A REMARK ON THE INVARIANTS OF W^*-ALGEBRAS[1)]

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The purpose of this paper is to supplement the theory of invariants of W^*-algebras accomplished by [1], [5], [6], [9] and [10] etc., with some considerations and to show some spatial isomorphism theorems.

Through the following discussion $M^\natural$ denotes the center of a W^*-algebra M on a Hilbert space H. We denote by $z(e)$ the central envelope of a projection e. By an isomorphism, we mean a *-preserving isomorphism. Ω denotes always the spectrum of $M^\natural$ and Ω_{eH} the spectrum of Me.

1. Definition of the Invariant. We employ the invariant of M in the same sense as in Pallu de la Barrière [1] if M is finite with finite commutator and denote it by $\overline{C}(t)$.

Next, let M be a W^*-algebra and α an infinite cardinal. A central projection e is called a homogeneous projection of type F_α if it is a "projection of type S_α" in the sense of Griffin [5] and "projection de uniforme d'ordre α" in [1] and e is called a homogeneous projection of type C_α if it is an "α-dimensional projection" in the sense of Griffin [6] and "homogeneous projection of order α" in Suzuki [13].

The following multiplicity theorem of a W^*-algebra is well known (cf. [1], [5], [6], [13]).

THEOREM 1. *Let M be a W^*-algebra on a Hilbert space H and π_1 (resp. π_2) the set of infinite cardinal α for which there exists a homogeneous projection of type F_α (resp. C_α). Then there exists a family of orthogonal central projections $\{e_\alpha\}_{\alpha\in\pi_1}$ (resp. $\{e_\beta\}_{\beta\in\pi_2}$) such that*

$$1 = e_1 + \sum\nolimits_{\alpha\in\pi_1} e_\alpha + \sum\nolimits_{\beta\in\pi_2} e_\beta$$

where $e_\alpha(\alpha \in \pi_1$ or $\pi_2)$ is a maximal homogeneous projection of type F_α or of type C_α and e_1 a maximal central finite projection. This decomposition is unique.

We denote by K_α an open and closed set in Ω corresponding to e_α in Theorem 1. A function $p(t)$ defined on a dense open set $\bigcup_{\alpha\in\{1\}\cup\pi_1\cup\pi_2} K_\alpha$ in Ω is called *the algebraic invariant of M*, if

$$p(t) = \alpha \qquad \text{for } t \in K_\alpha.$$

Now let $\{p_1, p_2\}$ be an orthogonal central decomposition where M_{p_1} is finite with finite commutator and M_{p_2} does not contain such coupled component. Denote by $\overline{C}(t)$, $p(t)$, and $p'(t)$ the invariant of M_{p_1}, the algebraic invariants

1) This is a part of the author's graduation thesis in March 1957.

of M_{p_2} and M'_{p_2} respectively. $p(t)$ and $p'(t)$ are defined on dense open sets Ω_1, Ω_2 in Ω_{p_2H}.

We define *the invariant of M* as follows;

$$C(t) = \overline{C}(t) \qquad \text{for } t \in \Omega_{p_1H}$$
$$= (p(t), p'(t)) \text{ for } t \in \Omega_1 \cap \Omega_2,$$

where $(p(t), p'(t))$ means a formal coupled function on $\Omega_1 \cap \Omega_2$. $C(t)$ is defined on a dense open set of Ω.

2. **Spatial Isomorphism Theorem.** The following lemma is due to Pallu de la Barrière [1].

LEMMA 1. *Let M be a finite W^*-algebra with finite commutator and $\overline{C}(t)$ the invariant of M. For any projection $e' \in M'$, let $\overline{C}_{e'H}(t)$ be the invariant of Me'. Then we have*

$$\overline{C}_{e'H}(t) = \overline{C}(t)\ e'^{\natural}(t) \text{ for all } t \in \Omega_{e'H} \text{ and } \overline{C}(t) \neq \infty.$$

LEMMA 2. *Let M_1 and M_2 be properly infinite W^*-algebras on each Hilbert space H_1 and H_2 with finite commutators and θ an isomorphism between them. There exists a continuous function $\gamma(t)$ defined on the common spectrum Ω of $M_i^{\natural}, i = 1, 2,$ and ranging over $[0, \infty]$ such that for any finite projection $e \in M_1$*

$$\overline{C}_{\theta(e)H_2}(t) = \gamma(t)\overline{C}_{eH_1}(t) \text{ for all } t \in \Omega_{eH_1} = \Omega_{\theta(e)H_2},$$

where $\overline{C}_{eH_1}(t)$ and $\overline{C}_{\theta(e)H_2}(t)$ are the invariants of M_{1e} and $M_{2\theta(e)}$.

PROOF. By J. Dixmier [4: Proposition 2] one can easily verify that there exists a W^*-algebra N on a Hilbert space K and two projections f_1', f_2' in N' whose central envelopes are the identity such that θ may be identified with the isomorphism

$$\theta : a'_{f_1} \to a'_{f_2} \qquad \text{for all } a \in N.$$

Thus, we can assume that $M_1 = N'_{f_1}$, $M_2 = N'_{f_2}$. Since f'_i $(i = 1, 2)$ are finite, $f'_0 = f'_1 \vee f'_2$ is finite, too. Take a finite projection $e \in N'_{f_1}$. Without loss of generality we may assume that $z(e) = 1$ in N'_{f_1} (i. e. $= f'_1$). We find a projection $e_0 \in N$ with $e = e_0 f'_1$; e_0 is finite in N with $z(e_0) = 1$. As $N'_{f_0'}$ and $N'_{f_0' e_0'}$ are finite, we can consider canonical applications $\natural_1$ and $\natural_2$ in $N'_{f_0'}$ and $N'_{f_0' e_0'}$, respectively. Then, applying lemma 1 to $N'_{f_0' e_0}$, we have

$$\overline{C}_{eH_1}(t) = \overline{C}_{e_0 f_1' K}(t) = \overline{C}_{e_0 f_0' K}(t)(e_0 f_1')^{\natural_2}(t)$$
$$\overline{C}_{\theta(e)H_2}(t) = \overline{C}_{e_0 f_2' K}(t) = \overline{C}_{e_0 f_0' K}(t)\,(e_0 f_2')^{\natural_2}(t)$$

for $t \in \Omega_{e_0 f_1' K} = \Omega_{e_0 f_2' K}$ such as $\overline{C}_{e_0 f_0' K}(t) \neq \infty$. Since $N'_{f_0'}$ is isomorphic to $N'_{e_0 f_0'}$ we get the identification $(e_0 f_i')^{\natural_2} = f_i'^{\natural_1}$ $(i = 1, 2)$. Moreover, by an isomorphism between $N'_{f_1'}$ and $N'_{f_0'}$, we get the further identification, that is, $\{f_i'^{\natural_1}(t)\}_{i=1,2}$ can be considered as continuous functions on $\Omega_{f_1' K}$. Therefore putting $\gamma(t) = f_2'^{\natural_1}(t)/f_1'^{\natural_1}(t)$, we have a continuous function on $\Omega_{f_1' K}$. Q. E. D.

We are now going to prove our main

THEOREM 2. *Let M_1 and M_2 be W^*-algebras and θ an isomorphism between them. By the isomorphism θ we may identify two spectra Ω_1, Ω_2 of $M_1^\natural$, $M_2^\natural$ respectively. Suppose M_1 and M_2 have the same invariant $C(t)$. Then*

1° *if $C(t) \neq (\alpha, 1)$ for all infinite cardinal α, θ is spatial,*

2° *if $C(t) = (\alpha, 1)$, and $\gamma(t) = 1$,* then *θ is spatial.*

PROOF. Following the same notations as in the proof of Lemma 2 we may assume

$$\theta: a'_{f_1} \rightarrow a'_{f_2} \qquad \text{for all } a \in N$$

where $z(f'_1) = z(f_2') = 1$. Hence, to prove θ being spatial is reduced to prove that $f_1' \sim f_2'$ mod N', so that it suffices to prove this fact in the following cases, separately.

Case (1°) $C(t) \neq (\alpha, 1)$.

a. f_1', f_2' are finite. In this case N_{f1}' is finite, otherwise this yields an excluded case. $f_0' = f_1' \vee f_2'$ is finite, too. Since N_{f0}' becomes finite we have, by lemma 1 applying to N_{f0}',

$$\overline{C}'_{f_1K}(t) = \overline{C}'_{f_0K}(t) f_1'^\natural(t) \text{ and } \overline{C}'_{f_2K}(t) = \overline{C}'_{f_0K}(t) f_2'^\natural(t)$$

for $t \in \Omega'_{f_1K}$ and $\overline{C}'_{f_0K}(t) \neq \infty$. Therefore $f_1'^\natural(t) = f_2'^\natural(t)$ for such t. But as N'_{f_0} is isomorphic to N'_{f_i}, we get $f_1'^\natural(t) = f_2'^\natural(t)$ over a dense open set in Ω'_{f_0K}, whence $f_1'^\natural = f_2'^\natural$. We get $f'_1 \sim f'_2$ mod N'_{f_0}, $f_1' \sim f_2'$ mod N'.

b. f_1', f'_2 are homogeneous projections of type $C_{\aleph_0}$ in N'_{f_1} aud N'_{f_2}. We may assume that $N'^\natural_{f_i}$ $(i = 1, 2)$ are countably decomposable, so that f_1' and f'_2 are countably decomposable in N'. Then we have

$$f'_1 = \sum_{n=1}^{\infty} e'_n \qquad \text{with } e'_n \prec f'_2 \text{ mod } N'.$$

On the other hand we may write $f'_2 = \sum_{n=1}^{\infty} f'_{2n}$ where $f'_2 \sim f'_{2n}$ for all n. Therefore $f'_1 \prec f'_2$; by symmetry it follows $f'_1 \sim f'_2$ mod N'.

c. f'_1, f'_2 are homogeneous projections of type F_α (or C_α) in N'_{f_i} $(i = 1, 2)$. (In case of type C_α we assume $\alpha > \aleph_0$). We may consider as in case *b*, that $N'^\natural_{f_i}$ are countably decomposable for $i = 1, 2$. Then we can find an infinite family $\{e'_i\}_{i \in I}$ of orthogonal, equivalent, finite (resp. cyclic) projections in N'_{f_1} such that $f'_1 = \sum_{i \in I} e'_i$, where the cardinal of I is α. Now take a fixed projection e'_{i_0} and a central projection g such that $ge'_{i_0} \succ gf_2'$ and $(1-g)e'_{i_0} \prec (1-g)$ f_2'. In both cases $ge'_{i_0} \succ gf_2'$ is impossible except for $g = 0$. Hence $e'_{i_0} \prec f_2'$. Take a maximal family $\{\overline{e}'_j\}_{j \in J}$ of orthogonal, equivalent, finite (resp. cyclic) projections such that $\overline{e}_j'$ is contained in f_2' and equivalent to e'_{i_0}. Notice that we may consider J as an infinite index set. Then we can choose a central projection h such that $\sum_{j \in J} h\overline{e}'_j \sim hf'_2 \neq 0$. Therefore the cardinal of J is α, which implies that $f'_1 = \sum_{i \in I} e'_i \sim \sum_{j \in J} \overline{e}'_j \leqq f'_2$. By symmetry, this shows $f'_1 \sim f'_2$

mod N'.

Case (2°), $C(t) = (\alpha, 1)$.

Put $f'_0 = f'_1 \vee f'_2$, then f'_0 is finite. If $\gamma(t) = 1$, $f'_1 \sim f'_2$ mod $N'_{f'_0}$ from the definition of $\gamma(t)$. Hence $f'_1 \sim f'_2$ mod N'. Q. E. D.

A spatial isomorphism between two W^*-algebras also induces an isomorphism between their commutators and these two isomorphisms coincide each other on the center. For the inverse statement we get

THEOREM 3. *Let M_1 and M_2 be W^*-algebras which do not contain such coupled components as* $(\mathrm{II}_\infty, \mathrm{II}_1)$ *and* $(\mathrm{II}_1, \mathrm{II}_1)$. *Suppose that θ and θ' are isomorphisms between M'_1 and M_2, M'_1 and M'_2 respectively and coincides each other on the center of M_1. Then θ is spatial.*

For the proof we need the following

LEMMA 3. *Let M_1 and M_2 be W^*-algebras with commutative commutators. Then any isomorphism between M_1 and M_2 is spatial.*

PROOF OF THEOREM 3. Without loss of generality we may restrict our proof to each separated coupled components which (M_1, M'_1) contain. But since, except for the cases of $(I_\infty$, finite) and (finite and type I, finite), the invariant of M_1 is constructed by couplings of algebraic invariants and theorem 2 may be applicable it is sufficient to deal with the above excluded cases. Now take an abelian projection e of M_1 with $z(e) = 1$. Then θ induces an isomorphism θ_1 between M_{1e} and $M_{2\theta(e)}$. For the commutator of M_{1e} we define θ'_1 as $\theta'_1(a'_e) = \theta'(a')\theta(e)$. Then one verifies easily that this is an isomorphism between M'_{1e} and $M'_{2\theta(e)}$. By lemma 3, θ'_1 is spatial. Moreover θ_1 is also spatial because $M_{1e} = M'^{\natural}_{1e}$. Hence $\overline{C}_{\theta(e)H_2}(t) = \overline{C}_{eH_1}(t)$ for $t \in \Omega_{eH_1} = \Omega_{\theta(e)H_2}$. Therefore, if it is the case of $(I_\infty$, finite) $\gamma(t) = 1$ on Ω_{eH_1} by Lemma 2. But, as $z(e) = 1$ we have $\gamma(t) = 1$ on Ω. That is, θ is spatial. If M_1 is finite and type I with finite commutator we have $\overline{C}'_{eH_1}(t) = \overline{C}'_{\theta(e)H_2}(t)$, where $\overline{C}'_{eH_1}(t)$, $\overline{C}'_{\theta(e)H_2}(t)$ denote the invariants of M'_{1e} and $M'_{2\theta(e)}$. By lemma 1 applying M'_1 and M'_2 we get

$$\overline{C}'_{eH_1}(t) = \overline{C}'_1(t)e^{\natural}(t) \quad \text{for} \quad t \in \Omega_{eH_1} \text{ and } \overline{C}'_1(t) \neq \infty,$$

$$\overline{C}'_{\theta(e)H_2}(t) = \overline{C}'_2(t)\theta(e)^{\natural}(t) \text{ for } t \in \Omega_{\theta(e)H_2} \text{ and } \overline{C}'_2(t) \neq \infty.$$

Therefore we get $\overline{C}'_1(t) = \overline{C}'_2(t)$ on Ω considering $z(e) = 1$, which implies $\overline{C}_1(t) = \overline{C}_2(t)$. Hence θ is spatial.

REMARK. Theorem 3 fails in the excluded cases. In the case of (II_1, II_1), take a standard approximately finite factor M on a separable Hilbert space H. Let R^2 be a two-dimensional Euclidean space and construct $H \otimes R^2$, tensor product of H and R^2. Consider an ampliation from M to $M \otimes 1$ over $H \otimes R^2$. Then $(M \otimes 1)' = M' \otimes B(R^2)$ is also an approximately finite factor by Misonou [10], and so $(M \otimes 1)'$ is isomorphic to M'. Thus M is coupled isomorphic to $M \otimes 1$ in the sense of theorem 3. But $\overline{C}(t) = 2\overline{C}_1(t) \neq \overline{C}_1(t)$ where $\overline{C}(t)$ and $\overline{C}_1(t)$ denote the invariants of $M \otimes 1$ and M respectively, whence M is not spatially isomorphic to $M \otimes 1$. In the case (II_∞, II_1), we shall only refer to the paper [14], [7] (an example of the W^*-algebra with a non-unitarily induced center-elementwise-invariant automorphism).

REFERENCES

[1] R. PALLU DE LA BARRIÈRE, Sur les algbères d'opérateurs dans les espaces hilbertiens, Bull. Soc. Math. France, 82 (1954), 1-52.

[2] J. DIXMIER, Sur certains espaces considérés par M. H. Stone, Summa Brad. Math., 11(1950), 151-182.

[3] ———, Applications ♮ dans les anneaux d'opérateurs, Compositio Math., 10 (1952), 1-55.

[4] ———, Sur les anne ux d'opérateurs dans les espaces hilbertiens, C. R. Acad. Sci. Paris, 238 (1954), 439-441.

[5] E. L. GRIFFIN, Some contributions to the theory of operator algebras I, Trans. Amer. Math. Soc., 75 (1953), 471-504.

[6] ———, Some contributions to the theory of operator algebras II, Trans. Amer. Math. Soc., 79 (1955), 386-400.

[7] R. V. KADISON, Isomorphisms of factors of infinite type, Canad. Journ. Math., 7 (1955), 322-327.

[8] I. KAPLANSKY, Quelques résultats sur les anneaux d'opérateurs, C. R. Acad. Sci. Paris, 231(1950), 485-486.

[9] Y. MISONOU, Unitary equivalence of factors of type III, Proc. Japan Acad., 29(1952), pp. 482-485.

[10] ———, On the direct product of W^*-algebras, Tôhoku Math. Journ., 6 (1954), 189-204.

[11] J. VON NEUMANN, On rings of operators, Ann. Math., 37(1936), 116-226.

[12] ———, On rings of operators IV, Ann. Math., vol. 44 (1943), 208-248.

[13] N. SUZUKI, On the invariants of W^*-algebras, Tôhoku Math. Journ., 7 (1955), 177-185.

[14] ———, On automorphisms of W^*-algebras leaving the center elementwise invariant, Tôhoku Math. Journ., 7(1955), 186-191.

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