\sum a_n$ is said to be absolutely summable (A) or briefly summable $|A|$ if the series $\sum a_n x^n$ is convergent for $0 \leqq x < 1$ and its sum-function $\phi(x)$ satisfies that

$$\int_0^1 |\phi'(x)|\, dx < \infty. \tag{1.1}$$

This summability has been studied by many writers. Recently T. M. Flett introduced an extension of this summability for $k \geqq 1$ replacing the condition (1.1) by the condition

$$\int_0^1 (1-x)^{k-1} |\phi'(x)|^k\, dx < \infty, \tag{1.2}$$

which has an important significance as well as (1.1) in the theory of Fourier series (cf. [1] where references are given). T. M. Flett called this summability "summability $|A|_k$" where $k \geqq 1$. At the same time he gave an extension for summability $|C|$ —— the absolute Cesàro summability——, that is, the series $\sum a_n$ is called summable $|C, \alpha|_k$, where $k \geqq 1$, $\alpha > -1$, if the series

$$\sum n^{k-1} |\sigma_{n-1}^{\alpha} - \sigma_n^{\alpha}|^k \tag{1.3}$$

is convergent, σ_n^{α} being the n th Cesàro mean of order α of the series $\sum a_n$. The summability $|C, \alpha|_1$ is the ordinary absolute summability $|C, \alpha|$.

Among the many theorems on the absolute summability, one of the most interesting is the so-called high-indices theorem. By the Zygmund high-indices theorem [5], if the series $\sum a_n$ is lacunary, that is, its terms are all zero except for the terms with indices $n_0 = 0 < n_1 < n_2 < \dots$ such that $n_{j+1}/n_j \geqq q > 1$ $(j = 1, 2, \dots)$, and if the series is summable $|A|$, then it turns out to be absolutely convergent or summable $|C, 0|$. Flett [2] studied an extension of this result to the summability $|A|_k$ and gave an inequality corresponding to that of the Zygmund theorem [5]. But he has left open the problem: If the series $\sum a_n$ is lacunary and summable $|A|_k$, then is it summable $|C, 1-1/k|_k$ where $k > 1$?

The main purpose of this paper is to give a negative answer to this problem. Therefore the Flett inequality ([2] Theorem 1) is the best possible one of this sense.

2. Our answer to the Flett problem is:

THEOREM 1. *For any $k > 1$ there exists a lacunary series $\sum c_n$ which is summable $|A|_k$ but is not summable $|C, 1-1/k|_k$.*

For the proof of this theorem we shall establish some preliminary theorems concerning the absolute summabilities of the Rademacher series.

Let us denote the Rademacher system by $\psi_0(x), \psi_1(x), \psi_2(x), \ldots. (0 \leqq x \leqq 1)$. The following theorem is a generalization of the Zygmund theorem [5] where $k = 1$.

THEOREM 2. *Let $k \geqq 1$. For the Rademacher series*

$$\sum_{j=0}^{\infty} a_j \psi_j(x), \tag{2.1}$$

if the series

$$\sum_{n=0}^{\infty} \Big(\sum_{j=2^n}^{2^{n+1}-1} a_j^2 \Big)^{k/2} \tag{2.2}$$

is convergent, then the series (2.1) *is summable $|A|_k$ almost everywhere. And, if the series* (2.2) *is divergent, then the series* (2.1) *is summable $|A|_k$ almost nowhere.*

PROOF. Suppose that the series (2.2) is convergent and that $k > 2$. We shall prove the convergence of the integral

$$\int_0^1 dx \int_0^1 (1-\rho)^{k-1} \Big| \sum_{n=1}^{\infty} n a_n \psi_n(x) \rho^n \Big|^k d\rho, \tag{2.3}$$

which leads us to the summability $|A|_k$ almost everywhere of the series (2.1). By the Khinchin inequality (cf. [3] p. 131, [4] Lemma 1) the integral (2.3) is not greater than

$$K \int_0^1 (1-\rho)^{k-1} \Big(\sum_{n=1}^{\infty} n^2 a_n^2 \rho^{2n} \Big)^{k/2} d\rho^{1)}$$

$$= K \sum_{i=0}^{\infty} \int_{\rho_i}^{\rho_{i+1}} (1-\rho)^{k-1} \Big(\sum_{n=1}^{\infty} n^2 a_n^2 \rho^{2n} \Big)^{k/2} d\rho$$

$$= K \sum_{i=0}^{\infty} J_i \tag{2.4}$$

say, where $\rho_i = 1 - 2^{-i}$ $(i = 0, 1, 2, \ldots.)$. Now easily we have

$$J_i \leqq \frac{K}{2^{i(k-1)}} \Big(\sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^2 a_n^2 \rho_{i+1}^{2n} \Big)^{k/2} \frac{1}{2^{i+1}}$$

1) K, K' are positive constants not necessarily the same in every occurrence.

$$\leqq \frac{K}{2^{ik+1}}\left(\sum_{j=0}^{i-1} 2^{2(j+1)} \rho_{i+1}^{2^{j+1}} \sum_{n=2^j}^{2^{j+1}-1} a_n^2\right)^{k/2}$$

$$+ \frac{K}{2^{ik+1}}\left(\sum_{j=i}^{\infty} 2^{2(j+1)} \rho_{i+1}^{2^{j+1}} \sum_{n=2^j}^{2^{j+1}-1} a_n^2\right)^{k/2}$$

$$= K\cdot P_i + K\cdot Q_i \tag{2.5}$$

say. We put for brevity

$$A_j^2 = \sum_{n=2^j}^{2^{j+1}-1} a_n^2 \qquad (j = 0, 1, 2, \ldots).$$

Then, as $\rho_i \leqq 1$ for $i = 0, 1, 2, \ldots,$

$$P_i \leqq \frac{K}{2^{ik+1}}\left(\sum_{j=0}^{i-1} 2^{2(j+1)} A_j^2\right)^{k/2},$$

and as $k > 2$ by the Hölder inequality with indices $k/2$ and $k/(k-2)$, we get

$$P_i \leqq \frac{K}{2^{ik+1}}\left(\sum_{j=0}^{i-1} 2^{\frac{k}{k-2}(j+1)}\right)^{\frac{k-2}{k}\cdot\frac{k}{2}}\left(\sum_{j=0}^{i-1} 2^{\frac{k}{2}(j+1)} A_j^k\right)^{\frac{2}{k}\cdot\frac{k}{2}}$$

$$\leqq \frac{K}{2^{ki/2+1}} \sum_{j=0}^{i-1} 2^{k(j+1)/2} A_j^k.$$

Hence, we have

$$\sum_{i=0}^{\infty} P_i \leqq K \sum_{i=0}^{\infty} \frac{1}{2^{ki/2}} \sum_{j=0}^{i-1} 2^{k(j+1)/2} A_j^k$$

$$\leqq K \sum_{j=0}^{\infty} 2^{k(j+1)/2} A_j^k \sum_{i=j+1}^{\infty} \frac{1}{2^{ki/2}}$$

$$\leqq K \sum_{j=0}^{\infty} A_j^k. \tag{2.6}$$

On the other hand,

$$Q_i = \frac{K}{2^{ik+1}}\left(\sum_{j=i}^{\infty} 2^{2(j+1)} \rho_{i+1}^{2^{j+1}} A_j^2\right)^{k/2}$$

$$\leqq \frac{K}{2^{ik}}\left(\sum_{j=i}^{\infty} 2^{2j} e^{-2^{j-i}} A_j^2\right)^{k/2},$$

since $\rho_{i+1}^{2^{j+1}} = \left(1 - \frac{1}{2^{i+1}}\right)^{2^{i+1}\cdot 2^{j-i}} \sim e^{-2^{j-i}}$. Let us put $\xi_m = 2^{2m} e^{-2^m}$, then by the Hölder inequality with the same indices as above,

$$Q_i \leqq \frac{K}{2^{ik}}\left(\sum_{j=i}^{\infty} \xi_{j-i}\, 2^{2i} A_j^2\right)^{k/2}$$

$$\leqq K\left(\sum_{j=i}^{\infty} \xi_{j-i}^{\frac{k}{2(k-2)}}\right)^{\frac{k-2}{k}} \left(\sum_{j=i}^{\infty} \xi_{j-i}^{\frac{k}{4}} A_j^k\right)$$

$$\leqq K \sum_{j=i}^{\infty} \xi_{j-i}^{k/4} A_j^k.$$

Hence we get

$$\sum_{i=0}^{\infty} Q_i \leqq K \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} \xi_{j-i}^{k/4} A_j^k = K \sum_{j=0}^{\infty} A_j^k \sum_{i=0}^{j} \xi_{j-i}^{k/4}$$

$$\leqq K \sum_{j=0}^{\infty} A_j^k. \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7) we can conclude the convergence of the integral (2.3).

The case $k > 2$ was proved. The case $1 \leqq k \leqq 2$ will be proved by showing the convergence of (2.3) with suitable application of the Hölder inequality along the similar way as above. But in this case the more will be proved in the next theorem, and we omit the proof here.

For the proof of the latter half of the theorem let us suppose that the series (2.1) is summable $|A|_k$ for all $x \in E$, $|E| > 0$. We may suppose that the summability is uniform in the set E. Hence we may suppose that

$$\int_E dx \left\{ \int_0 (1-\rho)^{k-1} \left| \sum_{n=1}^{\infty} n a_n \psi_n(x) \rho^{n-1} \right|^k d\rho \right\} < K.$$

By the Khinchin inequlity

$$K > \int_0^1 (1-\rho)^{k-1} d\rho \int_E \left| \sum_{n=1}^{\infty} n a_n \psi_n(x) \rho^{n-1} \right|^k dx$$

$$\geqq K' \int_0^1 (1-\rho)^{k-1} d\rho \left(\sum_{n=1}^{\infty} n^2 a_n^2 \rho^{2n} \right)^{k/2}.$$

We put $\rho_j = 1 - 2^{-j}$ and $I_j = (\rho_j, \ \rho_{j+1})$ then

$$K > \sum_{j=0}^{\infty} \int_{I_j} (1-\rho)^{k-1} d\rho \left(\sum_{n=1}^{\infty} n^2 a_n^2 \rho^{2n} \right)^{k/2}$$

$$\geqq \sum_{j=0}^{\infty} \left(\frac{1}{2^j}\right)^{k-1} \int_{\rho_j}^{\rho_{j+1}} \left(\sum_{n=2^j}^{2^{j+1}-1} n^2 a_n^2 \rho^{2n} \right)^{k/2} d\rho$$

$$\geqq \sum_{j=0}^{\infty} \frac{1}{2^{j(k-1)}} \rho_j^{2^{jk}} 2^{jk} \int_{\rho_j}^{\rho} \left(\sum_{n=2^j}^{2^{j+1}-1} a_n^2\right)^{k/2} d\rho$$

$$\geqq \sum_{j=0}^{\infty} \frac{1}{2^{j(k-1)}} \left(1-\frac{1}{2^j}\right)^{2^{jk}} 2^{jk} 2^{-(j+1)} A_j^k$$

$$\geqq \frac{1}{2e^k} \sum_{j=0}^{\infty} A_j^k.$$

Therefore the series (2. 2) is convergent. The theorem was proved.

Since the summability $|C, \alpha|_k$ implies the summability $|A|_k$, where $k \geqq 1$, $\alpha > -1$ (Flett [1] Theorem 2), the following theorem is a refinement of the first part of theorem 2 if the summability index k is restricted such as $1 \leqq k \leqq 2$.

THEOREM 3. *Let $\{\varphi_n(x)\}$ be an orthonormal system. If $1 \leqq k \leqq 2, \alpha > 1/2$ and if the series* (2. 2) *is convergent, then the orthonormal series $\sum a_n \varphi_n(x)$ is summable $|C, \alpha|_k$ almost everywhere.*

For the convenience we shall prove this theorem after the proof of the next two theorems.

THEOREM 4. *Let $k \geqq 1$ and $\alpha > -1$. For the Rademacher series* (2. 1), *if the series*

$$|\alpha|^k \sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{\sum_{j=1}^{n} (n-j+1)^{2(\alpha-1)} j^2 a_j^2\right\}^{k/2} + \sum_{n=1}^{\infty} \frac{|a_n|^k}{n^{k\alpha-k+1}} \tag{2.8}$$

is converegent, then the series (2. 1) *is summable $|C, \alpha|_k$ almost everywhere. And if the series* (2. 8) *is divergent the series* (2. 1) *is summable $|C, \alpha|_k$ almost nowhere.*

Since the second term of (2. 8) is a necessary condition for the summability $|C, \alpha|_k$ in a set of positive measure ([1] Theorem 3), it seems to us that the first term of (2. 8) should play an important rôle for the summability $|C, \alpha|_k$ if $\alpha \neq 0$.

PROOF OF THEOREM 4. Let us denote by $\sigma_n^\alpha(x)$ the n-th Cesàro mean of order α of the series (2. 1). Then by the well known formula we have

$$\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)$$
$$= \sum_{j=1}^{n} \frac{E_{n-j}^\alpha}{E_{n+1}^\alpha} \frac{\alpha j}{(n+1)(n-j+1)} a_j \psi_j(x) + \frac{1}{E_{n+1}^\alpha} a_{n+1} \psi_{n+1}(x),$$

where $E_n^\alpha = \binom{n+\alpha}{n} \sim n^\alpha$. Hence by the Khinchin inequality we get

$$\int_0^1 \sum_{n=1}^{\infty} n^{k-1} |\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)|^k dx$$

$$\leqq K\sum_{n=1}^{\infty} n^{k-1}\left\{\sum_{j=1}^{n}\left(\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}\right)^2 a_j^2+\left(\frac{1}{E_{n+1}^{\alpha}}\right)^2 a_{n+1}^2\right\}^{k/2}$$

$$\leqq K\sum_{n=1}^{\infty} n^{k-1}\left[\left\{\sum_{j=1}^{n}\left(\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}\right)^2 a_j^2\right\}^{k/2}+\left(\frac{1}{E_{n+1}^{\alpha}}\right)^k |a_{n+1}|^k\right]$$

which is easily majorated by K times the sum of the series (2.8). Hence the first part of Theorem was proved.

Now let us suppose that the $|C,\alpha|_k$ sum of the series (2.1) is uniformly bounded for $x\in E$, $|E|>0$. We have

$$K>\int_E\sum_{n=1}^{\infty} n^{k-1}|\sigma_{n+1}^{\alpha}(x)-\sigma_n^{\alpha}(x)|^k\,dx$$

$$=\int_E\sum_{n=1}^{\infty} n^{k-1}\left|\sum_{j=1}^{n}\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}a_j\psi_j(x)+\frac{1}{E_{n+1}^{\alpha}}a_{n+1}\psi_{n+1}(x)\right|^k dx.$$

Let N be a positive integer and replace $a_1, a_2, \ldots, a_{N-1}$ in the series (2.1) by zeros. This replacement has no influence on the summability, since

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sum_{j=1}^{N-1}\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}a_j\psi_j(x)\right|^k$$

$$\leqq K\sum_{n=1}^{\infty} n^{k-1}\left(\sum_{j=1}^{N}\frac{(n-j+1)^{\alpha-1}}{n^{\alpha+1}}j|a_j|\right)^k$$

$$\leqq K\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\sum_{j=1}^{N}j|a_j|$$

which is convergent. Therefore we may suppose that

$$K>\int_E\sum_{n=N}^{\infty} n^{k-1}\left|\sum_{j=N}^{n}\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}a_j\psi_j(x)+\frac{1}{E_{n+1}^{\alpha}}a_{n+1}\psi_{n+1}(x)\right|^k dx, \tag{2.9}$$

where $N=N(E)$ is determined by the Khinchin inequality:

$$\int_E\left|\sum_{j=N}^{n}\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}a_j\psi_j(x)+\frac{1}{E_{n+1}^{\alpha}}a_{n+1}\psi_{n+1}(x)\right|^k dx$$

$$\geqq K'\left\{\sum_{j=N}^{n}\left(\frac{E_{n-j}^{\alpha}}{E_{n+1}^{\alpha}}\frac{\alpha j}{(n+1)(n-j+1)}\right)^2 a_j^2+\left(\frac{1}{E_{n+1}^{\alpha}}\right)^2 a_{n+1}^2\right\}^{k/2} \tag{2.10}$$

From (2.9) and (2.10) we get immediately the convergence of the series (2.8), since the integer N may be replaced by 1 repeating the similar argument as above.

Thus the theorem was proved.

THEOREM 5. *Let $1 \leqq k \leqq 2$ and $\alpha > -1$. If the series (2.8) is convergent, then the orthonormal series $\sum a_n\varphi_n(x)$ is summable $|C,\alpha|_k$ almost everywhere.*

The proof is immediate by the same line as the proof of the first part of the preceding theorem using the Hölder inequality and the Parseval relation in place of using the Khinchin inequality. The detail may be omitted.

Now we are in a position to prove Theorem 3. For this purpose it is sufficient, by Theorem 5, to show that if one of the series (2.2) and (2.8) is convergent, so is the other under the condition $1 \leqq k \leqq 2$ and $\alpha > 1/2$.

Suppose that the series (2.2) is convergent. For the second term in (2.8),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|a_n|^k}{n^{k\alpha-k+1}} &= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \frac{|a_n|^k}{n^{k\alpha-k+1}} \\
&\leqq \sum_{j=0}^{\infty} \frac{1}{2^{j(k\alpha-k+1)}} \sum_{n=2^j}^{2^{j+1}-1} |a_n|^k \\
&\leqq \sum_{j=0}^{\infty} \frac{1}{2^{j(k\alpha-k+1)}} \left(\sum_{n=2^j}^{2^{j+1}-1} a_n^2 \right)^{k/2} 2^{j(1-k/2)} \\
&= \sum_{j=0}^{\infty} \frac{1}{2^{kj(\alpha-1/2)}} A_j^k \\
&\leqq \sum_{j=0}^{\infty} A_j^k
\end{aligned} \tag{2.11}$$

which is convergent as $\alpha > 1/2$. For the first term in (2.8),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{ \sum_{j=1}^{n} (n-j+1)^{2(\alpha-1)} j^2 a_j^2 \right\}^{k/2} \\
&\leqq K \sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{ \sum_{j=1}^{[n/2]} \cdots \right\}^{k/2} + K \sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{ \sum_{j=[n/2]+1}^{\infty} \cdots \right\}^{k/2} \\
&= K\cdot A + K\cdot B
\end{aligned} \tag{2.12}$$

say. Then by easy consideration

$$\begin{aligned}
A &\leqq \sum_{n=1}^{\infty} \frac{K}{n^{\alpha k+1}} \left\{ \sum_{j=1}^{[n/2]} n^{2(\alpha-1)} j^2 a_j^2 \right\}^{k/2} \\
&= K \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} \left\{ \sum_{j=1}^{[n/2]} j^2 a_j^2 \right\}^{k/2}
\end{aligned}$$

$$\leqq K\sum_{i=0}^{\infty}\frac{1}{2^{i(k+1)}}\sum_{n=2^i}^{2^{i+1}-1}\left\{\sum_{j=1}^{2^i} j^2 a_j^2\right\}^{k/2}$$

$$\leqq K\sum_{i=0}^{\infty}\frac{1}{2^{ki}}\left\{\sum_{j=1}^{2^i} j^2 a_j^2\right\}^{k/2}$$

$$= K\sum_{i=0}^{\infty}\frac{1}{2^{ki}}\left(\sum_{l=0}^{i-1}\sum_{j=2^l}^{2^{l+1}-1} j^2 a_j^2\right)^{k/2}$$

$$\leqq K\sum_{i=0}^{\infty}\frac{1}{2^{ki}}\left(\sum_{l=0}^{i-1} 2^{2l} A_l^2\right)^{k/2}$$

$$\leqq K\sum_{i=0}^{\infty}\frac{1}{2^{ki}}\sum_{l=0}^{i-1} 2^{kl} A_l^k \qquad \text{(as } 1\leqq k\leqq 2\text{)}$$

$$= K\sum_{l=0}^{\infty} A_l^k 2^{kl}\sum_{i=l+1}^{\infty}\frac{1}{2^{ki}}$$

$$\leqq K\sum_{l=0}^{\infty} A_l^k. \tag{2.13}$$

On the other hand, we get

$$B\leqq K\sum_{n=1}^{\infty}\frac{1}{n^{\alpha k+1}}\left\{\sum_{j=[n/2]+1}^{n}(n-j+1)^{2(\alpha-1)} j^2 a_j^2\right\}^{k/2}$$

$$\leqq K\sum_{n=1}^{\infty}\frac{1}{n^{\alpha k+1-k}}\left\{\sum_{j=[n/2]+1}^{n}(n-j+1)^{2(\alpha-1)} a_j^2\right\}^{k/2}$$

$$= K\sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}-1}\frac{1}{n^{\alpha k+1-k}}\left\{\sum_{j=[n/2]+1}^{n}(n-j+1)^{2(\alpha-1)} a_j^2\right\}^{k/2}$$

$$\leqq K\sum_{m=0}^{\infty}\frac{1}{2^{m(\alpha k+1-k)}}\sum_{n=2^m}^{2^{m+1}-1}\left\{\sum_{j=[n/2]+1}^{n}(n-j+1)^{2(\alpha-1)} a_j^2\right\}^{k/2}$$

$$\leqq K\sum_{m=0}^{\infty}\frac{1}{2^{m(\alpha k+1-k)}}\left\{\sum_{n=2^m}^{2^{m+1}-1}\sum_{j=[n/2]+1}^{n}(n-j+1)^{2(\alpha-1)} a_j^2\right\}^{k/2} 2^{m(1-k/2)}$$

(by the Hölder inequality)

$$\leqq K\sum_{m=0}^{\infty}\frac{1}{2^{m(\alpha k-k/2)}}\left\{\left(\sum_{j=2^{m-1}}^{2^m-1}\sum_{n=2^m}^{2j}+\sum_{j=2^m}^{2^{m+1}-1}\sum_{n=j}^{2^{m+1}-1}\right)(n-j+1)^{2(\alpha-1)} a_j^2\right\}^{k/2}$$

$$\leqq K\sum_{m=0}^{\infty}\frac{1}{2^{m(\alpha-1/2)k}}\{C_m+D_m\}^{k/2} \tag{2.14}$$

say. We have

$$C_m = \sum_{j=2^{m-1}}^{2^m-1} a_j^2 \sum_{n=2^m}^{2j} (n-j+1)^{2(\alpha-1)}$$

$$\leqq \sum_{j=2^{m-1}}^{2^m-1} a_j^2 \sum_{n=j}^{2j} (n-j+1)^{2(\alpha-1)}$$

$$\leqq K \sum_{j=2^{m-1}}^{2^{m+1}-1} a_j^2 j^{2\alpha-1} \qquad \text{(as } \alpha > 1/2\text{)}$$

$$\leqq K\, 2^{m(2\alpha-1)}(A_{m-1}^2 + A_m^2), \tag{2.15}$$

$$D_m = \sum_{j=2^m}^{2^{m+1}-1} a_j^2 \sum_{n=j}^{2^{m+1}-1} (n-j+1)^{2(\alpha-1)}$$

$$\leqq \sum_{j=2^m}^{2^{m+1}-1} a_j^2 \sum_{l=1}^{2^{m+1}-1} l^{2(\alpha-1)}$$

$$\leqq K \sum_{j=2^m}^{2^{m+1}-1} a_j^2\, 2^{m(2\alpha-1)}$$

$$= K\, 2^{m(2\alpha-1)} A_m^2. \tag{2.16}$$

Hence from (2.14), (2.15) and (2.16) we obtain

$$B \leqq K \sum_{m=0}^{\infty} \frac{1}{2^{m(\alpha-1/2)k}} \{2^{m(2\alpha-1)} A_{m-1}^2 + 2^{m(2\alpha-1)} A_m^2\}^{k/2}$$

$$\leqq K \sum_{m=0}^{\infty} \frac{1}{2^{m(\alpha-1/2)k}} 2^{m(\alpha-1/2)k} A_m^k$$

$$= K \sum_{m=0}^{\infty} A_m^k \tag{2.17}$$

which is also convergent.

Thus we get from (2.12), (2.13) and (2.17) the convergence of the first term in (2.8), that is, the convergence of (2.2) implies that of (2.8).

Now suppose that the series (2.8) is convergent. Then we have

$$\infty > \sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{ \sum_{j=1}^{n} (n-j+1)^{2(\alpha-1)} j^2 a_j^2 \right\}^{k/2}$$

$$\geqq K \sum_{n=1}^{\infty} \frac{1}{n^{\alpha k+1}} \left\{ \sum_{j=1}^{[n/2]+1} n^{2(\alpha-1)} j^2 a_j^2 \right\}^{k/2}$$

$$\geqq K\sum_{n=1}^{\infty}\frac{1}{n^{1+k}}\left(\sum_{j=1}^{[n/2]+1} j^2 a_j^2\right)^{k/2}$$

$$\geqq K\sum_{m=0}^{\infty}\frac{1}{2^{m(1+k)}}\sum_{n=2^m}^{2^{m+1}-1}\left(\sum_{j=1}^{[n/?]+1} j^2 a_j^2\right)^{k/2}$$

$$\geqq K\sum_{m=2}^{\infty}\frac{1}{2^{m(1+k)}}\,2^m\left(\sum_{j=1}^{2^{m-1}-1} j^2 a_j^2\right)^{k/2}$$

$$\geqq K\sum_{m=2}^{\infty}\frac{1}{2^{mk}}\,2^{mk}\left(\sum_{j=2^{m-2}}^{2^{m-1}-1} a_j^2\right)^{k/2}$$

$$= K\sum_{m=2}^{\infty} A_{m-2}^k.$$

Therefore the convergence of the series (2.8) implies the convergence of the series (2.2). The proof of Theorem 3 was completed.

Finally we shall give a proof of Theorem 1. Let us define

$$c_{2^n} = n^{-2/k} \qquad \text{for } n = 1, 2, 3, \dots,$$

and $\qquad c_j = 0 \qquad$ if j is not of the form 2^n.

The series

$$\sum c_j \psi_j(x) \tag{2.18}$$

is lacunary for every x. The series (2.18) is summable $|A|_k$ $(k \geqq 1)$ almost everywhere by Theorem 2, since we have

$$\sum_{n=0}^{\infty}\left(\sum_{j=2^n}^{2^{n+1}-1} c_j^2\right)^{k/2} = \sum_{n=1}^{\infty} c_{2^n}^k = \sum_{n=1}^{\infty} n^{-2}$$

which is convergent. And by Theorem 4 the series (2.18) is summable $|C, 1-1/k|_k$ almost nowhere for $k > 1$, since

$$\sum_{n=1}^{\infty}\frac{1}{n^k}\left\{\sum_{j=1}^{n}(n-j+1)^{-2/k} j^2 c_j^2\right\}^{k/2}$$

$$\geqq K\sum_{m=0}^{\infty}\frac{1}{2^{mk}}\sum_{n=2^m}^{2^{m+1}-1}\left\{\sum_{j=1}^{n}(n-j+1)^{-2k} j^2 c_j^2\right\}^{k/2}$$

$$= K\sum_{m=0}^{\infty}\frac{1}{2^{mk}}\sum_{n=2^m}^{2^{m+1}-1}\left\{\sum_{l=1}^{m}(n-2^l+1)^{-2/k}\,2^{2l}\,l^{-4/k}\right\}^{k/2}$$

$$\geqq K\sum_{n=0}^{\infty}\frac{1}{2^{mk}}\sum_{n=2^m}^{2^{m+1}-1}\left\{(n-2^m+1)^{-2/k}\,2^{2m}\,m^{-4/k}\right\}^{k/2}$$

$$\geqq K \sum_{m=0}^{\infty} \frac{1}{m^2} \sum_{n=2^m}^{2^{m+1}-1} (n-2^m+1)^{-1}$$

$$= K \sum_{m=0}^{\infty} \frac{1}{m^2} \sum_{n=1}^{2^m} n^{-1}$$

$$\geqq K \sum_{m=0}^{\infty} \frac{1}{m^2} \log 2^m$$

$$= K \sum_{m=0}^{\infty} \frac{1}{m}$$

which is divergent, and *a fortiori* the series (2. 8) is divergent for the indices $\alpha = 1 - 1/k$, and $k > 1$.
Therefore there exists a value of x such that the series (2. 18) is summable $|A|_k$ but not $|C, 1-1/k|_k$ $(k > 1)$.
This proved the theorem.

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