ON ORDER OF SUM OF THE SERIES OF ULTRASPHERICAL FUNCTIONS

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(Received October 2, 1957)

1. Let $f(x)$ be Lebesgue integrable in $(-1, 1)$.

The Legendre series for this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \tag{1.1}$$

where
$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Let
$$S_n(\cos\theta) = \sum_{\nu=0}^{n} a_\nu P_\nu(\cos\theta).$$

Wilson [1] has proved the following theorem:

If k denotes a number between 0 and 1/2 such that the integral $\int_0^\pi f(\cos\theta)(\sin\theta)^{k+1/2}\, d\theta$ exists and if $0 \leqq \alpha < \beta \leqq \pi$ then

$$\int_\alpha^\beta f(\cos\theta)\, P_n(\cos\theta) \sin\theta\, d\theta = o(n^{k-1/2}) \tag{1.2}$$

and

$$\left.\begin{aligned} S_n(\cos\theta) &= o(n^k) \\ &= o(\log n) \end{aligned}\right\} \begin{aligned} &0 < k \leqq 1/2, \\ &k = 0. \end{aligned} \tag{1.3}$$

We consider the generalised form of the series (1.1) as given by Kogbetliantz [2].

Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x), \tag{1.4}$$

where $P_n^{(\lambda)}(x)$ is the Gegenbauer's polynomial and

$$a_n = \frac{(n+\lambda)\Gamma(\lambda)}{\Gamma(1/2+\lambda)\Gamma(1/2)} \cdot \frac{\Gamma(n+1)\Gamma(2\lambda)}{\Gamma(n+2\lambda)} \int_{-1}^{1} (1-t^2)^{\lambda-1/2} f(t)\, P_n^{(\lambda)}(t)\, dt.$$

The series (1.4) reduces to (1.1) if $\lambda = 1/2$ and by putting $x = \cos\theta$ and making λ tend to 0, we get the trigonometric series as a limit-case.

Wilson's theorem will precisely be a particular case of the following more general theorem:

THEOREM: *If* $0 \leqq k \leqq \lambda$ *where* $0 < \lambda \leqq 2/3$ *and if the integral*

$$\int_{-1}^{1} \frac{f(x)}{(1-x^2)^{(1-\lambda-k)/2}} dx = \int_{0}^{\pi} f(\cos\theta)(\sin\theta)^{\lambda+k} d\theta$$

exists, then

$$S_n(\cos\theta) = \sum_{\nu=0}^{n} a_\nu P_\nu^{(\lambda)}(\cos\theta) = o(n^k) \quad 0 < k \leqq \lambda, \tag{1.5}$$

$$= o(\log n) \quad k = 0$$

for almost all θ *in* $(0, \pi)$ *excluding the end points.*

2. We require the following

LEMMA. *If* $0 \leqq \alpha < \beta \leqq \pi$, then,

$$\int_{\alpha}^{\beta} f(\cos\theta)\ P_n^{(\lambda)}(\cos\theta)\ (\sin\theta)^{2\lambda}\, d\theta = o(n^{\lambda+k-1}). \tag{2.1}$$

PROOF. Because

$$P_n^{(\lambda)}(-x) = (-1)^n P_n^{(\lambda)}(x), \tag{2.2}$$

it will be sufficient to prove (2.1) for the interval $0 \leqq \alpha < \beta \leqq \pi/2$. We may suppose $\alpha = 0$.

We write $$\int_0^{\beta} f(\cos\theta)\, P_n^{(\lambda)}(\cos\theta)(\sin\theta)^{2\lambda}\, d\theta = \int_0^{c/n} + \int_{c/n}^{\beta}$$

$$= J_1 + J_2, \text{ say.}$$

Now, Szegö [3] has derived the following orders for $P_n^{(\lambda)}(\cos\theta)$.

$$P_n^{(\lambda)}(\cos\theta) = \begin{cases} \theta^{-\lambda} O(n^{\lambda-1}) & c/n \leqq \theta \leqq \pi/2 \\ O(n^{2\lambda-1}) & 0 \leqq \theta \leqq c/n \end{cases} \tag{2.3}$$

λ being artitrary and real, $\lambda \neq 0, -1, -2, \ldots;\ c > 0$.
Therefore

$$|J_1| = \int_0^{c/n} |f(\cos\theta)(\sin\theta)^{\lambda+k}|(\sin\theta)^{\lambda-k} O(n^{2\lambda-1})\, d\theta, \qquad \text{using (2.3),}$$

$$= O(n^{2\lambda-1})\, o(n^{k-\lambda}), \qquad \text{using the property of } L\text{-integral,}$$

$$= o(n^{k+\lambda-1}). \tag{2.4}$$

It is well known [4] that

$$P_n^m(x) = \text{const.}\,(1-x^2)^{m/2}\, P_{n-m}^{(m+1/2)}(x), \tag{2.5}$$

where $P_n^m(x)$ is Legendre's associated function of order m, for which Hobson [5] has shown that, for $\eta \leqq \theta \leqq \pi - \eta$, (2.6) $$\frac{1}{n^m} P_n^m(\cos\theta) = \left(\frac{2}{n\pi\sin\theta}\right)^{1/2} \cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right] + O\left(\frac{1}{n^{3/2}}\right).$$

From (2.5) and (2.6) we find for the range $c/n \leqq \theta \leqq \pi/2$,

$$P_n^{(\lambda)}(\cos\theta) = \text{const.} (\sin\theta)^{-(\lambda-1/2)}(n+\lambda-1/2)^{\lambda-1/2}$$
$$\cdot\left[\left\{\frac{2}{(n+\lambda-1/2)\pi\sin\theta}\right\}^{1/2} \cos\left\{(n+\lambda)\theta - \frac{\pi}{4} + \frac{\pi}{2}(\lambda-1/2)\right\} + O\left(\frac{1}{n+\lambda-1/2}\right)^{3/2}\right]$$
$$= \text{const.} (\sin\theta)^{-(\lambda-1/2)}(n+\lambda-1/2)^{\lambda-1/2}.$$
$$\cdot\left[\left\{\frac{2}{(n+\lambda-1/2)\pi\sin\theta)}\right\}^{1/2} \sin\left\{(n+\lambda)\theta + \frac{\pi\lambda}{2}\right\} + O\left(\frac{1}{n+\lambda-1/2}\right)^{3/2}\right]$$
$$= O\left[(\sin\theta)^{-(\lambda-1/2)}\left\{n^{\lambda-1}(\sin\theta)^{-1/2}\sin[(n+\lambda)\theta + \pi\lambda/2] + O(n^{\lambda-2})\right\}\right]$$

(2.7) $$= O\left[(\sin\theta)^{-(\lambda-1/2)}\left\{n^{\lambda-1}(\sin\theta)^{1/2}\sin\omega_n + O(n^{\lambda-2})\right\}\right],$$

where $\omega_n = (n+\lambda)\theta + \pi\lambda/2$.

Henceforward, by A we will denote a constant, not always the same at each occurrence.

Now,

$$J_2 = A\int_{c/n}^{\beta} f(\cos\theta)(\sin\theta)^{2\lambda}(\sin\theta)^{-\lambda}\left(n+\lambda-\frac{1}{2}\right)^{\lambda-1}\sin\omega_n\, d\theta$$
$$+ O\left(n+\lambda-\frac{1}{2}\right)^{\lambda-2}\int_{c/n}^{\beta} f(\cos\theta)(\sin\theta)^{2\lambda}(\sin\theta)^{-(\lambda-1/2)}\, d\theta$$
$$= J_{2,1} + J_{2,2}, \text{ say.}$$

Consider $J_{2,1}$ first.

$$J_{2,1} = A\int_{c/n}^{\beta} f(\cos\theta)(\sin\theta)^{\lambda+k}\left(n+\lambda-\frac{1}{2}\right)^{\lambda-1}\sin\omega_n(\sin\theta)^{-k}\, d\theta.$$

Here, $(\sin\theta)^{-k}$ is a monotone decreasing function and so the second mean-value theorem can be applied to this integral. We get

$$J_{2,1} = A\cdot O(n^{k+\lambda-1})\int_{c/n}^{\eta} f(\cos\theta)(\sin\theta)^{\lambda+k}\sin\omega_n\, d\theta \qquad \text{where } c/n \leqq \eta \leqq \beta$$
$$= o(n^{k+\lambda-1}), \qquad \text{by Riemann-Lebesgue theorem.}$$

As regards $J_{2,2}$, it is $o(n^{k+\lambda-1})$ for all k, whether greater than or less than 1/2.

(2.8) Thus $J_2 = o(n^{k+\lambda-1})$.

Combining (2.4) and (2.8), the lemma is proved.

3. Proof of the Theorem. We have

$$S_n(x) = \sum_{\nu=0}^{n} a_\nu P_\nu^{(\lambda)}(x).$$

The recurrence formula satisfied by the ultraspherical functions is

$$n P_n^{(\lambda)}(x) = 2(n+\lambda-1)\, x P_{n-1}^{(\lambda)}(x) - (n+2\lambda-2)\, P_{n-2}^{(\lambda)}(x).$$

Using Christoffel's method, it is easy to show that

$$\sum_{\nu=0}^{n} \frac{(\nu+\lambda)\,\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)} \cdot P_\nu^{(\lambda)}(x)\, P_\nu^{(\lambda)}(t) = \frac{\Gamma(n+2)}{2\Gamma(n+2\lambda)} \cdot \frac{P_{n+1}^{(\lambda)}(x) P_n^{(\lambda)}(t) - P_{n+1}^{(\lambda)}(t) P_n^{(\lambda)}(x)}{x-t}$$

Now

$$S_n(\cos\theta) = \sum_{\nu=0}^{n} a_\nu P_\nu^{(\lambda)}(\cos\theta)$$

$$= \sum_{\nu=0}^{n} (\nu+\lambda) \frac{\Gamma(\lambda)\Gamma(\nu+1)\Gamma(2\lambda)}{\Gamma(1/2+\lambda)\Gamma(1/2)\Gamma(\nu+2\lambda)} \int_0^\pi f(\cos\phi)\,(\sin\phi)^{2\lambda} P_\nu^{(\lambda)}(\cos\theta)\, P_\nu^{(\lambda)}(\cos\phi)\, d\phi$$

$$= \frac{\Gamma(\lambda)\,\Gamma(2\lambda)}{\Gamma(1/2+\lambda)\Gamma(1/2)} \int_0^\pi f(\cos\phi)\,(\sin\phi)^{2\lambda} \sum_{\nu=0}^{n} \frac{(\nu+\lambda)\,\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)} P_\nu^{(\lambda)}(\cos\theta)\, P_\nu^{(\lambda)}(\cos\phi)\, d\phi.$$

We get

$$\frac{2\Gamma(1/2+\lambda)\,\Gamma(1/2)}{\Gamma(\lambda)\,\Gamma(2\lambda)} S_n(\cos\theta)$$

$$= \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} \int_0^\pi f(\cos\phi)\,(\sin\phi)^{2\lambda} \frac{P_{n+1}^{(\lambda)}(\cos\theta)\, P_n^{(\lambda)}(\cos\phi) - P_{n+1}^{(\lambda)}(\cos\phi)\, P_n^{(\lambda)}(\cos\theta)}{\cos\theta - \cos\phi}\, d\phi.$$

So,

$$(3.1) \quad \frac{2\Gamma(1/2+\lambda)\Gamma(1/2)\Gamma(n+2\lambda)}{\Gamma(\lambda)\,\Gamma(2\lambda)\,\Gamma(n+2)} S_n(\cos\theta) = \int_0^\pi f(\cos\phi)\,(\sin\phi)^{2\lambda} F(n,\theta,\phi)\, d\phi, \text{ say}$$

$$= \int_0^{\theta-\mu} + \int_{\theta-\mu}^{\theta+\mu} + \int_{\theta+\mu}^{\pi}$$

$$(3.2) \quad = I_1 + I_2 + I_3, \qquad \text{say.}$$

Write $\qquad h(\cos\phi) = \dfrac{f(\cos\phi)}{\cos\theta - \cos\phi}$.

If $0 \leqq \phi \leqq \theta - \mu$, we have $h(\cos\phi)\,(\sin\phi)^{\lambda+k}$ integrable.

Now,

$$I_1 = P_{n+1}^{(\lambda)}(\cos\theta) \int_0^{\theta-\mu} h(\cos\phi)\,(\sin\phi)^{2\lambda} P_n^{(\lambda)}(\cos\phi)\, d\phi$$

$$- P_n^{(\lambda)}(\cos\theta)\int_0^{\theta-\mu} h(\cos\phi)\, P_{n+1}^{(\lambda)}(\cos\phi)(\sin\phi)^{2\lambda}\, d\phi$$

$$= P_{n+1}^{(\lambda)}(\cos\theta)o(n^{k+\lambda-1}) - P_n^{(\lambda)}(\cos\theta)\, o(n^{\lambda+k-1}) \qquad \text{from the lemma.}$$

But for $0 < \lambda < 1$ and $0 \leqq \theta \leqq \pi$, we have [3, p. 166]

(3.3) $$(\sin\theta)^{\lambda} |P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1} n^{\lambda-1}.$$

Therefore $$|I_1| = O\{n^{\lambda-1}(\sin\theta)^{-\lambda}\}o(n^{\lambda+k-1})$$

(3.4) $$= o\{n^{2\lambda+k-2}\} \qquad \text{if } \theta \text{ is neither 0 nor } \pi.$$

(3.5) Similarly $I_3 = o\{n^{2\lambda+k-2}\}$.

The theorem will evidently be established if we simply show that $I_2 = o(n^{-2+2\lambda}\log n)$ for almost all θ in $(0, \pi)$.

We have

(3.6) $$I_2 = \int_{-\mu}^{\mu} F(n, \theta, \theta + t) f\{\cos(\theta + t)\}\{\sin(\theta + t)\}^{2\lambda}\, dt.$$

Further, it can be readily seen by taking $f(\cos\phi) \equiv 1$ in (3.1) that

(3.7) $$\int_{-\mu}^{\mu} F(n, \theta, \theta + t)\{\sin(\theta + t)\}^{2\lambda} dt \to \frac{2\Gamma(1/2 + \lambda)\Gamma(1/2)\Gamma(n + 2\lambda)}{\Gamma(\lambda)\Gamma(2\lambda)\Gamma(n + 2)}.$$

Let $$\psi(t) = \{f[\cos(\theta + t)] - f(\cos\theta)\}\{\sin(\theta + t)\}^{2\lambda}.$$

So, in view of (3.7), we have to prove that

$$I_2' = \int_{-\mu}^{\mu} F(n, \theta, \theta + t)\psi(t) dt = o\left(\frac{\log n}{n^{2-2\lambda}}\right) \qquad \text{almost everywhere.}$$

Suppose $\theta + \mu \leqq \pi/2$, which does in no way harm the generality of the proof.

Let $$I_2' = \left(\int_{-\mu}^{-1/n} + \int_{-1/n}^{1/n} + \int_{1/n}^{\mu}\right) F(n, \theta, \theta + t)\, \psi(t)\, dt$$

$$= L_1 + L_2 + L_3, \qquad \text{say.}$$

(3.8) μ may be so chosen that $\theta - 2\mu > 0$ and $\theta + 2\mu < \pi$.

Now, in view of (3.3) and (3.8)

$$\frac{\Gamma(n+2)}{2\Gamma(n+2\lambda)} F(n, \theta, \theta + t) = \sum_{\nu=0}^{n} \frac{(\nu + \lambda)\Gamma(\nu + 1)}{\Gamma(\nu + 2\lambda)} P_\nu^{(\lambda)}(\cos\theta) P_\nu^{(\lambda)}\{\cos(\theta + t)\}$$

$$= \sum_{\nu=0}^{n} O(\nu^{2-2\lambda}) O(\nu^{\lambda-1})\, O(\nu^{\lambda-1})$$

$$= \sum_{\nu=0}^{n} O(1) = O(n).$$

Therefore,

$$|L_2| = O\left(\int_{-1/n}^{1/n} n^{2\lambda-1}\psi(t)\,dt\right) = O(n^{2\lambda-1})o\left(\frac{1}{n}\right)$$

$$= o(n^{2\lambda-2}) \tag{3.9}$$

almost everywhere.

To evaluate L_1 and L_3, we use (2.7) again.

So, for $1/n \leqq |t| \leqq \mu$,

$$F(n,\theta,\theta+t) - \frac{A(n+1)^{\lambda-1}n^{\lambda-1}}{\pi(\sin\theta)^{\lambda}\{\sin(\theta+t)\}^{\lambda}}$$

$$\cdot[\sin(n+\lambda+1/2)t\,\mathrm{cosec}\,(t/2) - \mathrm{cosec}\,(\theta+t/2)\sin\{(2n+2\lambda+1)(\theta+t/2)+\lambda\pi\}]$$

$$= \mathrm{cosec}\,t/2\cdot O(n^{2\lambda-3}) = O(n^{2\lambda-2}).$$

Thus $\displaystyle L_3 = \int_{1/n}^{\mu} F(n,\theta,\theta+t)\psi(t)\,dt$

$$= \frac{A(n+1)^{\lambda-1}n^{\lambda-1}}{\pi(\sin\theta)^{\lambda}}\left\{\int_{1/n}^{\mu}\frac{\sin(n+\lambda+1/2)t}{\sin t/2}\cdot\frac{\psi(t)}{[\sin(\theta+t)]^{\lambda}}\,dt\right.$$

$$\left. - \int_{1/n}^{\mu}\frac{\sin[(2n+2\lambda+1)(\theta+t/2)+\pi\lambda]}{\sin(\theta+t/2)[\sin(\theta+t)]^{\lambda}}\psi(t)\,dt\right\} + O(n^{2\lambda-2}) \tag{3.10}$$

$$= A(n+1)^{\lambda-1}n^{\lambda-1}\{L_{3,1} - L_{3,2}\} + O(n^{2\lambda-2}). \tag{3.11}$$

Consider $L_{3,2}$ first. In this, $\dfrac{\psi(t)}{\sin(\theta+t/2)\{\sin(\theta+t)\}^{\lambda}}$ being integrable, $L_{3,2} = o(1)$ by Riemann-Lebesgue theorem.

Now,

$$L_{3,1} = \int_{1/n}^{\mu}\frac{\sin(n+\lambda+1/2)}{\sin(t/2)}\chi(t)\,dt$$

where $\chi(t) = \dfrac{\psi(t)}{\{\sin(\theta+t)\}^{\lambda}}$ is integrable in $(0,\mu)$.

Evidently, $L_{3,1}$ is a Dirichlet's integral and so according to Hardy's theorem

$$L_{3,1} = o(\log n) \text{ almost everywhere.}$$

From (3.11) we find that

$$L_3 = o\left(\frac{\log n}{n^{2-2\lambda}}\right). \tag{3.12}$$

L_1 behaves in the same way as L_3.

From (3.9) and (3.12) we see that the theorem is established.

It can be observed here that it shall be possible to further generalize the above theorem in the case of series of Jacobi polynomials.

I am highly indebted to Dr. B. N. Prasad for his kind guidance in the preparation of this paper.

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