HERMITIAN MANIFOLDS WITH QUATERNION STRUCTURE

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In previous papers [3, 4] we have studied affine connections in manifolds with almost complex, quaternion or Hermitian structure. As to the almost Hermitian manifold with quaternion structure, some considerations have been given [3], but results are rather complicated and a more explicit form of the connection has been required. In the present paper we shall determine, in an explicit form, all affine connections with respect to which the structures are all covariant constants in a Hermitian manifold with quaternion structure (§2). Complex coordinates are chosen and the determination of such connections is reduced to solving linear equations.

Possibility of introducing some special affine connection is in intimate relation with the integrability of the quaternion structure or with the Kähler's condition on the Hermitian metric. These relations are discussed in §3. Transformations preserving the quaternion structure are always considered as affine transformations with respect to some affine connection [4]. In the Hermitian case more precise results are obtained (§4).

Since we are considering a complex manifold, which is of complex n dimensions, we suppose that the Latin indices $a, b, c, \dots, i\ j, k, \dots$ run over the range $1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}$ and the Greek indices $\alpha, \beta, \gamma, \dots, \kappa, \lambda, \mu, \dots$ run over the values $1, 2, \dots, n$ and consequently the indices $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \dots, \bar{\kappa}, \bar{\lambda}, \bar{\mu}, \dots$ the range of symbols $\bar{1}, \bar{2}, \dots, \bar{n}$. In case of a complex manifold with quaternion structure n must be even.

1. Preliminaries. We consider a differentiable manifold with quaternion structure $(\phi_i{}^h, \psi_i{}^h)$, where a *quaternion structure* is, by definition, a pair of two almost complex structures $\phi_i{}^h$, $\psi_i{}^h$ such that

$$\phi_i{}^a\psi_a{}^h + \psi_i{}^a\phi_a{}^h = 0.$$

In a differentiable manifold there always exists a Riemannian metric γ_{ih}. Then the tensor defined by

$$h_{ih} = \frac{1}{2}(\gamma_{ih} + \phi_i{}^b\phi_h{}^a\gamma_{ba})$$

is also a Riemannian metric and we have

$$h_{ih} = \phi_i{}^b\phi_h{}^a h_{ba},$$

i. e. h_{ih} is an *almost Hermitian metric with respect to* $\phi_i{}^h$. Furthermore the tensor defined by

$$g_{ih} = \frac{1}{2}(h_{ih} + \phi_i{}^b\phi_h{}^a h_{ba})$$

is an almost Hermitian metric with respect to both $\phi_i{}^h$ and $\psi_i{}^h$:

$$g_{ih} = \phi_i{}^b\phi_h{}^a g_{ba} = \psi_i{}^b\psi_h{}^a g_{ba}. \tag{1.1}$$

We call a manifold with $\phi_i{}^h$, $\psi_i{}^h$, g_{ih} satisfying (1.1) a *Hermitian manifold with quaternion structure.* If, furthermore, g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$, such a manifold is called a *Kählerian manifold with quaternion structure.*

In this paper we assume that the almost complex structure $\phi_i{}^h$ gives a complex analytic structure and we choose a complex coordinate system $(z^\kappa, \bar{z}^\kappa)$. Then $\phi_i{}^h$, $\psi_i{}^h$ and g_{ih} take the special forms

$$(\phi_i{}^h) = \begin{pmatrix} i\delta_\lambda^\kappa & 0 \\ 0 & -i\delta_{\bar\lambda}^{\bar\kappa} \end{pmatrix}, \quad (\psi_i{}^h) = \begin{pmatrix} 0 & \psi_\lambda{}^{\bar\kappa} \\ \psi_{\bar\lambda}{}^\kappa & 0 \end{pmatrix}, \quad \psi_\lambda{}^{\bar\alpha}\psi_{\bar\alpha}{}^\kappa = -\delta_\lambda^\kappa,$$

$$(g_{ih}) = \begin{pmatrix} 0 & g_{\lambda\bar\kappa} \\ g_{\bar\lambda\kappa} & 0 \end{pmatrix}, \qquad g_{\lambda\bar\kappa} = \psi_\lambda{}^{\bar\beta}\psi_{\bar\kappa}{}^{\alpha} g_{\bar\beta\alpha}\,; \text{ conj.}^{1)},$$

and they are, of course, self-adjoint. *Throughout this paper all quantities are assumed to be self-adjoint and also to be real analytic.*

Now, on putting

$$\psi_{ih} = \psi_i{}^a g_{ah}, \qquad \psi^{ih} = g^{ia}\psi_a{}^h,$$

we have

$$\psi_{\bar\lambda\kappa} = \psi_{\lambda\bar\kappa} = 0, \qquad \psi_{\lambda\kappa} = -\psi_{\kappa\lambda}\,; \qquad \text{conj.}$$

The condition $\psi_i{}^a\psi_a{}^h = -\delta_i{}^h$ is equivalent to the condition

$$\psi_{ia}\psi^{ah} = \psi^{ha}\psi_{ai} = -\delta_i{}^h; \qquad \text{conj.}$$

Now, given a tensor $P_{ji}{}^h$ we put

$$\Pi_1 P_{ji}{}^h = \frac{1}{2}\,(P_{ji}{}^h - P_{jb}{}^a\psi_{ai}\psi^{bh}), \quad \Pi_2 P_{ji}{}^h = \frac{1}{2}\,(P_{ji}{}^h + P_{jb}{}^a\psi_{ai}\psi^{bh}).$$

By straightforward calculations we have

LEMMA 1. $\Pi_1\Pi_1 = \Pi_1,\ \Pi_2\Pi_2 = \Pi_2,\ \Pi_1\Pi_2 = \Pi_2\Pi_1 = 0,\ \Pi_1 + \Pi_2 =$ *identity.*

LEMMA 2. *Given a tensor $Q_{ji}{}^h$, we have $\Pi_1 Q_{ji}{}^h = 0$ if and only if there exists a tensor $P_{ji}{}^h$ such that $\Pi_2 P_{ji}{}^h = Q_{ji}{}^h$. We have $\Pi_2 Q_{ji}{}^h = 0$ if and only if there exists a tensor $P_{ji}{}^h$ such that $\Pi_1 P_{ji}{}^h = Q_{ji}{}^h$.*

PROOF. If $\Pi_2 P_{ji}{}^h = Q_{ji}{}^h$, by Lemma 1 we have $\Pi_1 Q_{ji}{}^h = \Pi_1\Pi_2 P_{ji}{}^h = 0$. If, conversely, $\Pi_1 Q_{ji}{}^h = 0$, by Lemma 1 we have

$$\Pi_2 Q_{ji}{}^h = \Pi_2 Q_{ji}{}^h + \Pi_1 Q_{ji}{}^h = (\Pi_1 + \Pi_2)\, Q_{ji}{}^h = Q_{ji}{}^h.$$

LEMMA 3. *If $\Pi_1 Q_{ji}{}^h = 0$, an equation $\Pi_2 P_{ji}{}^h = Q_{ji}{}^h$ ($P_{ji}{}^h$ unknown) has a solution and the general solution is given by*

$$P_{ji}{}^h = Q_{ji}{}^h + \Pi_1 A_{ji}{}^h,$$

where $A_{ji}{}^h$ is an arbitrary tensor.

PROOF. The condition $\Pi_1 Q_{ji}{}^h = 0$ implies that $Q_{ji}{}^h$ itself is a solution of the equation. The general solution $P_{ji}{}^h$ is written as

1) The sign "conj." denotes the complex conjugate of the formulas already written.

$$P_{ji}{}^{h} = Q_{ji}{}^{h} + B_{ji}{}^{h},$$

where $$\Pi_2 B_{ji}{}^{h} = 0.$$

By Lemma 2 $B_{ji}{}^{h}$ is written as $B_{ji}{}^{h} = \Pi_1 A_{ji}{}^{h}$ for some tensor $A_{ji}{}^{h}$. Furthermore for an arbitrary $A_{ji}{}^{h}$, $Q_{ji}{}^{h} + \Pi_1 A_{ji}{}^{h}$ is a solution of the equation. Thus the general solution is given by

$$P_{ji}{}^{h} = Q_{ji}{}^{h} + \Pi_1 A_{ji}{}^{h},$$

$A_{ji}{}^{h}$ being an arbitary tensor.

2. **Affine connections**. Let us assume that an affine connection Γ^{h}_{ji} is a metric (ϕ, ψ)-connection, i. e., $\nabla_j \phi_i{}^{h} = \nabla_j \psi_i{}^{h} = \nabla_j g_{ih} = 0$

The condition $\nabla_j \phi_i{}^{h} = 0$ implies $\Gamma^{\kappa}{}_{j\bar{\lambda}} = \Gamma^{\bar{\kappa}}_{j\lambda} = 0$, so that Γ^{h}_{ji} must have components $\Gamma^{h}_{ji} = (\Gamma^{\kappa}_{j\lambda}, \Gamma^{\bar{\kappa}}_{j\bar{\lambda}})$. It is to be remarked that the components $(\Gamma^{\kappa}_{\bar{\mu}\lambda}, \Gamma^{\bar{\kappa}}_{\mu\bar{\lambda}})$ define a self-adjoint tensor.

The condition $\nabla_j \psi_i{}^{h} = 0$ is written as

$$\nabla_{\mu} \psi_{\bar{\lambda}}{}^{\kappa} = \partial_{\mu} \psi_{\bar{\lambda}}{}^{\kappa} + \psi_{\bar{\lambda}}{}^{\alpha} \Gamma^{\kappa}_{\mu\alpha} - \Gamma^{\bar{\alpha}}_{\mu\bar{\lambda}} \psi_{\bar{\alpha}}{}^{\kappa} = 0\,; \text{ conj.},$$

$$\nabla_{\bar{\mu}} \psi_{\bar{\lambda}}{}^{\kappa} = \partial_{\bar{\mu}} \psi_{\bar{\lambda}}{}^{\kappa} + \psi_{\bar{\lambda}}{}^{\alpha} \Gamma^{\kappa}_{\bar{\mu}\alpha} - \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\lambda}} \psi_{\bar{\alpha}}{}^{\kappa} = 0\,; \text{ conj.},$$

from which we see

(2. 1) $$\Gamma^{\kappa}_{\mu\lambda} = -(\partial_{\mu} \psi_{\lambda}{}^{\bar{\alpha}}) \psi_{\bar{\alpha}}{}^{\kappa} - \psi_{\lambda}{}^{\bar{\beta}} \Gamma^{\bar{\alpha}}_{\mu\bar{\beta}} \psi_{\bar{\alpha}}{}^{\kappa}\,; \text{ conj..}$$

From $\nabla_j g_{ih} = 0$ we find

(2. 2) $$\nabla_{\mu} g_{\bar{\lambda}\kappa} = \partial_{\mu} g_{\bar{\lambda}\kappa} - \Gamma^{\alpha}_{\mu\kappa} g_{\bar{\lambda}\alpha} - \Gamma^{\bar{\alpha}}_{\mu\bar{\lambda}} g_{\bar{\alpha}\kappa} = 0\,; \text{ conj..}$$

Substituting (2. 1) into (2. 2) we get

$$\partial_{\bar{\mu}} g_{\bar{\lambda}\kappa} - \Gamma^{\alpha}_{\bar{\mu}\kappa} g_{\bar{\lambda}\alpha} + (\partial_{\mu} \psi_{\bar{\lambda}}{}^{\beta}) \psi_{\beta}{}^{\alpha} g_{\bar{\alpha}\kappa} + \psi_{\bar{\lambda}}{}^{\gamma} \Gamma^{\beta}_{\bar{\mu}\gamma} \psi_{\beta}{}^{\alpha} g_{\bar{\alpha}\kappa} = 0,$$

from which

(2. 3) $$(\partial_{\bar{\mu}} g_{\bar{\lambda}\alpha}) g^{\alpha\kappa} - \Gamma^{\kappa}_{\bar{\mu}\lambda} + (\partial_{\mu} \psi_{\beta}{}^{\gamma}) \psi_{\gamma}{}^{\alpha} g_{\alpha\lambda} g^{\beta\kappa} - \Gamma^{\alpha}_{\bar{\mu}\beta} \psi_{\alpha\lambda} \psi^{\beta\kappa} = 0.$$

Here from $\psi_{\bar{\beta}}{}^{\gamma} \psi_{\gamma}{}^{\alpha} = -\delta_{\bar{\beta}}{}^{\alpha}$ we have

(2 4) $$(\partial_{\mu} \psi_{\bar{\beta}}{}^{\gamma}) \psi_{\gamma}{}^{\alpha} = -\psi_{\bar{\beta}}{}^{\gamma} \partial_{\bar{\mu}} \psi_{\gamma}{}^{\alpha}$$

and from $\psi_{\gamma\lambda} = \psi_{\gamma}{}^{\alpha} g_{\bar{\alpha}\lambda}$ we have

(2. 5) $$(\partial_{\bar{\mu}} \psi_{\gamma}{}^{\alpha}) g_{\bar{\alpha}\lambda} = \partial_{\bar{\mu}} \psi_{\gamma\lambda} - \psi_{\gamma}{}^{\alpha} \partial_{\bar{\mu}} g_{\alpha\lambda}.$$

From (2. 4) and (2. 5) we get

$$\begin{aligned} (\partial_{\mu} \psi_{\beta}{}^{\gamma}) \psi_{\gamma}{}^{\alpha} g_{\alpha\lambda} g^{\beta\kappa} &= -\psi_{\beta}{}^{\gamma} (\partial_{\bar{\mu}} \psi_{\gamma}{}^{\alpha}) g_{\bar{\alpha}\lambda} g^{\bar{\beta}\kappa} \\ &= -\psi_{\bar{\beta}}{}^{\gamma} (\partial_{\bar{\mu}} \psi_{\gamma\lambda}) g^{\bar{\beta}\kappa} + \psi_{\beta}{}^{\gamma} \psi_{\gamma}{}^{\alpha} (\partial_{\bar{\mu}} g_{\bar{\alpha}\lambda}) g^{\bar{\beta}\kappa} \\ &= -(\partial_{\bar{\mu}} \psi_{\lambda\alpha}) \psi^{\alpha\kappa} - (\partial_{\bar{\mu}} g_{\lambda\bar{\alpha}}) g^{\alpha\kappa}. \end{aligned}$$

By using this formula we have from (2. 3)

(2. 6) $$\Gamma^{\kappa}_{\bar{\mu}\lambda} + \Gamma^{\alpha}_{\bar{\mu}\beta} \psi_{\alpha\lambda} \psi^{\beta\kappa} = -\partial_{\bar{\mu}} \psi_{\lambda\alpha} \psi^{\alpha\kappa}\,; \text{ conj..}$$

On putting $T^{\kappa}_{\bar{\mu}\lambda} = \Gamma^{\kappa}_{\bar{\mu}\lambda}$ and $P_{\bar{\mu}\lambda}{}^{\kappa} = -\dfrac{1}{2} (\partial_{\bar{\mu}} \psi_{\lambda\alpha}) \psi^{\alpha\kappa}$; conj., $T_{ji}{}^{h} = (T_{\bar{\mu}\lambda}{}^{\kappa},\ T_{\mu\bar{\lambda}}{}^{\bar{\kappa}})$

and $P_{ji}{}^{h} = (P_{\bar{\mu}\lambda}{}^{\kappa},\ P_{\mu\bar{\lambda}}{}^{\bar{\kappa}})$ are components of tensors. (2.6) is then written as

$$\Pi_2 T_{\bar{\mu}\lambda}{}^{\kappa} = P_{\bar{\mu}\lambda}{}^{\kappa}\text{; conj..} \tag{2.7}$$

We have, however, $\Pi_1 P_{\bar{\mu}\lambda}{}^{\kappa} = -\frac{1}{4}((\partial_{\bar{\mu}}\psi_{\lambda\alpha})\psi^{\alpha\kappa} - (\partial_{\mu}\psi_{\beta\alpha})\psi^{\alpha\gamma}\psi_{\gamma\lambda}\psi^{\beta\kappa})$

$$= -\frac{1}{4}((\partial_{\bar{\mu}}\psi_{\lambda\alpha})\psi^{\alpha\kappa} + (\partial_{\mu}\psi_{\alpha\lambda})\psi^{\alpha\kappa}) = 0\text{; conj.,}$$

and other components of $\Pi_1 P_{ji}{}^{h}$ also vanish. Therefore (1.7) implies

$$T_{\bar{\mu}\lambda}{}^{\kappa} = P_{\bar{\mu}\lambda}{}^{\kappa} + \Pi_1 A_{\bar{\mu}\lambda}{}^{\kappa}, \tag{2.8}$$

where $A_{ji}{}^{h} = (A_{\bar{\mu}\lambda}{}^{\kappa},\ A_{\mu\bar{\lambda}}{}^{\bar{\kappa}})$ is a tensor.

Substituting (2.8) into (2.1) we get

$$\Gamma^{\kappa}_{\bar{\mu}\lambda} = -(\partial_{\mu}\psi_{\lambda}{}^{\bar{\alpha}})\psi_{\bar{\alpha}}{}^{\kappa} - \psi_{\lambda}{}^{\beta}P_{\mu\beta}{}^{\bar{\alpha}}\psi_{\bar{\alpha}}{}^{\kappa} - \frac{1}{2}\psi_{\lambda}{}^{\beta}A_{\mu\beta}{}^{\alpha}\psi_{\alpha}{}^{\kappa} + \frac{1}{2}\psi_{\lambda}{}^{\beta}A_{\mu\gamma\bar{\rho}}\psi_{\rho\beta}\psi^{\bar{\gamma}\bar{\alpha}}\psi_{\alpha}{}^{\kappa}$$

$$= -(\partial_{\mu}\psi_{\lambda}{}^{\bar{\alpha}})\psi_{\bar{\alpha}}{}^{\kappa} + \frac{1}{2}\psi_{\lambda}{}^{\beta}(\partial_{\mu}\psi_{\bar{\beta}\bar{\gamma}})\psi^{\bar{\gamma}\bar{\alpha}}\psi_{\alpha}{}^{\kappa} - \frac{1}{2}\psi_{\lambda}{}^{\bar{\beta}}A_{\mu\bar{\beta}}{}^{\alpha}\psi_{\alpha}{}^{\kappa} = \frac{1}{2}A_{\mu\beta}{}^{\alpha}g_{\bar{\alpha}\lambda}g^{\bar{\beta}\kappa}.$$

By (2.4) and (2.5) we have

$$\psi_{\lambda}{}^{\bar{\beta}}(\partial_{\mu}\psi_{\bar{\beta}\bar{\gamma}})\psi^{\bar{\gamma}\alpha}\psi_{\alpha}{}^{\kappa} = \psi_{\lambda}{}^{\bar{\beta}}(\partial_{\mu}\psi_{\bar{\beta}}{}^{\gamma})g_{\gamma\bar{\rho}}\psi^{\bar{\rho}\alpha}\psi_{\alpha}{}^{\kappa} + \psi_{\lambda}{}^{\bar{\beta}}\psi_{\bar{\beta}}{}^{\gamma}(\partial_{\mu}g_{\gamma\bar{\rho}})\psi^{\bar{\rho}\alpha}\psi_{\bar{\alpha}}{}^{\kappa}$$

$$= (\partial_{\mu}\psi_{\lambda}{}^{\bar{\alpha}})\psi_{\alpha}{}^{\kappa} + (\partial_{\mu}g_{\lambda\alpha})g^{\bar{\alpha}\kappa},$$

so that we obtain

$$\Gamma^{\kappa}_{\mu\lambda} = \frac{1}{2}((\partial_{\mu}g_{\lambda\bar{\alpha}})g^{\bar{\alpha}\kappa} - (\partial_{\mu}\psi_{\lambda}{}^{\bar{\alpha}})\psi_{\bar{\alpha}}{}^{\kappa}) - \frac{1}{2}(\psi_{\lambda}{}^{\beta}A_{\mu\bar{\beta}}{}^{\alpha}\psi_{\alpha}{}^{\kappa} + g_{\lambda\bar{\alpha}}A_{\mu\beta}{}^{\alpha}g^{\bar{\beta}\kappa})\text{; conj.,} \tag{2.9}$$

$$\Gamma^{\kappa}_{\bar{\mu}\lambda} = -\frac{1}{2}(\partial_{\bar{\mu}}\psi_{\lambda\alpha})\psi^{\alpha\kappa} + \frac{1}{2}(A_{\bar{\mu}\lambda}{}^{\kappa} - A_{\mu\beta}{}^{\alpha}\psi_{\alpha\lambda}\psi^{\beta\kappa})\text{; conj..} \tag{2.10}$$

Thus we see that a metric (ϕ, ψ)-connection Γ^{h}_{ji} is given by (2.9) and (2.10).

Conversely, it is easy to verify that given any tensor field $A_{ji}{}^{h} = (A_{\bar{\mu}\lambda}{}^{\kappa},\ A_{\mu\bar{\lambda}}{}^{\bar{\kappa}})$, the quantities $\Gamma^{h}_{ji} = (\Gamma^{\kappa}_{\mu\lambda},\ \Gamma^{\kappa}_{\bar{\mu}\lambda}, \Gamma^{\bar{\kappa}}_{\mu\bar{\lambda}},\ \Gamma^{\bar{\kappa}}_{\bar{\mu}\bar{\lambda}})$ given by (2.9) and (2.10) define a metric (ϕ, ψ)-connection.

Thus we have

THEOREM 2.1. *In a Hermitian manifold with a quaternion structure in order that an affine connection Γ^{h}_{ji} be a metric (ϕ, ψ)-connection it is necessary and sufficient that Γ^{h}_{ji} be given by $\Gamma^{h}_{ji} = (\Gamma^{\kappa}_{j\lambda},\ \Gamma^{\bar{\kappa}}_{j\bar{\lambda}})$:*

$$\Gamma^{\kappa}_{\mu\lambda} = \frac{1}{2}((\partial_{\mu}g_{\lambda\bar{\alpha}})g^{\bar{\alpha}\kappa} - (\partial_{\mu}\psi_{\lambda}{}^{\bar{\alpha}})\psi_{\bar{\alpha}}{}^{\kappa}) - \frac{1}{2}(\psi_{\lambda}{}^{\bar{\beta}}A_{\mu\bar{\beta}}{}^{\alpha}\psi_{\bar{\alpha}}{}^{\kappa} + g_{\lambda\alpha}A_{\mu\bar{\beta}}{}^{\alpha}g^{\bar{\beta}\kappa})\text{; conj.,}$$

$$\Gamma^{\kappa}_{\bar{\mu}\lambda} = \frac{1}{2}(\partial_{\bar{\mu}}\psi_{\lambda\alpha})\psi^{\alpha\kappa} + \frac{1}{2}(A_{\bar{\mu}\lambda}{}^{\kappa} - A_{\bar{\mu}\beta}{}^{\alpha}\psi_{\alpha\lambda}\psi^{\beta\kappa})\text{; conj.,}$$

where $A_{ji}{}^{h} = (A_{\bar{\mu}\lambda}{}^{\kappa},\ A_{\mu\bar{\lambda}}{}^{\bar{\kappa}})$ is an arbitrary tensor field.

Since $A_{ji}{}^{h}$ is arbitrary, we may put $A_{ji}{}^{h} = 0$, and then we have

THEOREM 2.2. *In a Hermitian manifold an affine connection $\Gamma^{h}_{ji} = (\Gamma^{\kappa}_{j\lambda},\ \Gamma^{\bar{\kappa}}_{j\bar{\lambda}})$ given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \frac{1}{2}((\partial_\mu g_{\lambda\bar\alpha})\, g^{\bar\alpha\kappa} - (\partial_\mu \psi_\lambda{}^{\alpha})\psi_{\bar\alpha}{}^{\kappa})\,;\ \text{conj.},$$

$$\Gamma^{\kappa}_{\bar\mu\lambda} = -\frac{1}{2}(\partial_\mu \psi_{\lambda\alpha})\psi^{\alpha\kappa}\,;\ \text{conj.}$$

is a metric (ϕ, ψ)*-connection.*

3. Integrability of the quaternion structure and affine connections. In a Hermitian manifold, an affine connection $\overset{0}{\Gamma}{}^{h}_{ji} = (\overset{0}{\Gamma}{}^{\kappa}_{\mu\lambda}, \overset{0}{\Gamma}{}^{\bar\kappa}_{\mu\bar\lambda})$ defined by

$$\overset{0}{\Gamma}{}^{\kappa}_{\lambda\mu} = (\partial_\mu g_{\lambda\bar\alpha})\, g^{\bar\alpha\kappa}\,;\ \text{conj.}$$

is a metric ϕ-connection, i. e. $\overset{0}{\nabla}_j g_{ih} = \overset{0}{\nabla}_j \phi_i{}^h = 0$. The Hermitian metric g_{ih} is then Kählerian if and only if the connection $\overset{0}{\Gamma}{}^{h}_{ji}$ is symmetric, i. e. $\partial_\mu g_{\lambda\bar\kappa} = \partial_\lambda g_{\mu\bar\kappa}$.

In a quaternion manifold, an affine connection $\overset{1}{\Gamma}{}^{h}_{ji} = (\overset{1}{\Gamma}{}^{\kappa}_{\mu\lambda}, \overset{1}{\Gamma}{}^{\bar\kappa}_{\mu\lambda})$ defined by

$$\overset{1}{\Gamma}{}^{\kappa}_{\mu\lambda} = -(\partial_\mu \psi_\lambda{}^{\bar\alpha})\psi_{\bar\alpha}{}^{\kappa}\,;\ \text{conj.}$$

is a (ϕ, ψ)-connection, i. e. $\overset{1}{\nabla}_j \phi_i{}^h = \overset{1}{\nabla}_j \psi_i{}^h = 0$. The almost complex structure $\psi_i{}^h$ is integrable, i. e. it gives another complex structure, if and only if the connection $\overset{1}{\Gamma}{}^{h}_{ji}$ is symmetric, i. e. $\partial_\mu \psi_\lambda{}^\kappa = \partial_\lambda \psi_\mu{}^\kappa = 0$ [3, 4].

On the other hand in a Hermitian manifold with quaternion structure the condition

$$\partial_{\bar\mu} \psi_{\lambda\kappa} = 0\,;\ \text{conj.} \tag{3.1}$$

is equivalent to the condition that the tensor field $\psi_{ih} = (\psi_{\lambda\kappa}, \psi_{\bar\lambda\bar\kappa})$ is complex analytic. Furthermore (3. 1) is equivalent to

$$(\partial_{\bar\mu} \psi_\lambda{}^{\alpha})\, g_{\alpha\kappa} + \psi_\lambda{}^{\alpha}\, \partial_{\bar\mu}\, g_{\alpha\kappa} = 0$$

or to

$$(\partial_\mu\, g_{\lambda\bar\alpha})\, g^{\bar\alpha\kappa} = -(\partial_\mu \psi_\lambda{}^{\alpha})\psi_\alpha{}^{\kappa}. \tag{3.2}$$

Since in a Hermitian manifold with quaternion structure it is possible to introduce the metric ϕ-connection $\overset{0}{\Gamma}{}^{h}_{ji}$ and (ϕ, ψ)-connection $\overset{1}{\Gamma}{}^{h}_{ji}$, (3. 2) means that the two connections coincide with each other.

THEOREM 3. 1. *In a Hermitian manifold with quaternion structure we introduce a metric* ϕ*-connection* $\overset{0}{\Gamma}{}^{h}_{ji}$ *and a* (ϕ, ψ)*-connection* $\overset{1}{\Gamma}{}^{h}_{ji}$*, each defined by*

$$\overset{0}{\Gamma}{}^{\kappa}_{\mu\lambda} = (\partial_\mu\, g_{\lambda\bar\alpha})\, g^{\bar\alpha\kappa}\,;\ \text{conj.},\quad \overset{1}{\Gamma}{}^{\kappa}_{\mu\lambda} = -(\partial_\mu \psi_\lambda{}^{\alpha})\,\psi_\alpha{}^{\kappa}\,;\ \text{conj.},$$

other components being zero. Then the following four conditions are equivalent with each other:

1) *The tensor* $\psi_{ih} = (\psi_{\lambda\kappa}, \psi_{\bar\lambda\bar\kappa})$ *is complex analytic*: $\partial_{\bar\mu}\, \psi_{\lambda\kappa} = 0$; conj..

2) *The two connections coincide with each other*: $\overset{0}{\Gamma}{}^{h}_{ji} = \overset{1}{\Gamma}{}^{h}_{ji}$.

3) *The metric ϕ-connection $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection*: $\overset{0}{\nabla}_j\psi_i{}^h = 0$.

4) *The (ϕ, ψ)-connection $\overset{0}{\Gamma}{}^h_{ji}$ is a metric connection*: $\overset{1}{\nabla}_j g_{ih} = 0$.

PROOF. The equivalence of 1) and 2) has been established above.

That of 2) and 3) will be seen as follows. Since $\overset{0}{\Gamma}{}^h_{ji}$ is a metric ϕ-connection and $\overset{1}{\Gamma}{}^h_{ji}$ is a (ϕ, ψ)-connection, the condition $\overset{0}{\Gamma}{}^h_{ji} = \overset{1}{\Gamma}{}^h_{ji}$ implies that $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection. If, conversely, $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection, from the special form of $\overset{0}{\Gamma}{}^h_{ji} = (\overset{0}{\Gamma}{}^\kappa_{\mu\lambda}, \overset{0}{\Gamma}{}^{\bar\kappa}_{\bar\mu\bar\lambda})$ we have

$$\overset{0}{\nabla}_\mu\psi_\lambda{}^\kappa = \partial_\mu\psi_\lambda{}^\kappa - \overset{0}{\Gamma}{}^\alpha_{\mu\lambda}\psi_\alpha{}^\kappa = 0,$$

from which we see

$$\overset{0}{\Gamma}{}^\kappa_{\mu\lambda} = -(\partial_\mu\psi_\lambda{}^\alpha)\psi_\alpha{}^\kappa = \overset{1}{\Gamma}{}^\kappa_{\mu\lambda}.$$

In an analogous way, the equivalence of 2) and 4) can be established.

Next we consider the relations between the integrability of $\psi_i{}^h$ and the Kähler's condition on g_{ih} with respect to both $\phi_i{}^h$ and $\psi_i{}^h$.

THEOREM 3.2. *In a Hermitian manifold with quaternion structure the following six conditions are equivalent with each other*:

1) *The Riemannian metric g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$.*

2) *The tensors $\phi_{jih} = \partial_{[j}\phi_{ih]}$ and $\psi_{jih} = \partial_{[j}\psi_{ih]}$ vanish identically and the almost complex structure $\psi_i{}^h$ is integrable*: $\partial_\mu\psi_\lambda{}^\kappa = \partial_\lambda\psi_\mu{}^\kappa$.

3) *The connection $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection without torsion.*

4) *The connection $\overset{1}{\Gamma}{}^h_{ji}$ is a metric connection without torsion, i. e. Riemannian connection.*

5) *The tensor $\psi_{ih} = (\psi_{\lambda\kappa}, \psi_{\bar\lambda\bar\kappa})$ is complex analytic and the almost complex structure $\psi_i{}^h$ is integrable.*

6) *The metric g_{ih} is Kählerian with respect to $\phi_i{}^h$ and the tensor ψ_{ih} is complex analytic.*

PROOF. The equivalence of 1) and 2) is well-known.

If g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$, the Riemannian connection Γ^h_{ji}, i. e. one defined by Christoffel symbols, is a (ϕ, ψ)-connection. Since g_{ih} is Kählerian with respect to $\phi_i{}^h$, Γ^h_{ji} coincides with $\overset{0}{\Gamma}{}^h_{ji}$, so that $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection without torsion. If, conversely, $\overset{0}{\Gamma}{}^h_{ji}$ is a ψ-connection without torsion, it is a metric (ϕ, ψ)-connection without torsion and must coincide with the Riemannian connection. Hence g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$. Thus the equivalence of 1) and 3) is established. In an analogous way the equivalence of 1) and 4) will be proved.

The equivalence of 1) and 5) is proved by use of Theorem 3.1. In fact, if g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$, the Riemannian connection coincides with both $\overset{0}{\Gamma}{}^h_{ji}$ and $\overset{1}{\Gamma}{}^h_{ji}$. It follows from Theorem 3.1 that the tensor $\psi_{ih} = (\psi_{\lambda\kappa}, \psi_{\bar\lambda\bar\kappa})$ is complex analytic. Since $\overset{1}{\Gamma}{}^h_{ji}$ must be symmetric, the integrability condition of $\psi_i{}^h$ is satisfied. If, conversely, the condition 5) is satisfied, the complex-analyticity of ψ_{ih} implies that $\overset{1}{\Gamma}{}^h_{ji}$ is a metric (ϕ, ψ)-connection by Theorem 3.1. The symmetry of $\overset{1}{\Gamma}{}^h_{ji}$ follows from the integrability of $\psi_i{}^h$.

1) $\rightarrow$ 6) is obvious. If 6) is satisfied, the analyticity of ψ_{ih} implies $(\partial_\mu g_{\lambda\bar\alpha})g^{\bar\alpha\kappa} = -(\partial_\mu \psi_\lambda{}^{\alpha})\psi_\alpha{}^\kappa$. Since g_{ih} is Kählerian, the Riemannian connection coincides with $\overset{0}{\Gamma}{}^h_{ji} = \overset{1}{\Gamma}{}^h_{ji}$ and is a ψ-connection, i.e. g_{ih} is Kählerian with respect to both $\phi_i{}^h$ and $\psi_i{}^h$.

4. Transformations preserving the quaternion structure. We consider a differentiable transformation f preserving the quaternion structure: $f\phi_i{}^h = \phi_i{}^h$ and $f\psi_i{}^h = \psi_i{}^h$ [2]. The former condition means that f is complex analytic (with respect to $\phi_i{}^h$). The latter condition implies that the field of partial derivatives of $\psi_i{}^h$ is also invariant by f. If the tensor $\psi_{ih} = (\psi_{\lambda\kappa}, \psi_{\bar\lambda\bar\kappa})$ is complex analytic, the metric (ϕ, ψ)-connection $\overset{0}{\Gamma}{}^h_{ji} = \overset{1}{\Gamma}{}^h_{ji}$ is defined only by $\psi_i{}^h$ and its partial derivatives by complex coordinates: $\overset{1}{\Gamma}{}^\kappa_{\mu\lambda} = -(\partial_\mu \psi_\lambda{}^\alpha)\psi_\alpha{}^\kappa$; conj., others being zero. Therefore $\overset{1}{\Gamma}{}^h_{ji}$ is remained invariant by f. Thus we have

THEOREM 4.1. *In a Hermitian manifold with quaternion structure we assume that the tensor ψ_{ih} is complex analytic. Then a differentiable transformation preserving the quaternion structure is always an affine transformation with respect to the metric (ϕ, ψ)-connection $\overset{0}{\Gamma}{}^h_{ji} (= \overset{1}{\Gamma}{}^h_{ji})$.*

If a Hermitian manifold with quaternion structure is Kählerian, the assumptions of Theorem 4.1 are satisfied, so that we have

THEOREM 4.2. *In a Kählerian manifold with quaternion structure, a differentiable transformation preserving the quaternion structure is always an affine transformation with respect to the Riemannian connection.*

Since, in a complete, connected irreducible Riemannian manifold, an affine transformation is always an isometry [1, 2], we have

THEOREM 4.3. *In a complete, connected irreducible Kählerian manifold with quaternion structure, a differentiable transformation preserving the quaternion stucture is always an isometry.*

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2) As to the notation see [4].

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