ON THE CONJUGATE SPACE OF OPERATOR ALGEBRA

MASAMICHI TAKESAKI

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The purpose of this paper is to study the structure of operator algebras as Banach spaces that has been developed by J. Dixmier, Z. Takeda, S. Sakai and others. § 1 is devoted to study the structure of the conjugate spaces of an operator algebra, in which a certain decomposition theorem on the conjugate space is proved with some applications. In § 2, we prove the decomposition theorem of a homomorphism from an operator algebra into the other, corresponding to the decomposition of the conjugate space in §1. Then, using this result we study the continuity property of a homomorphism and find the alternative proof of those results which are discussed in J. Feldman and J. Fell [7].

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1. The conjugate space of operator algebra. We denote always by $\mathbf{M}^*$ for the conjugate space of a Banach space $\mathbf{M}$ and if $\mathbf{M}$ is a W^*-algebra we write the space of all σ-weakly continuous linear functionals on $\mathbf{M}$ by $\mathbf{M}_*$. Let $\mathbf{M}$ be a C^*-algebra, then the second conjugate space $\mathbf{M}^{**}$ of $\mathbf{M}$ is a W^*-algebra and its σ-weak topology coincides with $\sigma(\mathbf{M}^{**}, \mathbf{M}^*)$-topology by Z. Takeda [11]. $\mathbf{M}^{**}$ has further properties such as any $*$-representation of $\mathbf{M}$ on some Hilbert space has a unique σ-weakly continuous extension to a $*$-representation of $\mathbf{M}^{**}$. Thus we call this W^*-algebra *universal enveloping algebra* of $\mathbf{M}$ and denote by $\widetilde{\mathbf{M}}$.

Next, we define the operators L_a, R_a on $\mathbf{M}^*$, the conjugate space of C^*-algebra $\mathbf{M}$, for $a \in \mathbf{M}$ as follows;

$< x, L_a\varphi > = < ax, \varphi >$ and $< x, R_a\varphi > = < xa, \varphi >$ for all $x \in \mathbf{M}$, $\varphi \in \mathbf{M}^*$. Then the following properties are easily verified.

$$L_{(\lambda a+\mu b)} = \lambda L_a + \mu L_b, \qquad R_{(\lambda a+\mu b)} = \lambda R_a + \mu R_b$$

$$L_{(ab)} = L_b L_a, \qquad R_{(ab)} = R_a R_b,$$

where a and b are arbitrary elements of $\mathbf{M}$, λ and μ arbitrary complex numbers. We denote the family of all L_a (resp. R_a) by $\mathbf{L}_{(\mathbf{M})}$ (resp. $\mathbf{R}_{(\mathbf{M})}$). Next we call a subspace V of $\mathbf{M}^*$ left invariant (resp. right invariant) if it is invariant under $\mathbf{L}_{(\mathbf{M})}$ (resp. $\mathbf{R}_{(\mathbf{M})}$). Especially, we call a two-sided invariant subspace invariant simply. Then we have the duality of left ideal and left invariant subspace in the following

THEOREM 1. *Let* $\mathbf{M}$ *be a* C^**-algebra, then there exists a one-to-one corres-*

pondence between the σ-weakly closed left ideal (resp. right ideal) $\mathfrak{m}$ of $\widetilde{\mathbf{M}}$ and the closed left invariant (resp. right invariant) subspace V of $\mathbf{M}^*$ such that

$$\mathfrak{m} = V^\circ \quad \text{and} \quad V = \mathfrak{m}^\circ$$

where V° and $\mathfrak{m}^\circ$ are the polar of V and $\mathfrak{m}$ in $\widetilde{\mathbf{M}}$ and $\mathbf{M}^*$ respectively.

Especially, if $\mathbf{M}$ is a W^*-algebra, then there exists a one-to-one correspondence between the σ-weakly closed left ideal (resp. right ideal) and closed left invariant (resp. right invariant) subspace of $\mathbf{M}_*$.

PROOF. Let V be a left-invariant closed subspace of $\mathbf{M}^*$ and $\mathfrak{m} = V^\circ$. Since $L_aV \subset V$ for all $a \in \mathbf{M}$, we have $< ax, V > = < x, L_aV > = < x, V > = 0$ for all $x \in \mathfrak{m}$. Hence we have $ax \in \mathfrak{m}$ for all $a \in \mathbf{M}$, $x \in \mathfrak{m}$. Therefore $\mathfrak{m}$ is a left ideal of $\widetilde{\mathbf{M}}$, for $\mathfrak{m}$ is σ-weakly closed and $\mathbf{M}$ σ-weakly dense in $\widetilde{\mathbf{M}}$.

The converse correspondence and the second part of our theorem are clear by the above arguments. This concludes the proof.

Using the similar argument we shall study the maximal left ideal of a C^*-algebra. Let $\mathbf{M}$ be a C^*-algebra and φ a positive linear functional on $\mathbf{M}$. We call a subset of $\mathbf{M}$ $\mathfrak{m}_\varphi = \{x \in \mathbf{M} : < x^*x, \varphi > = 0\}$ the *left kernel* of φ by the terminology of R. Kadison [9]. Then we have the following

THEOREM 2. *Let $\mathbf{M}$ be a C^*-algebra, φ a pure state on $\mathbf{M}$, then $\mathbf{M}/\mathfrak{m}_\varphi$, the factor space of $\mathbf{M}$ by the left kernel $\mathfrak{m}_\varphi$ becomes a Hilbert space as quotient space and its norm coincides with the one canonically induced by φ.*

PROOF. (1) Case where $\mathbf{M}$ is a W^*-algebra and φ σ-weakly continuous: By the continuity of φ, there exists a minimal projection e of $\mathbf{M}$ such that $\mathfrak{m}_\varphi = \mathbf{M}(1-e)$ where e is the carrier projection of φ. Hence $\mathbf{M}/\mathfrak{m}_\varphi$ is algebraically isomorphic to $\mathbf{M}e$ by the natural correspondence.

If we denote by $\bar{x}$ the element of $\mathbf{M}/\mathfrak{m}_\varphi$ corresponding to $x \in \mathbf{M}e$, we get

$$\begin{aligned}\|\bar{x}\| &= \inf[\|x+y\|;\ y \in \mathfrak{m}_\varphi] = \inf[\|xe + y(1-e)\|;\ y \in \mathbf{M}] \\ &\geqq \|xe\| = \|x\| \geqq \|\bar{x}\|.\end{aligned}$$

Therefore, the mapping $x \to \bar{x}$ is an isometry from $\mathbf{M}e$ onto $\mathbf{M}/\mathfrak{m}_\varphi$.

Now, we consider the canonical representation π_φ of $\mathbf{M}$ on the Hilbert space H_φ induced by φ in the sense of I. E. Segal. Then there exists a cyclic vector ξ_φ of H_φ such that $< x, \varphi > = (\pi_\varphi(x)\xi_\varphi, \xi_\varphi)$ for $x \in \mathbf{M}$. Since $\pi_\varphi(e)$ is the projection to the one-dimensional subspace of H_φ spanned by ξ_φ, we have

$$\begin{aligned}\|xe\| &= \|ex^*xe\|^{1/2} = \|\pi_\varphi(e)\pi_\varphi(x)^*\pi_\varphi(x)\pi_\varphi(e)\|^{1/2} \\ &= (\pi_\varphi(xe)\xi_\varphi, \pi_\varphi(xe)\xi_\varphi)^{1/2} = \|\pi_\varphi(xe)\xi_\varphi\|.\end{aligned}$$

Moreover, $\pi_\varphi(\mathbf{M})$ is the full operator algebra on H_φ, for π_φ is a σ-weakly continuous irreducible representation. Therefore, the mapping $x \to \pi_\varphi(x)\xi_\varphi$ is an isometry from $\mathbf{M}e$ onto H_φ. Hence $\mathbf{M}/\mathfrak{m}_\varphi$ is isometric to H_φ by the canonical correspondence.

(2) General case: Let $\widetilde{\mathbf{M}}$ be the universal enveloping algebra of $\mathbf{M}$,

then φ is a σ-weakly continuous pure state on $\widetilde{\mathbf{M}}$. Let $\widetilde{\mathfrak{m}}_\varphi$ be the left kernel of φ in $\widetilde{\mathbf{M}}$, then the factor space $\widetilde{\mathbf{M}}/\widetilde{\mathfrak{m}}_\varphi$ coincides with the Hilbert space canonically constructed by φ in case (1).

Put $V = \widetilde{\mathfrak{m}}_\varphi{}^0$, then V is a left invariant subspace of M^* by Theorem 1 and $V^* = \widetilde{\mathbf{M}}/\widetilde{\mathfrak{m}}_\varphi$. Hence V is a Hilbert space, so that V is $\sigma(\mathbf{M}^*, \mathbf{M})$-closed by its reflexivity, and $V = \mathfrak{m}_\varphi{}^0$. Thus we have

$$(\mathbf{M}/\mathfrak{m}_\varphi)^* = V \quad \text{and} \quad (\mathbf{M}/\mathfrak{m}_\varphi)^{**} = \widetilde{\mathbf{M}}/\widetilde{\mathfrak{m}}_\varphi,$$

that is, $\mathbf{M}/\mathfrak{m}_\varphi$ is a Hilbert space and its norm coincides with the one induced by φ. This concludes the proof.

Applying this result for irreducible $*$-representation of C^*-algebra, we can give the alternative proof of the Theorem due to R. V. Kadison [9]. But we omit the detail.

DEFINITION 1. A positive linear functional φ on a W^*-algebra $\mathbf{M}$ is called *singular* if there exists no non-zero σ-weakly continuous positive linear functional ψ on $\mathbf{M}$ such as $\psi \leqq \varphi$. Moreover we call a linear functional ψ on $\mathbf{M}$ *singular* too, if φ is a linear combination of singular positive linear functionals as defined above. We denote by $\mathbf{M}_*^+$ the space of all singular elements of $\mathbf{M}^*$.

THEOREM 3. *Let $\mathbf{M}$ be a W^*-algebra, then the left (or right) invariant closed subspace V of $\mathbf{M}^*$ is uniquely decomposed into the l^1-direct sum of its σ-weakly continuous part $V \cap M_*$ and singular part $V \cap \mathbf{M}_*^+$, i.e.*

$$V = (V \cap \mathbf{M}_*) \oplus_{l^1} (V \cap \mathbf{M}_*^+).$$

Moreover $V \cap \mathbf{M}_*$ and $V \cap \mathbf{M}_*^+$ are written by the central projection z_0 of $\widetilde{\mathbf{M}}$ as follows:

$$V \cap \mathbf{M}_* = R_{z_0}V \quad and \quad V \cap \mathbf{M}_* = R_{(1-z_0)}V.$$

PROOF. As $\mathbf{M}_*$ is an invariant subspace of $\mathbf{M}^*$, There exists a central projection z_0 of $\widetilde{\mathbf{M}}$ such that $\mathbf{M}_*^0 = \widetilde{\mathbf{M}}(1 - z_0)$ by Theorem 1. Hence we have easily $M_* = R_{z_0}\mathbf{M}^*$.

Next we shall show $\mathbf{M}_*^+ = R_{(1-z_0)}\mathbf{M}^*$. In fact, if φ is a positive linear functional of $R_{(1-z_0)}\mathbf{M}^*$ and there exists a σ-weakly continuous positive linear functional ψ such as $\psi \leqq \varphi$, then we have $< x^*xz_0, \psi > \leqq < x^*xz_0, \varphi > = 0$ for all $x \in \widetilde{\mathbf{M}}$, which implies $R_{z_0}\psi = 0$. On the other hand, we have $R_{(1-z_0)}\psi = 0$ by the continuity of ψ and the above argument, so that we have $\psi = 0$. Hence φ is singular. Conversely, if φ is singular positive, then $R_{z_0}\varphi = 0$ because $R_{z_0}\varphi \leqq \varphi$ and $R_{z_0}\varphi \in \mathbf{M}_*$. Hence we have $\mathbf{M}_*^+ = R_{(1-z_0)}\mathbf{M}^*$, for $\mathbf{M}_*^+$ and $R_{(1-z_0)}\mathbf{M}^*$ are spanned by their positive parts respectively.

Therefore, we have $\mathbf{M}_* = R_{z_0}\mathbf{M}^*$, $\mathbf{M}_*^+ = R_{(1-z_0)}\mathbf{M}^*$ and $\mathbf{M}^* = \mathbf{M}_* \oplus_{l^1} \mathbf{M}_*^+$. Moreover, since V is invariant under R_{z_0} and $R_{(1-z_0)}$, one may easily verify that $V = (V \cap \mathbf{M}_*) \oplus_{l^1} (V \cap \mathbf{M}_*^+)$. This concludes the proof.

This theorem includes the decomposition of a finitely additive measure into the completely additive part and the purely finitely additive part in the commutative case due to Yosida-Hewitt [16].

Using Theorem 3, we can show the following generalization of the well known Dixmier's result.

COROLLARY 1. *Let* $\mathbf{M}$ *be a* W^**-algebra, then a bounded linear functional* φ *on* $\mathbf{M}$ *is* σ*-weakly continuous if and only if* φ *is* σ*-weakly continuous on every maximal abelian* W^**-subalgebra of* $\mathbf{M}$.

PROOF. It suffices to prove only the sufficiency. By Theorem 3, it is sufficient to show that there exists no non-zero singular linear functional satisfying the hypothesis. Suppose there exists a non-zero singular linear functional φ satisfying the hypothesis. Moreover we may assume, without loss of generality, that φ is self-adjoint (i.e. φ is real valued on the self-adjoint part of $\mathbf{M}$).

There exists a projection e of $\mathbf{M}$ such as $< e, \varphi > \ > 0$ (or < 0). Hence we may assume $< 1, \varphi > \ > 0$, considering the restriction of φ on $e\mathbf{M}e$.

Now, if there exists a non-zero projection e of $\mathbf{M}$ such that $< f, \varphi > \ > 0$ for all non-zero projection $f \leqq e$, $L_e R_e \varphi$ is a non-zero σ-weakly continuous linear functional by its positiveness. This contradicts to the singularity of φ. Therefore, for any non-zero projection e of $\mathbf{M}$, there exists a non-zero projection $f \leqq e$ such as $< f, \varphi > \ \leqq 0$.

Hence, if $\{e_\alpha\}$ is a maximal family of orthogonal projections such as $< e_\alpha, \varphi > \ \leqq 0$, we have $\Sigma_\alpha e_\alpha = 1$. Considering the maximal abelian W^*-subalgebra generated by $\{e_\alpha\}$, we get $< 1, \varphi > \ \leqq 0$ by the continuity of φ on this subalgebra. This contradicts to the hypothesis $< 1, \varphi > \ > 0$. This concludes the proof.

Applying this result, we can show the following generalization of Umegaki's result in the case of semi-finite type [14].

COROLLARY 2. *Let* $\mathbf{M}$ *be a* W^**-algebra, then a subset* K *of* $\mathbf{M}_*$ *is relatively* $\sigma(\mathbf{M}_*, \mathbf{M})$*-compact if and only if, for each maximal abelian* W^**-subalgebra* $\mathbf{A}$ *of* $\mathbf{M}$*, the restriction of* K *on* $\mathbf{A}$ *is relatively* $\sigma(\mathbf{A}_*, \mathbf{A})$*-compact in* $\mathbf{A}_*$.

PFOOF. It suffices to prove only the sufficiency. Since $\mathbf{M}$ is the linear span of the self-adjoint part of $\mathbf{M}$, K is simply bounded on $\mathbf{M}$, so that K is bounded by Banach-Steinhaus' Theorem.

Hence, if we imbed canonically K into $\mathbf{M}^*$ and denote the $\sigma(\mathbf{M}^*, \mathbf{M})$-closure of K by $\widetilde{K}, \widetilde{K}$ is $\sigma(\mathbf{M}^*, \mathbf{M})$-compact. We shall show $\widetilde{K} \subset \mathbf{M}_*$. In fact, for each maximal abelian W^*-subalgebra $\mathbf{A}$ of $\mathbf{M}$, the restriction of $\widetilde{K}$ on $\mathbf{A}$ is contained in the $\sigma(\mathbf{A}^*, \mathbf{A})$-closure of the restriction of K on $\mathbf{A}$. Hence the restriction of $\widetilde{K}$ on $\mathbf{A}$ is contained in $\mathbf{A}_*$ by the hypothesis. Therefore, for each $\varphi \in \widetilde{K}$, φ is σ-weakly continuous on $\mathbf{A}$. Thus we have $\widetilde{K} \subset \mathbf{M}_*$ by Corollary 1, that is, K is relatively $\sigma(\mathbf{M}_*, \mathbf{M})$-compact. This concludes the proof.

We conclude this section to prove the characterization of a singular positive linear functional in the commutative case.

THEOREM 4. *Let* $\mathbf{A}$ *be an abelian* W^**-algebra,* Ω *the spectrum space of* $\mathbf{A}$*. Then, in order that the position linear functional on* $\mathbf{A}$ *induced by a Radon measure* μ *on* Ω *is singular it is necessary and sufficient that its support is non-dense.*

PROOF. Sufficiency is clear from the definition of singularity so that it suffices to prove only the necessity. Let $F = \{N_\alpha\}$ be a maximal family of mutually disjoint non dense Borel sets such as $\mu(N_\alpha) \neq 0$, then F is at most enumerable, for $\mu(\bigcup_\alpha N_\alpha) = \Sigma_\alpha \mu(N_\alpha) \leqq \mu(\Omega) < +\infty$. Since an enumerable sum of non-dense sets is also non-dense in a hyperstonean space by Dixmier's result [3], $N_0 = \bigcup_\alpha N_\alpha$ is non-dense.

Let N be any non-dense Borel set of Ω, then $\mu(N - N_0) = 0$ by the maximality of F. Put $< f, \mu_1 > = < \chi_{N_0} f, \mu >$ for $f \in C(\Omega) \cong \mathbf{A}$ and $\mu_2 = \mu - \mu_1$, where χ_{N_0} is the characteristic function of N_0, then any non-dense Borel set is μ-measure zero, so that μ_2 is a normal measure by Dixmier's result [3]. Therefore, if the positive linear functional on $\mathbf{A}$ induced by μ is singular, we have $\mu_2 = 0$, so that $\mu = \mu_1$ has a non-dense support. This concludes the proof.

2. Homomorphism of operator algebra. We start with the following

DEFINITION 2. A bounded linear mapping θ from a W^*-algebra $\mathbf{M}$ to another W^*-algebra $\mathbf{N}$ is called *singular* if ${}^t\theta(\mathbf{N}_*) \subset \mathbf{M}_*^\perp$, where ${}^t\theta$ is the transpose of θ.

We get the following theorem corresponding to the decomposition theorem for the conjugate spaces of operator algebras.

THEOREM 5. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two* W^**-algebras and* π *a* $*$*-homomorphism from* $\mathbf{M}$ *into* $\mathbf{N}$*, then there exists a central projection* z *in the* σ*-weak closure of* $\pi(\mathbf{M})$ *such that if we define the* $*$*-homomorphisms* π_1 *and* π_2 *from* $\mathbf{M}$ *into* $\mathbf{N}$ *as follows:*

$$\pi_1(x) = \pi(x)z$$
$$\pi_2(x) = \pi(x)(1-z) \quad \textit{for all } x \in \mathbf{M},$$

then π_1 *is* σ*-weakly continuous and* π_2 *is singular.*

PROOF. We may assume, without loss of generality, that $\pi(\mathbf{M})$ is σ-weakly dense in $\mathbf{N}$.

Let z_0 be the central projection of $\widetilde{\mathbf{M}}$ as in Theorem 3. π is uniquelly extended to a σ-weakly continuous $*$-homomorphism $\widetilde{\pi}$ from $\widetilde{\mathbf{M}}$ onto $\mathbf{N}$ by A. Grothendieck [8]. If we put $z = \widetilde{\pi}(z_0)$, then z is clearly a central projection of $\mathbf{N}$ and has the properties in our theorem.

In fact, let φ be an arbitrary element of $\mathbf{N}_*$, then for all $x \in \mathbf{M}$ we have

$$\begin{aligned} < x, \ {}^t\pi_1(\varphi) > &= < \pi_1(x), \ \varphi > = < \pi(x)z, \ \varphi > \\ &= < \tilde{\pi}(xz_0), \ \varphi > = < xz_0, \ {}^t\pi(\varphi) > \\ &= < x, \ R_{z_0}{}^t\pi(\varphi) > \end{aligned}$$

where ${}^t\pi_1$ is the transpose of π_1. Since $R_{z_0}\mathbf{M}^* = \mathbf{M}_*$ by Theorem 3, we have ${}^t\pi_1(\varphi) \in \mathbf{M}_*$, so that π_1 is σ-weakly continuous. By similar computation we have ${}^t\pi_2(\varphi) = R_{(1-z_0)}{}^t\pi(\varphi)$ for all $\varphi \in \mathbf{N}_*$, so that ${}^t\pi_2(\mathbf{N}_*) \subset \mathbf{M}_*^\perp$, i. e. π_2 is singular. This concludes the proof.

Combining the above result with the characterization of a singular functional in §1, we study the σ-weak continuity of the $*$-homomorphism from a W^*-algebra into another σ-finite one.

LEMMA 1. *Let $\mathbf{A}$ be an abelian W^*-algebra, $\mathbf{B}$ a σ-finite W^*-algebra and π a $*$-isomorphism from $\mathbf{A}$ into $\mathbf{B}$, then there exists a non-zero projection e of $\mathbf{A}$ such that π is σ-weakly continuous on $\mathbf{A}e$. Moreover, the σ-weakly continuous part π_1 of π is a $*$-isomorphism form $\mathbf{A}$ into $\mathbf{B}$ and the singular part π_2 of π is a non-faithful $*$-homomorphism from $\mathbf{A}$ into $\mathbf{B}$.*

PROOF. $\pi_1^{-1}(0)$ is a σ-weakly closed ideal, so that there exists a projection z of A such that $\pi_1^{-1}(0) = \mathbf{A}z$. If $z \neq 0$, then π_2 is a singular $*$-isomorphism from $\mathbf{A}z$ into $\mathbf{B}$.

Now, there exists a faithful σ-weakly continuous state φ on $\mathbf{B}$ by σ-finiteness of $\mathbf{B}$. Put $\psi = {}^t\pi_2(\varphi)$, then ψ is a singular positive linear functional on $\mathbf{A}$.

Hence there exists a non-zero projection $e \leqq z$ such as $< e, \ \psi > = 0$ by Theorem 4, so that we have

$$< \pi_2(e), \varphi > = < e, {}^t\pi_2(\varphi) > = < e, \psi > = 0,$$

which implies $\pi_2(e) = 0$. This contradicts to the faithfulness of π_2 on $\mathbf{A}z$. Therefore we have $z = 0$, i. e. π_1 is faithful on $\mathbf{A}$.

Finally, there exists a non-zero projection e of $\mathbf{A}$ such as $\pi_2(e) = 0$ by the above argument, so that $\mathbf{A}e \subset \pi_2^{-1}(0)$. Then, π and π_1 coincide on $\mathbf{A}e$ and π_2 is not faithful. Therefore, π is σ-weakly continuous on $\mathbf{A}e$. This concludes the proof.

LEMMA 2. *Let $\mathbf{M}$ be a σ-finite W^*-algebra, then there exists a maximal abelian W^*-subalgebra $\mathbf{A}_0$ of $\mathbf{M}$ as follows: if a $*$-homomorphism π from $\mathbf{M}$ to another one $\mathbf{N}$ is σ-weakly continuous on $\mathbf{A}_0$, then π is σ-weakly continuous on $\mathbf{M}$. Moreover, if $\mathbf{M}$ is of finite type or type $\mathbf{III}$, any maximal abelian W^*-subalgebra of $\mathbf{M}$ has this property.*

PROOF. It is sufficient to show that π is σ-weakly continuous on all abelian W^*-subalgebras which are generated by the family of orthogonal projections.

(1) Type $\mathbf{I}_n$ case: Let $\mathbf{A}_0$ be a maximal abelian W^*-sub lgebra of $\mathbf{M}$, $\{e_n\}$ a family of orthogonal projections and $\mathbf{A}$ be a maximal abelian W^*-sub-

algebra containing $\{e_n\}$, then $u^*\mathbf{A}_0u = \mathbf{A}$ for some unitary $u \in \mathbf{M}$. From the assumption, the mappings

$$x \to uxu^* \to \pi(uxu^*) = \pi(u)\pi(x)\pi(u^*)$$
$$\to \pi(u^*)\pi(u)\pi(x)\pi(u^*)\pi(u) = \pi(x)$$

are continuous on $\mathbf{A}$, hence π is continuous on $\mathbf{M}$.

(2) Type $\mathbf{II}_1$ case: Let $\mathbf{A}_0$ be any maximal abelian W^*-subalgebra of $\mathbf{M}$, then for any projection of $\mathbf{M}$ there exists a projection of $\mathbf{A}_0$ equivalent to this projection. Moreover, one easily verifies that for any projection $p \precsim e$ ($e \in \mathbf{A}_0$, $p \in \mathbf{M}$) there exists also a projection q of $\mathbf{A}$, such as $p \sim q \leqq e$. Therefore, for an arbitrary family of orthogonal projections $\{p_n\}$ in $\mathbf{M}$, we can get the family of orthogonal projections $\{q_n\}$ in $\mathbf{A}_0$ such as $p_n \sim q_n$. Hence there exists a partial isometry u in $\mathbf{M}$ as u induces the equivalency for all p_n and q_n. Furthermore we can choose u as a unitary of $\mathbf{M}$ by the finiteness of $\mathbf{M}$. That is, $up_nu^* = q_n$ for all n.

Now we consider the mappings as follows:

$$x \to uxu^* \to \pi(uxu^*) = \pi(u)\pi(x)\pi(u^*)$$
$$\to \pi(u)^*\pi(u)\pi(x)\pi(u^*)\pi(u) = \pi(x),$$

for $x \in \mathbf{M}$. The continuity of π on $\mathbf{A}_0$ implies that of π on an abelian W^*-subalgebra generated by $\{p_n\}$. This implies the σ-weak continuity of π, for $\{p_n\}$ is an arbitrary family of orthogonal projections in $\mathbf{M}$.

(3) Type $\mathbf{I}_\infty$ or $\mathbf{II}_\infty$ case: Let $\{e_n\}$ be a family of orthogonal finite projections such as $\Sigma e_n = 1$ and A_0 an arbitrary maximal abelian W^*-subalgebra containing them.

For a projection $p \in \mathbf{M}$, we shall show that $\mathbf{A}_0$ contains a finite projection q as $q \precsim p$. By the assumption on $\{e_n\}$ there exists a number n_0 such that $c(p)e_{n_0} \neq 0$ where $c(p)$ means the central envelope of p. Hence we get a central projection z such as $zp \precsim ze_{n_0}$, $(1-z)p \succsim (1-z)e_{n_0}$ and $zp \neq 0$ or $(1-z)e_{n_0} \neq 0$. If $(1-z)e_{n_0} \neq 0$, then $q = (1-z)e_{n_0}$ is the desired one. If $(1-z)e_{n_0} = 0$ we have $zp \neq 0$, which implies the existence of a finite projection e of $\mathbf{A}_0$ as $zp \precsim e$. Applying the argument in case (1) and (2) for a finite W^*-algebra $e\mathbf{M}e$ and its maximal abelian W^*-subalgebra $e\mathbf{A}_0$, we get a projection $q \in \mathbf{A}_0$ as $zp \sim q$.

Now suppose a family F such that $F = \{q \in \mathbf{A}_0\,;\, q \precsim p\}$ for a finite projection p of $\mathbf{M}$. The above argument shows that F is a non-empty inductive set, so that there exists a maximal projection q of F and p becomes equivalent to q. Therefore, for a finite projection such as $p \precsim e$ for some projection e of $\mathbf{A}_0$ we get a projection $q \in \mathbf{A}_0$, such as $p \sim q \leqq e$ considering with a W^*-algebra $e\mathbf{M}e$ and its maximal abelian W^*-subalgebra $e\mathbf{A}_0$. Next, if p is an arbitrary projection of $\mathbf{M}$ there exists a family of orthogonal finite projections $\{p_n\}$ such as $p = \Sigma p_n$. Hence, we have a family of orthogonal projection $\{q_n\}$ of $\mathbf{A}_0$ such that $p_n \sim q_n$ for all n. Put $q = \Sigma q_n$, then $q \in \mathbf{A}_0$ and $p \sim q$. Therefore the same computation for $\mathbf{M}$ and $\mathbf{A}_0$ used above shows that $p \precsim e$, $e \in \mathbf{A}_0$ implies the existence of a projection $q \in \mathbf{A}_0$ such as $p\sim q\leqq e$.

After all, the above arguments show that $\mathbf{A}_0$ contains a projection e with $e\sim(1-e)$. Take an arbitrary family $\{p_n\}$ of orthogonal projections of $e\mathbf{M}e$, then $(1-e)\mathbf{A}_0$ contains a family $\{q_n\}$ of orthogonal projections such as $q_n\sim p_n$ for all n. Let v_n be a partial isometry of $\mathbf{M}$ that gives an equivalency of p_n and q_n, that is, $v_n^*v_n=p_n$ and $v_nv_n^*=q_n$. Put $u=\Sigma_n(v_n+v_n^*)+(1-\Sigma_n(p_n+q_n))$, then u is a unitary and induces all equivalencies between $\{p_n\}$ and $\{q_n\}$. Considering the decomposition of π by this unitary u as in case (1) and (2), we get the σ-weak continuity of π on the abelian W^*-subalgebra of $e\mathbf{M}e$ generated by family $\{p_n\}$. Therefore π is σ-weakly continuous on $e\mathbf{M}e$.

If we decompose π into its continuous part and singular one, the singular part vanishes on $e\mathbf{M}e$, hence vanishes on $\mathbf{M}$ because of the equivalency of e and $(1-e)$. This completes the proof of case (3).

(4) Type **III** case: Take an arbitrary maximal abelian W^*-subalgebra $\mathbf{A}_0$ and a projection p of $\mathbf{M}$. Denoting by $c(p)$ the central envelope of p we consider the maximal family $\{q_n\}$ of orthogonal projections of $\mathbf{A}_0$ whose central envelopes are orthogonal each other and covered by $c(p)$. Put $q=\Sigma_nq_n$, then, by the maximality of $\{q_n\}$, we have $c(p)=c(q)$, which implies $p\sim q$. Therefore the same computation as in case (3) shows the continuity of π on $\mathbf{M}$ if π is continuous on $\mathbf{A}_0$. This concludes the proof.

REMARK. Though we restricted our arguments to the case of σ-finite W^*-algebra, the above proposition holds without the assumption of σ-finiteness. We omit the proof because this is immaterial for our further arguments.

THEOREM 6. *Let π be a $*$-isomorphism from a σ-finite W^*-algebra $\mathbf{M}$ into a σ-finite W^*-algebra $\mathbf{N}$, then the continuous part π_1 of π is a $*$-isomorphism and the singular part π_2 of π is always non-faithful $*$-homomorphism.*

PROOF. We can find a central projection z of $\mathbf{M}$ such as $\pi_1^{-1}(0)=\mathbf{M}z$. If $z\neq 0$, π_2 is singular faithful on $\mathbf{M}z$. So we may assume, without loss of generality, that π_2 is faithful on $\mathbf{M}$. Let $\mathbf{A}_0$ be a maximal abelian W^*-subalgebra in the sense of Lemma 2. By Lemma 1 there exists some non-zero projection e of $\mathbf{A}_0$ such that π_2 is σ-weakly continuous on $e\mathbf{A}_0$. Hence π_2 is σ-weakly continuous on $e\mathbf{M}e$ as $e\mathbf{A}_0$ has the same property in $e\mathbf{M}e$ as $\mathbf{A}_0$ does in $\mathbf{M}$. Since π_2 is singular we have $\pi_2(e\mathbf{M}e)=0$. This yields a contradiction.

Therefore π_1 is faithful and π_2 is not. This concludes the proof.

The above discussions have some applications. We shall show in the following that some of the results of J. Feldman and J. Fell [7] are proved by this method.

At first, we start with the lemma which is somewhat well known (cf. J. Calkin [1]).

LEMMA 3. *If the kernel of a $*$-homomorphism π from a σ-finite, properly infinite W^*-algebra $\mathbf{M}$ into a W^*-algebra $\mathbf{N}$ is not σ-weakly closed, then there exists a family of orthogonal projections in $\mathbf{N}$ with the cardinal of at least the continum.*

PROOF. By our assumption there exists a monotone increasing sequence $\{e_n\}$ in $\mathfrak{m}=\pi^{-1}(0)$ with $e=\bigvee_{n=1}^{\infty} e_n \notin \mathfrak{m}$.

Take a family of orthogonal projections $\{p_n\}$ in $\mathbf{M}$ and partial isometries $\{v_n\}$ such that $p_n \sim 1$, $v_n^* v_n = p_n$ and $v_n v_n^* = 1$ for all n. Define the family $\{q_n\}$ of orthogonal projections by $q_n = v_n^* e_n v_n$, then we have

$$\sum_{i=1}^{\infty} q_{n_i} \gtrsim e_{n_1} + \sum_{i=1}^{\infty}(e_{n_{i+1}} - e_{n_i}) = \bigvee_{n=1}^{\infty} e_n = e \notin \mathfrak{m}.$$

Next, let $\{r_n\}$ be a countable set of all rational numbers. For any real number s we can choose an infinite subsequence of $\{r_{n_i}\}$ so that they coverge to s. Correspond s to an index set $\{n_i\}$ of the above sequence $\{r_{n_i}\}$. We have, if $s \neq s'$ for real numbers s and s', $\{n_i\} \cap \{n_i'\}$ as at most a finite set where $\{n_i\}$ and $\{n_i'\}$ are corresponding index sets of s and s' respectively. Therefore, if we set $q_s = \sum_{i=1}^{\infty} q_{n_i}$ for a real number s where $\{n_i\}$ is an index set corresponding to s, we get $q_s \notin \mathfrak{m}$ and $q_s q_{s'} \in \mathfrak{m}$ for $s \neq s'$. Hence the family $\{\pi(q_s)\}$ of orthogonal non-zero projections of $\mathbf{N}$ is of the power of continuum at least.

THEOREM 7. *Let $\mathbf{M}$ be a finite factor or σ-finite, properly infinite W^*-algebra, then any $*$-homomorphism π from $\mathbf{M}$ into σ-finite W^*-algebra $\mathbf{N}$ is σ-weakly continuous.*

PROOF. Since a finite factor $\mathbf{M}$ is simple, there exists no non-trivial non-faithful homomorphism on $\mathbf{M}$. Therefore π is σ-weakly continuous by Theorem 6.

Let $\mathbf{M}$ be properly infinite and suppose the kernel of the singular part π_2 of π is σ-weakly closed, then there exists a central projection z of $\mathbf{M}$ such that $\pi_2^{-1}(0) = \mathbf{M}(1-z)$. Hence π_2 is a $*$-isomorphism from $\mathbf{M}z$ to $\mathbf{N}$, which is impossible by Theorem 6. Therefore we have $z=0$, i.e. $\pi_2 = 0$. On the other hand, if $\pi_2^{-1}(0)$ is not σ-weakly closed, then $\mathbf{N}$ is not σ-finite by Lemma 3, which is a contradiction. We get, after all, $\pi_2 = 0$. This completes the proof.

COROLLARY. *Let $\mathbf{M}$ be a finite factor or σ-finite, properly infinite W^*-algebra and $\mathbf{N}$ a W^*-algebra on some separable Hilbert space, then any bounded homomorphism (not necessarily $*$-preserving) π from $\mathbf{M}$ into $\mathbf{N}$ is σ-weakly continuous.*

PROOF. Let V be the closure of ${}^t\pi(\mathbf{N}_*)$ in $\mathbf{M}^*$, then V is a closed invariant subspace of $\mathbf{M}^*$. Since $\mathbf{N}_*$ is separable by our assumption for $\mathbf{N}$, V is also separable. Therefore, $V \cap \mathbf{M}_*^+$ is also a closed invariant separable subspace of $\mathbf{M}^*$. Hence there exists a central projection z of $\widetilde{\mathbf{M}}$ such as $(V \cap \mathbf{M}_*^+)^0 = \widetilde{M}z$ and $(V \cap \mathbf{M}_*^+)^* \cong \widetilde{\mathbf{M}}(1-z)$. By the separability of $V \cap \mathbf{M}_*^+$, $\widetilde{\mathbf{M}}(1-z)$ is a σ-finite W^*-algebra. Therefore, by Theorem 7, the $*$-homomorphism from $\mathbf{M}$ to $\widetilde{\mathbf{M}}(1-z)$ such as $x \to x(1-z)$ for $x \in \mathbf{M}$ must be σ-weakly continuous. Hence $V \cap \mathbf{M}_*^+ = 0$ and by the equality $V = (V \cap \mathbf{M}_*) \oplus_{l^1} (V \cap \mathbf{M}_*^+)$ we get $V \subset \mathbf{M}_*$. That is, π is σ-weakly continuous.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.