A CHARACTERIZATION OF DISTRIBUTIVE LATTICES

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1. Introduction. Iseki [5] has shown that a lattice L is distributive if and only if every meet irreducible ideal of L is prime. A trivial extension of Iseki's proof yields the result that a lattice L is distributive if and only if every completely irreducible ideal is prime. The main purpose of this note is to obtain a more general form of the latter theorem in which we do not assume the existence of any lattice operations in L: i. e., we obtain a necessary and sufficient condition that an arbitrary partially ordered set be a distributive lattice. Our criterion reduces to that of Iseki when the existence of the lattice operations in P is assumed. Our main tool is the concept of a "dual imbedding operator" (for the basic properties of imbedding operators see [2, 4, 6, 7]) which we use to obtain a generalization of a representation theorem of Birkhoff and Frink [1]. Our main result is then deduced as a consequence of this representation theorem. As a corollary of our result, we obtain, without employing the lattice operations, a characterization of a Boolean algebra in terms of its order properties.

2. Preliminaries. Let P be a partially ordered set with respect to a relation $\leqq$. We assume always that P contains least and greatest elements 0 and I respectively. We shall make fundamental use of the following concept.

DEFINITION. Let ψ be a function on the set 2^P of all subsets of P into 2^P. We say that ψ is a *dual imbedding operator* on P if and only if

(i) $A \subset \psi(A)$ for all $A \subset P$,

(ii) $A \subset B$ implies $\psi(A) \subset \psi(B)$,

(iii) $\psi[\psi(A)] = \psi(A)$ for all $A \subset P$,

(iv) If $\{x\}$ denotes the set consisting only of the element x of P, then $\psi(\{x\}) = \{y \in P | y \geqq x\}$.

If $\psi(A) = A$, we say that A is a *ψ-ideal* of P. The set $\{y \in P | y \geqq x\}$ is a ψ-ideal for any dual imbedding operator ψ, and will be denoted by J_x. It is readily verified that if A is a ψ-ideal of P and $x \in A$, then $J_x \subset A$. A ψ-ideal A is *completely irreducible* if and only if $A \neq P$ and A is not the intersection of any set of ψ-ideals, each distinct from A. A dual imbedding operator ψ is called *inductive* if and only if for any chain $\mathfrak{A}$ of ψ-ideals, $\bigcup\{A | A \in \mathfrak{A}\}$ is a ψ-ideal (properties of such operators are discussed in [6]). A standard application of Zorn's lemma yields the following result.

LEMMA 1. *If ψ is an inductive dual imbedding operator on P, and K is a ψ-ideal not containing the element x of P, then there exists a ψ-ideal M such that $K \subset M$ and M is maximal with respect to not containing x.*

We also need the following lemma, the dual of which is proved in [6 Lemma G].

LEMMA 2. *A ψ-ideal M in P is completely irreducible if and only if there exists $x \in P$ such that M is maximal with respect to not containing x.*

If $S \subset P$, let $S^* = \{y \in P | y \geqq x \text{ for all } x \in S\}$, and let $S^+ = \{y \in P | y \leqq x \text{ for all } x \in S\}$. We write S^{+*} for the set $(S^+)^*$. If S consists only of the two elements $\{x, y\}$, we write $\{x, y\}^*$ for S^*.

We define an operator λ on an arbitrary partially ordered set P as follows:

$$\lambda(A) = \bigcup \{F^{+*} | F \text{ is a finite subset of } A\}.$$

It is readily verified that λ is an inductive dual imbedding operator (the λ-ideals are precisely the dual ideals of Frink [3]). If P is a lattice, its λ-ideals are simply its dual ideals in the usual sense; i.e., $\lambda(A)$ is the dual ideal generated by A.

We say that two partially ordered sets P and $\bar{P}$ are *isomorphic* if and only if there exists a 1:1 mapping f of P onto $\bar{P}$ such that $f(a) \leqq f(b)$ if and only if $a \leqq b$.

We now obtain a representation theorem which generalizes a result of Birkhoff and Frink [1; Theorem 11, p. 307]. Let $\Omega(\psi)$, or simply Ω, denote the set of all completely irreducible ψ-ideals of P. Let the set 2^Ω of all subsets of Ω be partially ordered by set inclusion. For each $x \in P$, let $T_\psi(x) = \{M \in \Omega | x \in M\}$. Then we have

THEOREM 1. *If ψ is an inductive dual imbedding operator on P, then the mapping $x \to T_\psi(x)$ is an isomorphism of P into 2^Ω.*

PROOF. To show that the above mapping is 1:1, suppose that $x \in P$, $y \in P$, and $x \neq y$. Then either $x \notin J_y$ or $y \notin J_x$. Suppose the latter: then by Lemma 1 there exists a ψ-ideal $M \supset J_x$ such that M is maximal with respect to not containing y. By Lemma 2, $M \in \Omega(\psi)$. Thus $M \in T_\psi(x)$ but $M \notin T_\psi(y)$, and hence $T_\psi(x) \neq T_\psi(y)$.

We now show that this mapping is an isomorphism. Clearly $x \leqq y$ implies $T_\psi(x) \subset T_\psi(y)$. Furthermore, suppose that $T_\psi(x) \subset T_\psi(y)$, and that $x \not\leqq y$. Then $y \notin J_x$, and there exists $M \in \Omega(\psi)$ such that $M \supset J_x$ and $y \notin M$. But then $M \in T_\psi(x)$, $M \notin T_\psi(y)$; a contradiction. Hence the theorem is proved.

We shall also need the following definition.

DEFINITION. A subset S of a partially ordered set P is *prime* if and only if $\{x, y\}^* \subset S$ implies $x \in S$ or $y \in S$. S is *coprime* if and only if $\{x, y\}^+ \subset S$ implies $x \in S$ or $y \in S$.

3. Main theorem. We are now ready to state our main result. If $M \subset P$, let M' denote the complement of M with respect to P.

THEOREM 2. *A partially ordered set P is a distributive lattice if and only if there exists an inductive dual imbedding operator ψ on P such that every* *set* $\mathfrak{M}$ *of completely irreducible* ψ*-ideals satisfies*

(*i*) $\bigcup \{M \mid M \in \mathfrak{M}\}$ *is prime, and*

(*ii*) $\bigcup \{M' \mid M \in \mathfrak{M}\}$ *is coprime.*

PROOF. To prove the necessity of the condition, we need only note that if P is a distributive lattice the inductive dual imbedding operator λ, defined above, satisfies (i) and (ii).

To prove the sufficiency, let ψ be an inductive dual imbedding operator on P satisfying (i) and (ii), and let Ω denote the set of all completely irreducible ψ-ideals. For each $x \in P$, let $T(x) = \{M \in \Omega \mid x \in M\}$. By Theorem 1, the mapping $x \to T(x)$ maps P isomorphically onto a subset $\bar{P}$ of 2^{Ω}. Let us denote set union and interesection by $\bigcup$ and $\bigcap$ respectively.

We first show that $\bar{P}$ is closed with respect to $\bigcup$. Let $x \in P$, $y \in P$, and let $K = \{x, y\}^*$. We shall show that there exists $r \in K$ such that $T(r) = T(x) \bigcup T(y)$. Clearly the relation $T(r) \supset T(x) \bigcup T(y)$ holds for all $r \in K$. Let us assume that the reverse inclusion, $T(r) \subset T(x) \bigcup T(y)$ fails to hold for all $r \in K$. Then for each $r \in K$, there exists $M_r \in T(r)$ such that $x \notin M_r$ and $y \notin M_r$. Let $S = \bigcup \{M_r \mid r \in K\}$. Then $K = \{x, y\}^* \subset S$. Since by hypothesis S is prime, we must have $x \in S$ or $y \in S$. But then $x \in M_r$ for some r, or $y \in M_r$ for some r; a contradiction.

We now show that $\bar{P}$ is also closed with respect to $\bigcap$. Let $x \in P$, $y \in P$, and let $J = \{x, y\}^+$. We shall show that there exists $r \in J$ such that $T(r) = T(x) \bigcap T(y)$. The relation $T(r) \subset T(x) \bigcap T(y)$ holds for all $r \in J$. Let us assume that the reverse inclusion fails to hold for all $r \in J$. Then for each $r \in J$, there exists $M_r \in T(x) \bigcap T(y)$ with $r \notin M_r$. Hence the set $J \bigcap \bigcap \{M_r \mid r \in J\}$ is empty. Then $J' \bigcup (\bigcap M_r)' = J' \bigcup \bigcup M_r' = P$. Hence $J = \{x, y\}^+ \subset \bigcup M_r'$. But by hypothesis $\bigcup M_r$ is coprime, and hence $x \in \bigcup M_r'$ or $y \in \bigcup M_r'$. But then $x \notin M_r$ for some r or $y \notin M_r$ for some r; a contradiction.

$\bar{P}$ is thus a sublattice of the distributive lattice 2^{Ω}, and hence P is a distributive lattice. This completes the proof of the theorem.

Let us say that a partially ordered set P satisfies the *descending chain condition* if and only if every infinite descending chain $z_1 > z_2 > z_3 > \dots$ in P has the element 0 as a greatest lower bound. We then have the following corollary (the obvious dual formulation is left to the reader).

COROLLARY. *Let* P *be a partially ordered set with the descending chain condition. Then* P *is a distributive lattice if and only if there exists an inductive dual imbedding operator* ψ *on* P *such that for every set* $\mathfrak{M}$ *of completely irreducible* ψ*-ideals,* $\bigcup \{M' \mid M \in \mathfrak{M}\}$ *is coprime.*

PROOF. The necessity follows as before. To prove the sufficiency, we note that, as in the above proof, $\bar{P}$ is closed with respect to $\bigcap$. To conclude the proof we need, therefore, only to apply the following simple lemma.

LEMMA 3. *A partially ordered set P which satisfies the descending chain condition, and which is closed with respect to* $\cap$, *is also closed with respect to* $\cup$.

PROOF. Suppose that $a \in P$, $b \in P$ and that $a \cup b$ does not exist in P. Let $c \in \{a, b\}^*$. Since $c \neq a \cup b$, there exists $d \in \{a, b\}^*$ such that $d \not\geq c$. But $c \cap d$ exists in P, and $c \cap d \in \{a, b\}^*$. Hence $a < c \cap d < c$ and $b < c \cap d < c$. Thus for each $c \in \{a, b\}^*$, there exists $c' \in \{a, b\}^*$ with $c' < c$. We can therefore construct an infinite descending chain $Z \subset \{a, b\}^*$; a contradiction.

4. Characterization of Boolean algebras.

DEFINITION. A partially ordered set P is *complemented* if and only if for each $x \in P$ there exists $x' \in P$ such that $\{x, x'\}^* = \{I\}$ and $\{x, x'\}^+ = \{0\}$. The element x' need not be unique.

LEMMA 4. *Let P be a complemented partially ordered set and K a prime* λ*-ideal in P with* $K \neq P$. *Then (i)* $x \in K$ *implies* $x' \notin K$ *for all* x', *and (ii)* $x \notin K$ *implies* $x' \in K$ *for all* x'.

PROOF. (i). If $x \in K$ and there exists $x' \in K$, then $\{x, x'\}^{+*} = P$, and hence $K = P$.

(ii). Since $\{x, x'\}^* = \{I\}$, we have $\{x, x'\}^* \subset K$ for all λ-ideals K. Hence, if K is prime, we have $x \in K$ or $x' \in K$.

LEMMA 5. *If P is a complemented partially ordered set, and every completely irreducible* λ*-ideal of P is prime, then* x' *is unique for each* $x \in P$.

PROOF. Suppose that there exist two distinct complements, x_1' and x_2', for some $x \in P$. Let us suppose that $x_2' \not\geq x_1'$. Since λ is inductive, there exists a λ-ideal K containing x_1' such that K is maximal with respect to not containing x_2'. K is completely irreducible, and hence prime. But $x \notin K$, by Lemma 4. Since also $x_2' \notin K$, we have a contradiction.

Let Λ denote the set of all completely irreducible λ-ideals of P, and let $R(x) = \{M \in \Lambda \mid x \in M\}$ for each $x \in P$.

LEMMA 6. *If P satisfies the hypothesis of Lemma* 5, *then* $R(x') = [R(x)]'$ *for all* $x \in P$.

PROOF. Follows immediately from Lemma 4.

THEOREM 3. *A partially ordered set P is a Boolean algebra if and only if (i) P is complemented, and (ii) the union of any set of completely irreducible* λ*-ideals in P is prime.*

PROOF. Assume that (i) and (ii) hold. The mapping $x \to R(x)$ maps P isomorphically onto a subset $\bar{P} \subset 2^{\Lambda}$. As in the proof of Theorem 2, $\bar{P}$ is closed with respect to $\cup$. Since, by Lemma 6, $\bar{P}$ is also closed with respect to complementation, it follows that $\bar{P}$ is a Boolean algebra. The necessity of the conditions is well-known.

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