INDECOMPOSABLE TRAJECTORIES

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Whether the motion of any one of the various fundamental particles of physics is as a wave or as a projectile or neither we have in Set Theory many allied questions. Basic among these is that concerning the type of point set over which this motion takes place. Obviously any fundamental particle in a science based upon observation must at any moment cover an uncountable point set. Is this point set a nicely behaving one or a very peculiar type? Is it connected or not, closed or not, dense in a domain or not? Observation can never tell. It is the task of mathematics to develop all interesting possibilities.

Our interest here is partly this: if a particle moves in an arc-wise path, or its image on an arc-wise trajectory in phase space, with no recrossing but densely in a domain, what type of connected set can result? Especially we are interested here when an indecomposable connected set results, but we do note in Theorem 6 that even a locally connected connexe can result. We are also interested in disjoint sums of these paths.

Let the m-dimensional imbedding space be S_m. We need one where we can show the existence of an arc densely in a connected domain: a separable Moore space [5] satisfying Axioms 1 and 2 is such a space; or S_m can be any equivalent metric space. For ease in explaining we will speak of a region about $p, \in S$, as the interior of an $(m-1)$-sphere with center at p.

DEFINITIONS. Let p be a point of a path of a particle at time t_0 and p' be at any time t: if pp' is an arc we say the path is arc-wise connected[1]. The points of the path, even through infinite time, will then be called an *arc-wise path* or *trajectory* for the image in phase space. For phase space see [2: p 13 and 1: pp. 8-13]. A connected set C is said to be an *indecomposable connexe*; if it is not the sum of two connected subsets each with a different closure than that of C; if C is an indecomposable connexe and arc-wise connected from some point p, then C will be called an *arc-wise trajectory* or *connexe*. The closure of C is denoted by $\overline{C}$.

NOTATION. By $\{T_i\}$ we mean an infinite class of T_i $(i=1,2,\ldots.)$. By

1) If M is *disconnected* we will write $M=H+K$ *separate*, meaning M is the point set sum of two mutually exclusive, non-vacuous subsets H and K, neither of which contains a limit point of the other. If M is not disconnected it is *connected*. We call a non-degenerate connected set a *connexe*. By *arc* we mean simple continuous arc, i.e. a set topologically equivalent to the graph of a continuous $f(x)$ from $x=a$ to $x=b$. A *simple* chain is always one with a finite number of regions as *links*. Definitions of the terms used can be found in [5, 6, or 4] usually.

the point set sum, $\bigcup C_i$, of $\{C_i\}$ we mean the set of points contained in the sum of all the elements C_i. The set T is *chain-wise constructed* if T is the point set sum of a class $\{C_i\}$, where each C_i is a simple chain between some two points, as is also $C_1 + C_2 + \ldots + C_g$ ($g = 1, 2, \ldots$). Each link-region of C_i will be the interior of an $(m-1)$-sphere of S_m: the small radius of C_i will be the radius of the smallest of these sphere and the large radius that of the largest.

THEOREM 1. *In any connected domain D of S_m there exists an arc-wise indecomposable trajectory dense in D.*

PROOF. This consists in combining two familar processes: (*a*) The "tunneling" process of Wada for the construction of an indecomposable connexe as used in [10: Th. 1, pp. 178-179]; (*b*) The method of constructing an arc as in [5: Th. 1, pp. 86-88 or 6: Th. 3.9, p. 80]

In the proof of Theorem 1 of [10] we have: 1*a*) A sequence of connected domains, i.e. "tunnels", $T_1, T_2, \ldots, T_j, \ldots$ where each T_j contains T_{j+1}, i.e. $T_j \supset T_{j+1}$; 2*a*) Each T_j is dense in $D - T_j$; 3*a*) Each T_j is chain-wise constructed by $\{C_i\}_j$, where each C_i is a simple chain as above; 4*a*) Where r_i is the smallest radius of $C_i \in \{C_i\}_j$, $\lim r_i = 0$ for each T_j; 5*a*) If, for a fixed j, H is the point set sum of the first h elements of $\{C_i\}_j$, then $T_{j-1} \supset \bar{H}$; 6*a*) If r'_j is the largest of the large radii of the C_i of $\{C_i\}_j$, then $\lim r'_j = 0$. In the construction of T_j we will say the simple chain C_i of $\{C_i\}_j$ is the *i-stage* of T_j.

In the arc-wise connected Theorem 1 of [5: pp. 86-88] we have: 1*b*) A sequence of simple chains, $C'_1, C'_2, \ldots, C'_j, \ldots$ from point p to point q; 2*b*) Each closure of a link-region of C'_j is contained in a link-region of C'_{j-1}; 3*b*) If r''_j is the large radius of C'_j, then $\lim r''_j = 0$.

We combine these two proofs, to give that the set intersection $T = \bigcap T_j$ ($j = 1, 2, \ldots$) is an arc-wise indecomposable connexe as follows: 1) We take densely in D points $p_1, p_2, \ldots p_g, \ldots$ and the g-stage of each T_j is taken to give an arc $p_g p_{g+1}$ where T finally will be $p_1p_2 + p_2p_3 + \ldots$; 2) Thus to obtain p_1p_2 we take C'_j of 1*b*) above as $C_1 \in \{C_i\}_j$ for each $j = 1, 2, \ldots$ and each simple chain C'_j here joins p_1 to p_2; 3) To permit the "looping back" into T_1 needed for T to be dense in D we take $p_3 \in T_1 - p_1p_2$ and take C'_j of 1*b*) as $C_2 \in \{C_i\}_j$ for each $j = 2, 3, \ldots$ and each C'_j is taken joining p_2 to p_3; 4) Similarly we take $p_4 \in T_2 - p_1p_3$ and C'_j, joining p_3 to p_4, as $C_3 \in \{C_i\}_j$ for each $j = 3, 4, \ldots$; 5) The arc $p_1p_3 = p_1p_2 + p_2p_3$; 6) We continue the construction of the arc p_1p_g by obvious induction: 7) After the g-stage of construction, having obtained p_1p_g, then for this g the $\{C_i\}_g$ for T_g only has to satisfy 1*a*)-6*a*) above and not 1*b*)-3*b*). It is well known that all the above can be done so as to obtain the arc-wise connected set T.

That T is an indecomposable connexe can be shown by the usual method for the Wada process: suppose T is the sum of two connexes H and K, neither of whose closure is that of T. Then there exist regions, H', K', such that

$\overline{H'}\cdot K = 0 = \overline{H}\cdot\overline{K'}$ and $H'\cdot H \neq 0 \neq K'\cdot K$, where '0' is the null set. By 2*a*) and 6*a*) there exists a simple chain, composed of links of some of the elements of some $\{C_i\}_j$, whose point set sum $\supset$ a connected domain D', with boundary B, such that $D' + B$ joins H' to K' to H' and $(B - \overline{H'})\cdot T = 0$; also D' does not contain all of $K'\cdot T$. Thus B separates K and so K is not connected. Hence T is the desired arc-wise indecomposable trajectory dense in D.

Let Z_j, which we will call a "cylinder", be the part of the boundary of T_j which is also in the point set sum of the $(m-1)$-spheres giving the link-regions of $\{C_i\}_j$. Thus above we have that T is enclosed in a descending tower $\{Z_j\}$ of cylinders and $T\cdot Z_j = 0$. When we have this situation we will say T is *$\mathcal{E}$-densely looped* by the tower $\{Z_j\}$, and so by the complement of T. We mean by this that for every $p, q \in T$ and regions H', K', where $\overline{H'}\cdot\overline{K'} = 0$, $p \in H', q \in K'$, there exists a Z_j joining H' to K' to H' so that Z_j separates $K'\cdot T$. Since $T\cdot Z_j = 0$ for all j, we also will say that T is *$\mathcal{E}$-shielded* by the descending tower $\{Z_j\}$. It follows by the above type of argument that: (I) *A connexe T, dense in a connected domain D, which is $\mathcal{E}$-densely looped, or $\mathcal{E}$-shielded, by a descending tower $\{Z_j\}$ of cylinders is an indecomposable connexe.*

If, in the plane, $\{P_v\}$ $(v = 1, 2, \ldots, n;\ n > 1)$ is a set of mutually exclusive arc-wise paths each contained, and dense, in a domain D, then P_v is an indecomposable connexe; this is true because P_w is in the complement of P_v, $v \neq w$, and so P_v is $\mathcal{E}$-densely looped by its complement, i. e. P_w plays the role of Z_j above. The sum of any $n-1$ of the P_v is an indecomposable connexe by the same reasoning. We also show below in Theorem 7 that, if $n = 1$, the arc-wise path P_1 in the plane is an indecomposable connexe, but in S_m, $m > 2$, by Theorems 5 and 6, it may not be unless it has $\mathcal{E}$-shielding, as in the proof of Theorem 4 below.

THEOREM 2. *For any finite n there exists in any connected domain D of S_m a set of n mutually exclusive arc-wise indecomposable trajectories each dense in D.*

PROOF. For $n = 1$ this is true by Theorem 1. Suppose we have constructed k mutually exclusive arc-wise indecomposable trajectories each dense in D. Then P_g $(g = 1, 2, \ldots, k)$ is the point set sum of $\{f_i\}_g$, where f_i is an arc having nothing common with f_h, $i \neq h$, except an end point if h is $i+1$.

Consider the case $m > 2$. We wish to obtain an arc-wise indecomposable trajectory T having nothing common with a P_g. We do this by using the well known method of constructing an arc missing a countable number of continua, no sum of which separates a domain in S_m, to modify the proof of Theorem 1 as follows: 1) As above $T = \bigcap T_j$ $(j = 1, 2, \ldots)$ where T_j is chain-wise constructed by $\{C_i\}_j$, the g-stage of T_j gives an arc $p_g p_{g+1}$, and T is the point set sum of $\{p_g p_{g+1}\}$; 2) Let F_1 be the sum of the 1-stage $f_1 \in \{f_i\}_g$, for $g = 1, 2, \ldots, k$ and, by induction, let F_h be $F_1 + F_2 + \ldots + F_{h-1}$ plus the sum of the h-stage $f_h, \in \{f_i\}_g$ for g above; 3) Take now the link-regions of $C_1 \in \{C_i\}_1$ so that their closures have nothing common with F_1, and in general take the link-regions of $C_1, C_2, \ldots . C_h \in \{C_i\}_h$ so that their closures have nothing common with F_h. Thus it follows that no arc $p_g p_{g+1}$ has a point common with $P_1 + P_2 + \ldots . + P_k$, and so T does not. For the case $m = 2$, an arc can separate a domain and so there the above method must be modified to construct the n arc simultaneously.

COROLLARY 2.1. *If I is an m'-dimensional hereditarily indecomposable continuum which is imbeddable in a connected domain D of the Cantorian manifold S_m, $m' < m - 1$, then for any finite n there exists a set of n mutually exclusive arc-wise paths P_v $(v = 1, 2, \ldots . n)$, each of which is an indecomposable connexe dense in D and does not contain a point of I; also, if $I' \subset I, I' + P_1 + P_2 + \ldots . + P_n = V$ is an indecomposable connexe.*

PROOF. This follows from Theorem 2, since, for $m > m' + 1$, $D - I$ is a connected domain.

Here V is 1-indecomposable but not n-indecomposable[2], since each P_i is dense in D and so does not itself have an essential part. In the case where there are n particles in motion with mutually exclusive paths B_i, each indecomposable and alone dense in some domain D_i, the D_i mutually exclusive, where each B_i contains a limit point of some other B_j however, then $B = B_1 + B_2 + \ldots . + B_n$ is an n-indecomposable connexe; the D_i could be tori with common boundary point p and then $B + p$ is n-indecomposable. We call V above an *n-indecomposable path* or *trajectory fusion* and B an *n-indecomposable trajectory union*.

THEOREM 3. *For any finite n there exists in any connected domain D of S_m an n-indecomposable arc-wise trajectory union $P = P_1 + P_2 + \ldots . + P_n$ dense in D; if $E \subset \bar{D} - P$, then $P + E$ is an n-indecomposable trajectory union, locally n-indecomposable at each point of E, and locally 1-indecomposable at each point of P_i.*

PROOF. Each P_i is constructed in a domain D_i from Theorem 1. To obtain the n mutually exclusive D_i one uses Wada's process as in Theorem 1 modified as follows to give a set of D_i with common boundary: (1) One constructs a chain C_1 for each P_i in steps; 1.1) First one obtains a simple chain of regions for P_1 from points p_{11} to p_{12}; 1.2) Then from a covering of regions with $(m - 1)$-sphere as boundary whose closures have nothing common with the chain of 1.1) one obtains a similar chain for P_2 between points p_{21} and p_{22}, repeating for $P_3, \ldots, P_n$; 1.3) Having obtained these n mutually exclusive chains, one continues to extend them by a similar process. This entire process

2) In [11] the following are defined for a connexe W: If $W = \bigcup W_i$, $(i = 1, 2, \ldots . n)$, W_i is connected, and $E(W_i)$, $\neq 0$, is the part of W_i not contained in the sum of the closures of the other W_k, then $E(W_i)$ is called the *essential part* of W_i; If W is the sum of n, but not $n+1$, W_i each with an essential part, then W is said to be an *n-indecomposable connexe*; If $p \in W$ and for each region $R, p \in R, R \cdot W$ is contained in n, but not $n+1$, W_i of W, each with an essential part containing points of $R \cdot W$, then W is said to be *locally n-indecomposable at p*.

is well known. We note E can be the widely connected or biconnected set of [10 : p. 181].

THEOREM 4. *If a particle, or image, Q moves in S_3 densely interior to a torus U always in a counter clock-wise direction about the center, then the path P can be an arc-wise indecomposable connexe.*

PROOF. By a cylinder Z with open ends we mean Z is the boundary of the point set sum of the regions of a simple chain C, except Z does not contain the part of this boundary which also is part of the boundary of the two end regions. Let, at time t_0, Q be at p_0 and at t_i be at p_i, where lim $t_i = \infty$. Let $\{Z_i\}$ be a descending tower of open end cylinders in U giving an $\mathcal{E}$-shielding of P as follows: 1) $P \cdot Z_i = 0$ for $i = 1, 2, \ldots$; 2) Q passes densely through each end of Z_i; 3) Where r_i is the large radius of Z_i, $\lim r_i = 0$; 4) P is interior to Z_1 from t_0 to t_1 and interior to Z_k from t_0 to t_k; 5) Q moves densely through U. Thus from (I) above P is an arc-wise indecomposable trajectory. This motion conserves angular momentum.

COROLLARY 4. 1. *The n particles Q_i can move densely interior to a torus U of S_3 in mutually exclusive paths P_i such that $P_1 + P_2 + \ldots + P_n$ is an n-indecomposable path fusion.*

LEMMA 1. *There exists in the coordinate plane a bounded connected set N which is the sum of a countable number of mutually exclusive arcs.*

PROOF. Let $a_i = (1/i, 1)$, $b_i = (1/i, 0), c_i = (0, -1/i), a'_i = (-1/i, -1), b'_i = (-1/i, 0), c'_i = (0, 1/i)$ for $i = 1, 2, \ldots$: let f_i be the straight line interval a_ib_i plus the circular arc b_ic_i on the circle with center at $(0, 0)$; similarly let f'_i be the straight line interval $a'_ib'_i$ plus the circular arc $b'_ic'_i$. Let $N = (f_1 + f_2 + \ldots) + (f'_1 + f'_2 + \ldots)$. Suppose $N = H + K$ separate, where say $H \supset f_1$. Hence $H \supset c_1$ and so $\supset$ infinitely many f'_i; thus $H \supset$ all c'_i and so all f_i. Hence $K = 0$ and N is connected.

THEOREM 5. *There exists in any connected domain D of a euclidean S_3 an arc-wise connected path P, dense in D, which is a decomposable connexe.*

PROOF. One can take the origin and the unit so that the interior R of $x^2 + y^2 + z^2 = 4$ is in D; thus $R \supset N$ of Lemma 1. Let an arc-wise path or trajectory T, dense in D, be taken as follows: 1) The particle will travel so that T finally will contain each f_i and f'_i of Lemma 1 as subarcs; 2) Whenever the image enters R it will not leave until it passes through a point $(0, y)$ where $-1 < y < 1$, i. e. every arc of $R \cdot T \supset$ one of these points; To assure T is dense in D the part of the method for this of [10 : pp. 178-179] may be used. Thus 2) and Lemma 1 gives that $R \cdot T$ is connected. Suppose T is indecomposable. Hence, by Lemma A' of [9 : p. 799], T is the sum of the connexes $R \cdot T$ and $T - R \cdot T$, neither of which have the same closure as T. Therefore T is decomposable.

THEOREM 6. *Let U be the interior of a torus in S_3, where if R is any* *spherical region in S_3, p, $q \in R$, then $R \supset$ a geodesic pq. Then there exists an arc-wise path P of a particle p, which moves counter clock-wise and densely in U, but P is locally connected.*

PROOF. Since U is completely separable there exists a countable set s_1, $s_2, \ldots, s_i, \ldots$, of spheres in U such that any domain D', of U, $\supset$ an s_i. Let $\{p_i\}$ $(i = 1, 2, \ldots.)$ be the class of all possible pairs $p_i = (s_g, s_h)$, where $g \neq h$, of s_g, s_h. Let $\{f_k\}$ be a class of mutually exclusive arcs in U where f_k joins s, s' of $p_k = (s, s')$ as follows: a) by a geodesic (straight) line interval if possible; *b*) if not, then by a curved line: in either case f_k must be such that finally in one revolution of p through U it can move counter clockwise over f_k. Thus p moves densely through U until finally $P \supset$ each f_k. Let R be any region as above in U and suppose $R \cdot P = H + K$ separate. Let $h \in H$ and $k \in K$. Then there exists a $p_k = (s, s')$, where s bounds Q and s' bounds Q', $h \in Q$, $k \in Q'$, and $R \supset Q + Q'$. Hence, by *a*), there exists a geodesic arc $h'k'$ in $P \cdot R$, where $h' \in H \cdot Q$ and $k' \in K \cdot Q'$. As $h'k'$ is connected it lies entirely in H or K. Thus $R \cdot P$ is connected and P is locally.

THEOREM 7. *If P is an arc-wise trajectory contained, and dense, in a locally compact, connected domain D of a subspace S_2 of S_m, then P is an indecomposable trajectory.*

PROOF. Suppose $P = H + K$, where H and K are connexes neither with the same closure as P. Then there exist regions H', K' such that $H' \cdot K = 0 = \overline{H} \cdot \overline{K'}$, $H \supset H' \cdot P$, $K \supset K' \cdot P$; H' can be taken with a simple closed curve h as boundary and K' likewise with k. For the image of a particle p to move densely in D on an arc-wise trajectory P we have: 1) $P \supset$ an arc f_1 and $h \supset$ an arc h_1 with common end points and $f_1 + h_1$ bound a domain D_1, which contains a subarc of k; 2) $D_1 \cdot P \supset$ a similar arc f_2 which divides D_1 into two connected domains D_{11} and D_{12}, 3) For $j = 11$ or 12, $P \cdot D_j \supset$ a similar arc f_j which divides D_j into two connected domains D_{j1} and D_{j2} and this process can be continued by induction; 4) If, on h_1, f_j has end points a, b and f_k has end points c, d, then one can take the f_i such that always on h_1 one of the arcs ab and cd contains the other. We thus obtain a countable class $\{D_j\}$ of domains whose boundaries are contained in $P + h_1$. Consider every sequence, $D'_1, D'_2, \ldots. D'_g$, $\ldots.$, where $D'_g \supset \overline{D}'_{g+1}$: the class of possible $\bigcap \overline{D}'_g$ is uncountable and almost all of them cut K'; at most a countable number of them can contain subarcs of P. Hence there exist uncountable many of these, and so one, which does not contain a point of P and thus separates K, because of [5: Theorem 42; p. 28]. Hence P is an indecomposable connexe.

THEOREM 8. *If the image of a particle p moves densely in a locally compact phase subspace S_2 entirely under forces independent of time on an arc-wise indecomposable trajectory P, then S_2 is the sum of uncountably many mutually exclusive indecomposable trajectories, each dense in S_2.*

PROOF. [We assume: if any image q moves in time t over path Q', then

Q' is closed and $x \in Q'$ implies x is a limit point of $S_2 - Q'$]. If in time from t_0 to t' p moves from p_0 to p', then $p_0 \neq p'$, for if not the trajectory P would repeat this thereafter and not be indecomposable. If q is another particle image moving over trajectory Q, then if $P \cdot Q \neq 0$, P and Q thereafter would have to be the same trajectory; thus $P = Q$, unless $P \cdot Q = 0$. Hence, for $P \cdot Q = 0$, P is in the complement of Q and so, as noted above, Q is an indecomposable connexe. For time t_i such that $\lim t_i = \infty$, if Q_i is the trajectory of image q from t_0 to t_i, Q is the sum of a countable number of continua, each dense in its complement. Thus by Theorem 15 of [5 : p. 11] or by the Theorem of Baire [4 : p. 320], S_2 must be the sum of uncountably many indecomposable connexes, mutually exclusive, and each dense in S_2.

It is to be noted that by the methods of proof used above one could have both a decomposable and an indecomposable trajectory dense in a domain D of S_m, $m > 2$: thus the question arises whether conditions could be put on the phase space in order to make all the trajectories indecomposable; there is also a similar question in Theorem 8 in order to make all the trajectories arc-wise, when one is.

One sees that if a particle moves densely in a subspace S of a conservative phase space S_m as in Khinchin's "Statistical Mechanics", S is an invariant part, hence is metrically indecomposable, and thus is a surface of constant energy [2 : pp. 15, 29, 46]. The construction of the indecomposable connexes above is related to "random walk": If $\bar{D}$, of D above is compact, it has a finite covering, giving rise to a random choice of a chain C_1; the closure of the point set sum D_1 of C_1 has a finite covering, giving rise to a random choice of a chain C_2; continuing by induction one has the statistical question concerning the probability $\bigcap D_i$ will be an indecomposable connexe[3]. Compactness could be omitted.

THEOREM 9. *If D is a connected domain in S_2, then there exists a non-widely connected, hereditarily indecomposable connexe I contained, and dense, in D.*

PROOF. To show this we use the process of Wada to obtain denseness, as in Theorem 1 above, and combine it with that of Bing in [7] to obtain here hereditability. Let the pairs $p_k = (s_i, s_j)$ be as in the proof of Theorem 6. In combining these processes we obtain: 1) By Wada's a set of chains C_k $(k = 1, 2, \dots)$ joining s_i and s_j of p_k and a set of domains D_k, the point set sum of the links of C_k; 2) By Bing's [7 : pp. 268-270] a set of ε_k-crooked chains C'_k contained in D_k and joining s_i and s_j of p_k; 3) These links of C'_k are of diameter ε_k and $\lim \varepsilon_k = 0$; 4) The chain C'_{k+1} has a subchain in D'_k joining s_i and s_j of p_k; 5) Where Z_k is the cylinder without ends of the proof of Theorem 4 of C'_k, the chains are taken so that $Z_g \cdot Z_h = 0$ for $g \neq h$; 6) Here, contrary to [7 : p. 268], D'_k does not separate S_2.

3) A related answer is given by R. H. Bing in *Concerning hereditarily indecomposable continua*, Pacific Journal of Mathematics, v. 1, 1951, p. 46.

Let $I = (\overline{D'_1 \cdot D'_2 \cdot \ldots \cdot D'_k}) + (\overline{D'_2 \cdot D'_3 \cdot \ldots}) + (\overline{D'_3 \cdot D'_4 \ldots}) + \ldots$. Since the D'_k are connected, I is also. Because of 4), $(\overline{D'_g \cdot D'_{g+1}}) \neq 0 \neq I$ and by 1) I is dense in D. The descending tower $\{Z_k\}$ ε-shields I by 5) and so any connected dense subset of I is indecomposable. Suppose $I' = H + K$, where I', H, K are connexes with different closures. By 2) and 4) there exist $s'_j, \in \{s_i\}$, for $j = 1, 2, 3, 4$ so that each C'_k, for $k >$ some g, has a subchain joining s'_1, s'_2, s'_3, s'_4 in that order, where $(s'_1 + s'_4) \cdot I' = 0, s'_2 \cdot K = 0 \neq s'_2 \cdot H$, and $s'_3 \cdot H = 0 \neq s'_3 \cdot K$. Thus each D'_k contains $H + K = I'$. Then following the argument of Condiotin (4) in [7 : p. 268] we see that there must be an arc pxq in a C'_j which is not ε_k-crooked, which is a contradiction. Thus I is a hereditarily indecomposable connexe.

The arc-wise indecomposable trajectories are nicely behaving compared with these very peculiar hereditarily indecomposable connexes. If motion takes place on these peculiar connexes, it would not seem to be either as a projectile or as a wave. The question concerning the types of peculiar sets which can exist is mostly unexplored; it would seem that to understand the nature of space they must be explored; perhaps finally this must be done even to understand the nature of matter.

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