ON INCOMPLETE INFINITE DIRECT PRODUCT OF W^*-ALGEBRAS

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(Received March 5, 1958)

The purpose of this paper is to show some results on incomplete infinite direct products of W^*-algebras. As we can see from [7: Part **IV**, Chap. 7], the behaviour of complete infinite direct product of W^*-algebras is not caught. Hence we shall do not pursue the complete infinite direct product of W^*-algebras. [1]

1. Preliminaries. Throughout this paper, let $\mathbf{I}$ be a set of indices of arbitrary size and let for each $i \in \mathbf{I}$ a W^*-algebra $\mathbf{M}_i$ on a Hilbert space $\mathfrak{H}_i$ (or, sometimes $\mathfrak{H}$) be given. By $\mathbf{B}(\mathfrak{H})$ we denote the full operator algebra on a Hilbert space $\mathfrak{H}$. We owe other notations and terminology used here to [5] and [7]. J. von Neumann introduced the concept of infinite direct product of full operator algebras, in which the infinite direct product of underlying Hilbert spaces has important meaning. For convenience, we shall sketch the outlines of general theory in [7]. For any pair of C-sequences

$$X = (x_i : i \in \mathbf{I}),\ \ Y = (y_i : i \in \mathbf{I})\ (x_i,\ y_i \in \mathfrak{H}_i \text{ for each } i \in \mathbf{I})$$

we associate

$$(X, Y) = \prod_{i \in \mathbf{I}} (x_i,\ y_i).$$

Then, for linear combinations $\sum_{j=1}^{m} \lambda_j X_j,\ \sum_{k=1}^{n} \mu_k Y_k$ we define the inner product

$$\left(\sum_{j=1}^{m} \lambda_j X_j,\ \sum_{k=1}^{n} \mu_k Y_k\right) = \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_j \bar{\mu}_k (X_j,\ Y_k),$$

and get a prehilbert space. Its completion is called the complete infinite direct product of $\mathfrak{H}_i$ and denoted by $\prod \bigotimes_{i \in \mathbf{I}} \mathfrak{H}_i$. An element in $\prod \bigotimes_{i \in \mathbf{I}} \mathfrak{H}_i$ determined by $(x_i : i \in \mathbf{I})$ is denoted by $\Pi \otimes_{i \in \mathbf{I}} x_i$. Next, the family of all C_0-sequences is divided into equivalence-classes by the relation "$\approx$" ([7: Chap. 3]). We denote equivalence-class of a given C_0-sequence $(x_i : i \in \mathbf{I})$ by $\mathfrak{C}(x_i : i \in \mathbf{I})$ (or, simply by $\mathfrak{C}$ if there is no confusion). Then $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathfrak{H}_i$ means the closed linear

1) In preparing this paper, the author has received valuable advice and suggestions from Dr. Z. Takeda and Dr. Y. Misonou which he gratefully acknowledges.

set determined by all $\Pi \underset{i\in\mathbf{I}}{\otimes} x_i$, where $(x_i : i \in \mathbf{I}) \in \mathfrak{C}$ and is called the $\mathfrak{C}$-adic incomplete infinite direct product. Then the following results are obtained.

THEOREM A ([7: LEMMA 4.1.4]). *Let $(x_i^0 : i \in \mathbf{I})$ be a C_0-sequence with $\|x_i^0\| = 1$ for all $i \in \mathbf{I}$ and d_i be the dimension of $\mathfrak{H}_i$ for each $i \in \mathbf{I}$. Choosing a complete orthonormal set $\{x_{i,j} : j \in \mathbf{J}_i\}$ of $\mathfrak{H}_i$ such as $0 \in \mathbf{J}_i$ and $x_{i,0} = x_i^0$, where the cardinal of the set of indices $\mathbf{J}_i$ is equal to d_i for each $i \in \mathbf{I}$. If we construct $\Pi \underset{i\in\mathbf{I}}{\otimes} x_{i,j}$ where $j (\in \mathbf{J}_i) = 0$ except for a finite number of i's, then the totality of such $\Pi \underset{i\in\mathbf{I}}{\otimes} x_{i,j}$ is a complete orthonormal set in $\prod \underset{i\in\mathbf{I}}{\bigotimes}^{\mathfrak{C}} \mathfrak{H}_i$ for $\mathfrak{C} = \mathfrak{C}(x_0^i : i \in \mathbf{I})$.*

THEOREM B ([7: THEOREM VI]). *Let $\mathbf{J}$ be a set of indices. If $\mathbf{I}$ be divided into mutually disjoint sets $\mathbf{I}_j (j \in \mathbf{J})$, and if $(x_i^0 : i \in \mathbf{I})$ be a C_0-sequence, we can form the equivalence classes $\mathfrak{C} = \mathfrak{C}(x_i^0 : i \in \mathbf{I})$ $\left(in \prod \underset{i\in\mathbf{I}}{\bigotimes} \mathfrak{H}_i\right)$, $\mathfrak{C}_j = \mathfrak{C}(x_i : i \in \mathbf{I}_j)$ (in $\prod \underset{i\in\mathbf{I}_j}{\bigotimes} \mathfrak{H}_i$) and $\mathfrak{C}_0 = \mathfrak{C}\ (\Pi \underset{i\in\mathbf{I}_j}{\otimes} x_i^0 : j \in \mathbf{J})$ $\left(in \prod \underset{j\in J}{\bigotimes} \left(\prod \underset{i\in\mathbf{I}_j}{\bigotimes} \mathfrak{H}_i\right)\right)$. Then the classes $\mathfrak{C}_j$, $\mathfrak{C}_0$ depend on $\mathfrak{C}$ only and there exists a unique isomorphism of $\prod \underset{i\in\mathbf{I}}{\bigotimes}^{\mathfrak{C}} \mathfrak{H}_i$ and $\prod \underset{j\in J}{\bigotimes}^{\mathfrak{C}_0} \left(\prod \underset{i\in\mathbf{I}_j}{\bigotimes}^{\mathfrak{C}_j} \mathfrak{H}_i\right)$ such that $\Pi \underset{i\in\mathbf{I}}{\otimes} x_i$ corresponds to $\Pi \underset{j\in J}{\otimes} (\Pi \underset{i\in\mathbf{I}_j}{\otimes} x_i)$ for all C_0-sequences $(x_i : i \in \mathbf{I}) \in \mathfrak{C}$.*

2. Infinite direct product of W^*-algebras. First we define the infinite direct product of W^*-algebras $\mathbf{M}_i$ $(i \in \mathbf{I})$ as J. von Neumann has done for full operator algebras. For any fixed $i_0 \in \mathbf{I}$ every $A_{i_0} \in \mathbf{M}_{i_0}$ is extended to an operator $\bar{A}_{i_0}$ on $\prod \underset{i\in\mathbf{I}}{\bigotimes} \mathfrak{H}_i$ such that for each $\Pi \underset{i\in\mathbf{I}}{\otimes} x_i \in \prod \underset{i\in\mathbf{I}}{\bigotimes} \mathfrak{H}_i$,

$$\bar{A}_{i_0} (\Pi \underset{i\in\mathbf{I}}{\otimes} x_i) = A_{i_0} x_{i_0} \otimes (\Pi \underset{i\in\mathbf{I}\ i\neq i_0}{\otimes} x_i).$$

The set of all extended operators $\bar{A}_{i_0}$ of elements of $\mathbf{M}_{i_0}$ is denoted by $\bar{\mathbf{M}}_{i_0}$. Then we have the following

DEFINITION 1. Let $\mathbf{I}$, $\mathbf{M}_i$ and $\mathfrak{H}_i$ $(i \in \mathbf{I})$ be as above. Then the W^*-algebra on $\prod \underset{i\in\mathbf{I}}{\bigotimes} \mathfrak{H}_i$ generated by all $\bar{A}_i \in \bar{\mathbf{M}}_i$ with $i \in \mathbf{I}$ is called *the complete infinite direct product of* $\mathbf{M}_i$ *on* $\mathfrak{H}_i$ and denoted by $\prod \underset{i\in\mathbf{I}}{\bigotimes} \mathbf{M}_i$.

Since every $\bar{A}_{i_0}$ $(i_0 \in \mathbf{I})$ commutes with the projection on $\prod \underset{i\in\mathbf{I}}{\bigotimes}^{\mathfrak{C}} \mathfrak{H}_i$, $\prod \underset{i\in\mathbf{I}}{\bigotimes}^{\mathfrak{C}} \mathfrak{H}_i$ is stable under $\prod \underset{i\in\mathbf{I}}{\bigotimes} \mathbf{M}_i$; so any operator $\mathbf{A} \in \prod \underset{i\in\mathbf{I}}{\bigotimes} \mathbf{M}_i$ may be considered as an operator in each $\prod \underset{i\in\mathbf{I}}{\bigotimes}^{\mathfrak{C}} \mathfrak{H}_i$ for each equivalence-class $\mathfrak{C}$ (cf. [7]). Then we have:

DEFINITION 2. For any equivalence-class $\mathfrak{C}$ the W^*-algebra $\prod\bigotimes_{i\in\mathbf{I}} \mathbf{M}_i$ restricted in $\prod\bigotimes_{i\in\mathbf{I}}^{\mathfrak{C}} \mathfrak{H}_i$ is denoted by $\prod\bigotimes_{i\in\mathbf{I}}^{\mathfrak{C}} \mathbf{M}_i$ and we call it $\mathfrak{C}$-*adic incomplete infinite direct product of* $\mathbf{M}_i$.

Now we have the following

THEOREM 1. *If all* $\mathbf{M}_i$ *are factors, then for each equivalence-class* $\mathfrak{C}$ *the* $\mathfrak{C}$-*adic incomplete infinite direct product* $\prod\bigotimes_{i\in\mathbf{I}}^{\mathfrak{C}} \mathbf{M}_i$ *is a factor.*

To prove this theorem we provide following lemmas.

LEMMA 1. *If* $\mathbf{M}_i$ $(i \in \mathbf{I})$ *are* W^*-*algebras on a Hilbert space* $\mathfrak{H}$ *satisfying the following conditions*:

(i) $\mathbf{M}_i \subset \mathbf{M}_j'$ *for* $i \neq j,\ i,\ j \in \mathbf{I}$;

(ii) $\mathbf{R}(\mathbf{M}_i : i \in \mathbf{I}) = \mathbf{B}(\mathfrak{H})$.

Then every $\mathbf{M}_i$ *is a factor.*

PROOF. $\mathbf{M}_i \cap \mathbf{M}_i' \subset \Big[\bigcap_{j\in\mathbf{I}\ j\neq i} \mathbf{M}_j'\Big] \cap \mathbf{M}_i' = \bigcap_{j\in\mathbf{I}} \mathbf{M}_j'$

$= \mathbf{R}(\mathbf{M}_j : j \in \mathbf{I})' = \mathbf{B}(\mathfrak{H})' = (\alpha\mathbf{I})$ for each $i \in \mathbf{I}$.

Hence we have $\mathbf{M}_i \cap \mathbf{M}_i' = (\alpha\mathbf{I})$ for each $i \in \mathbf{I}$.

LEMMA 2. *Let* $\mathbf{M}_i$ *be a factor on a Hilbert space* $\mathfrak{H}$ *for each* $i \in \mathbf{I}$ *satisfying the following conditions*: *For each* $i \in \mathbf{I}$ *there exists* $\mathbf{P}_i$, *a* W^*-*algebra on* $\mathfrak{H}$, *such that*

(i) $\mathbf{M}_i \subset \mathbf{P}_i,\ \mathbf{M}_j \subset \mathbf{P}_i'$ *for any* $j \neq i,\ j \in \mathbf{I}$,

(ii) $\mathbf{P}_i$ *and* $\mathbf{P}_i'$ *are both normal, and*

(iii) $\mathbf{R}(\mathbf{P}_i : i \in \mathbf{I}) = \mathbf{B}(\mathfrak{H})$.

Then $\mathbf{R}(\mathbf{M}_i : i \in \mathbf{I})$ *is a factor.*

PROOF. From (i), for each $i, j \in \mathbf{I},\ i \neq j,\ \mathbf{M}_i \subset \mathbf{M}_j'$. We have, by (ii),

$$\mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i' \cap \mathbf{P}_i)' \cap \mathbf{P}_i = \mathbf{M}_i' \cap [(\mathbf{M}_i' \cap \mathbf{P}_i)' \cap \mathbf{P}_i] = \mathbf{M}_i' \cap \mathbf{M}_i = (\alpha\mathbf{I}),$$

for each $i \in \mathbf{I}$. Therefore, considering $\mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i' \cap \mathbf{P}_i) \subset \mathbf{P}_i$, we get

$$\mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i' \cap \mathbf{P}_i) = [\mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i' \cap \mathbf{P}_i)' \cap \mathbf{P}_i]' \cap \mathbf{P}_i = (\alpha\mathbf{I})' \cap \mathbf{P}_i = \mathbf{P}_i,$$

for each $i \in \mathbf{I}$. Hence

$$\begin{aligned} &\mathbf{R}(\mathbf{R}(\mathbf{M}_i : i \in \mathbf{I}),\ \mathbf{R}(\mathbf{M}_i' \cap \mathbf{P}_i : i \in \mathbf{I})) \\ &= \mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i' \cap \mathbf{P}_i : i \in \mathbf{I}) = \mathbf{R}(\mathbf{R}(\mathbf{M}_i,\ \mathbf{M}_i \cap \mathbf{P}_i) : i \in \mathbf{I}) \\ &= \mathbf{R}(\mathbf{P}_i : i \in \mathbf{I}) = \mathbf{B}(\mathfrak{H}). \end{aligned}$$

Now, for any $i \in \mathbf{I}$, $\mathbf{M}_i$ commutes with all $\mathbf{M}_j' \cap \mathbf{P}_j$, $(j \in \mathbf{I})$ by (i). Thus $\mathbf{R}(\mathbf{M}_i : i \in \mathbf{I})$ commutes with $\mathbf{R}(\mathbf{M}_i' \cap \mathbf{P}_i : i \in \mathbf{I})$. Hence, by Lemma 1, $\mathbf{R}(\mathbf{M}_i : i \in \mathbf{I})$ is a factor.

From [7] we quote the following result ([7: Theorems **VIII** and **X**]).

LEMMA 3. *Let* $\mathbf{B}_i$ *be a full operator algebra on a Hilbert space* $\mathfrak{H}_i$ *for each* $i \in \mathbf{I}$. *Then* $\bar{\mathbf{B}}_i$ *is isomorphic to* $\mathbf{B}_i$ *for each* $i \in \mathbf{I}$, *and moreover, each* $\mathfrak{C}$*-adic incomplete infinite direct product* $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathbf{B}_i$ *coincides with* $\mathbf{B}(\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathfrak{H}_i)$.

PROOF OF THEOREM 1. Let each $\mathbf{B}_i$ be the full operator algebra on each $\mathfrak{H}_i$. Put $\bar{\mathbf{B}}_i = \mathbf{P}_i$. By Lemma 3 and [5: Lemma 11.2.2] all $\mathbf{P}_i, \mathbf{P}_i'$ are normal. It is clear that if $i, j \in \mathbf{I}$, $i \neq j$, $\bar{\mathbf{M}}_i \subset \mathbf{P}_i$ and $\bar{\mathbf{M}}_j \subset \mathbf{P}_i'$. Thus, by Lemmas 2 and 3, we see that $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathbf{M}_i$ is a factor for each $\mathfrak{C}$.

Now, let $\mathbf{M}_i$ be a W^*-algebra on a Hilbert space $\mathfrak{H}_i$ with normal state σ_i for each $i \in \mathbf{I}$. Z. Takeda [10] defined the restricted infinite direct product of $\mathbf{M}_i$ with σ_i denoted by $\bigotimes_{i \in I} (\mathbf{M}_i, \sigma_i)$ as follows: We construct a positive functional σ_0 on the algebraical direct product of $\mathbf{M}_i$ such that

$$\sigma_G(\dots \otimes I_{i_p} \otimes A_{iq} \otimes \dots \otimes A_{i_r} \otimes I_{i_s} \otimes \dots) = \sigma_{iq}(A_{iq}) \dots \sigma_{i_r}(A_{i_r}).$$

The weak closure of the representation of the algebraical direct product of $\mathbf{M}_i$ by σ_0 is called the restricted infinite direct product of $\mathbf{M}_i$ with σ_i. Then the next theorem states the relation between the restricted infinite direct product and the incomplete infinite direct product of W^*-algebras.

THEOREM 2. *Let* $\mathbf{M}_i$ *be a* W^**-algebra on a Hilbert space* $\mathfrak{H}_i$ *having a generating and separating vector* x_i *with unity norm for each* $i \in \mathbf{I}$. *Define a normal state* σ_i *by* $\sigma_i(A_i) = (A_i x_i, x_i)$ *for every* $A_i \in \mathbf{M}_i$ *for each* $i \in \mathbf{I}$. *Then* $\bigotimes_{i \in \mathbf{I}} (\mathbf{M}_i, \sigma_i)$ *and* $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathbf{M}_i$ *are spatially isomorphic, where* $\mathfrak{C} = \mathfrak{C}(x_i : i \in \mathbf{I})$.

PROOF. If we denote the representative space of $\bigotimes_{i \in \mathbf{I}} (\mathbf{M}_i, \sigma_i)$ by $\mathfrak{H}$, the image of the algebraical direct product $\mathbf{M}_0$ of $\mathbf{M}_i (i \in \mathbf{I})$ is dense in $\mathfrak{H}$. If we denote the natural mapping from $\bigotimes_{i \in \mathbf{I}} (\mathbf{M}_i, \sigma_i)$ into $\mathfrak{H}$ by V, we have

$$\|V(\dots \otimes I_{i_p} \otimes A_{iq} \otimes \dots \otimes A_{i_r} \otimes I_{i_s} \otimes \dots)\| \\ = \|A_{iq} x_{iq}\| \dots \|A_{i_r} x_{i_r}\|$$

for any $(\dots \otimes I_{i_p} \otimes A_{iq} \otimes \dots \otimes A_{i_r} \otimes I_{i_s} \otimes \dots) \in \mathbf{M}_0$. Now, consider the mapping W from $V(\mathbf{M}_0)$ into $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathfrak{H}_i$ determined by

$$W\{V(\dots \otimes I_{i_p} \otimes A_{iq} \otimes \dots \otimes A_{i_r} \otimes I_{i_s} \otimes \dots) \\ = (\dots x_{i_p} \otimes A_{iq} x_{iq} \otimes \dots \otimes A_{i_r} x_{i_r} \otimes x_{i_s} \otimes \dots)$$

Then we see that this mapping is linear and isometric, for we have

$$\|(\dots \otimes x_{ip} \otimes A_{iq} x_{iq} \otimes \dots \otimes A_{i_r} x_{i_r} \otimes x_{i_s} \otimes \dots)\| \\ = \|A_{iq} x_{iq}\| \dots \|A_{i_r} x_{i_r}\|.$$

As each x_i is a generating vector for $\mathbf{M}_i$, we know that W-image of $V(\mathbf{M}_0)$ is dense in $\prod \bigotimes_{i \in \mathbf{I}}^{\mathfrak{C}} \mathfrak{H}_i$ by [7: Lemma 4.1.2.]. Hence the mapping W can be extended to a linear isometric mapping U from $\mathfrak{H}$ onto $\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathfrak{H}_i$.

For any $A = (\ldots\otimes I_{i_p}\otimes A_{i_q}\otimes \ldots \otimes A_{i_r}\otimes I_{i_s}\ldots) \in \mathbf{M}_0$ and $y = \Pi\bigotimes_{i\in I} y_i \in \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathfrak{H}_i$, we have

$$(UAU^{-1})y = Ay.$$

Therefore it suffices to show that if $A_\lambda \in \mathbf{M}_0$ is a directed set which converges to $A \in \bigotimes_{i\in I}(\mathbf{M}_i, \sigma_i)$ in the weak topology, $UA_\lambda U^{-1}$ converges to UAU^{-1} in the weak topology. For any $\varepsilon > 0$ and $f\in \mathfrak{H}$ there exists a λ_0 such that

$$|(Af, f) - (A_\lambda f, f)| < \varepsilon \text{ for } \lambda \geqq \lambda_0.$$

On the other hand, for any $x\in \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathfrak{H}_i$ there exists an $f\in \mathfrak{H}$ such that $x = Uf$. Thus we have

$$|(UAU^{-1}x, x) - (UA_\lambda U^{-1}x, x)| = |(AU^{-1}x, U^{-1}x) - (A_\lambda U^{-1}x, U^{-1}x)| < \varepsilon$$

for $\lambda \geqq \lambda_0(x, \varepsilon)$. This shows that $UA_\lambda U^{-1}$ converges to UAU^{-1} in the weak topology.

3. The type of the incomplete infinite direct product. The type of the incomplete infinite direct product of W^*-algebras of the same type varies according as the choice of equivalence-class ([7 : Part **IV**, Chap. 7]). The most favourable result on this type problem is to make the equivalence-class correspond to the type of the incomplete infinite direct product, and *vice versa*, but we have only a partial results. Namely, we have merely the following two results.

THEOREM 3. *If $\mathbf{M}_i$ be a factor of type I on a Hilbert space $\mathfrak{H}_i$ for each $i \in \mathbf{I}$, there exists an equivalence-class $\mathfrak{C}$ of C_0-sequences such that $\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i$ is a factor of type I.*

THEOREM 4. *If $\mathbf{M}_i$ be a factor which is not of type I on a Hilbert space $\mathfrak{H}_i$ for each $i \in \mathbf{I}$, there exists an equivalence-class $\mathfrak{C}$ of C_0-sequences such that $\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i$ is a factor of type III.*

To prove these theorems we need the following lemma.

LEMMA 4. *Let $\mathbf{M}_{1i}$, $\mathbf{M}_{2i}$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_{1i}$, $\mathfrak{H}_{2i}$, respectively. Consider three equivalence-classes of C_0-sequences $\mathfrak{C}_1 = \mathfrak{C}(x_{1i} : i \in \mathbf{I})$, $\mathfrak{C}_2 = \mathfrak{C}(x_{2i} : i \in \mathbf{I})$ and $\mathfrak{C}_0 = \mathfrak{C}(x_{1i} \otimes x_{2i} : i \in \mathbf{I})$ where $x_{1i} \in \mathfrak{H}_{1i}$, $x_{2i} \in \mathfrak{H}_{2i}$, then $\prod \bigotimes_{i\in I}^{\mathfrak{C}_0} (\mathbf{M}_{1i} \otimes \mathbf{M}_{2i})$ is spatially isomorphic to $\left(\prod \bigotimes_{i\in I}^{\mathfrak{C}_1} \mathbf{M}_{1i}\right)\otimes\left(\prod \bigotimes_{i\in I}^{\mathfrak{C}_2} \mathbf{M}_{2i}\right)$.*

PROOF. Considering $\prod \bigotimes_{i\in I, j=1,2}^{\mathfrak{C}} \mathfrak{H}_{ji}$ for $\mathfrak{C} = \mathfrak{C}(x_{ji} : i \in \mathbf{I},\ j = 1,2)$, we can find a linear isometric mapping U of $\prod\bigotimes_{i\in I}^{\mathfrak{C}_0}(\mathfrak{H}_{1i}\otimes\mathfrak{H}_{2i})$ onto $\left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_1}\mathfrak{H}_{1i}\right)\otimes\left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_2}\mathfrak{H}_{2i}\right)$ such that

$$U\Pi\otimes(y_{1i}\otimes_{i\in I} y_{2i})=(\Pi\otimes_{i\in I} y_{1i})\otimes(\Pi\otimes_{i\in I} y_{2i})$$

for all C_0-sequences $(y_{1i}\otimes y_{2i}: i\in I)\in\mathfrak{C}_0$ (cf. Theorem A).

For any $A=\Pi\otimes_{i\in I}(A_{1i}\otimes A_{2i})\in\prod\bigotimes_{i\in I}^{\mathfrak{C}_0}(\mathbf{M}_{1i}\otimes\mathbf{M}_{2i})$ and $z=(\Pi\otimes_{i\in I} z_{1i})\otimes(\Pi\otimes_{i\in I} z_{2i})\in\left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_1}\mathfrak{H}_{1i}\right)\otimes\left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_2}\mathfrak{H}_{2i}\right)$ we have $UAU^{-1}z=U(\Pi\otimes_{i\in I}(A_{1i}\otimes A_{2i}))U^{-1}z=[\Pi\otimes_{i\in I} A_{1i})\otimes(\Pi\otimes_{i\in I} A_{2i})]z$.

Hence it is easily seen that U provides an isomorphism between

$$\prod\bigotimes_{i\in I}^{\mathfrak{C}_0}(\mathbf{M}_{1i}\otimes\mathbf{M}_{2i}) \text{ and } \left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_1}\mathbf{M}_{1i}\right)\otimes\left(\prod\bigotimes_{i\in I}^{\mathfrak{C}_2}\mathbf{M}_{2i}\right).$$

PROOF OF THEOREM 3. By the assumption, there exist Hilbert spaces $\mathfrak{H}_{1i}$, $\mathfrak{H}_{2i}$ such that $\mathbf{M}_i$ is spatially isomorphic to $\mathbf{B}(\mathfrak{H}_{1i})\otimes I_{\mathfrak{H}_{2i}}$ for each $i\in I$. So it suffices to consider $\prod\bigotimes_{i\in I}(\mathbf{B}(\mathfrak{H}_{1i})\otimes I_{\mathfrak{H}_{2i}})$ for $\prod\bigotimes_{i\in I}\mathbf{M}_i$. Choosing a C_0-sequence $(x_{ij}: i\in I, j=1,2)$ in $\prod\bigotimes_{i\in I, j=1,2}\mathfrak{H}_{j,i}$, we put $\mathfrak{C}_1=\mathfrak{C}(x_{1i}: i\in I)$, $\mathfrak{C}_2=\mathfrak{C}(x_{2i}: i\in I)$ and $\mathfrak{C}=\mathfrak{C}(x_{1i}\otimes x_{2i}: i\in I)$. By Lemma 4 $\left[\prod\bigotimes_{i\in I}^{\mathfrak{C}_1}\mathbf{B}(\mathfrak{H}_{1i})\right]\otimes I_{\mathfrak{C}_2}$ is spatially isomorphic to $\prod\bigotimes_{i\in I}^{\mathfrak{C}}(\mathbf{B}(\mathfrak{H}_{1i})\otimes I_{\mathfrak{H}_{2i}})$ and the assertion of the theorem is clear by Lemma 3.

PROOF OF THEOREM 4. It is known that a W^*-algebra not of type I is the direct product of two W^*-algebras, one of which is of type I_p for any positive integer p. Accordingly we represent each $\mathbf{M}_i$ by the direct product of factors $\mathbf{M}_{1i}$, $\mathbf{M}_{2i}$ acting on $\mathfrak{H}_{1i}$, $\mathfrak{H}_{2i}$ respectively, the latter being of type I_2. Now there exists an equivalence-class $\mathfrak{C}$ for which $\prod\bigotimes_{i\in I}^{\mathfrak{C}}\mathbf{M}_{2i}$ is a factor of type III [8]. Hence if we choose an equivalence-class $\mathfrak{C}=\mathfrak{C}(x_{1i}\otimes x_{2i}: i\in I)$ (in $\prod\bigotimes_{i\in I}(\mathfrak{H}_{1i}\otimes\mathfrak{H}_{2i})$) such that $\prod\bigotimes_{i\in I}^{\mathfrak{C}_2}\mathbf{M}_{2i}$ is of type III for $\mathfrak{C}_2=\mathfrak{C}(x_{2i}: i\in I)$, we get a factor $\prod\bigotimes_{i\in I}^{\mathfrak{C}}(\mathbf{M}_{1i}\otimes\mathbf{M}_{2i})$ of type III by Lemma 4 and [9].

By Lemma 4 and [6: Lemma 5.2.1] we can easily show the following:

COROLLARY ([1]. [2]. [3]). *The direct product of two approximately finite factors is also an approximately finite factor.*

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2) $I_{\mathfrak{C}_2}$ means the identity operator on $\Pi\otimes_{i\in I}^{\mathfrak{C}_2}\mathfrak{H}_{2i}$

4. Normalcy of the infinite direct product. Concerning normalcy of W^*-algebra we have the following

THEOREM 5 (*cf.* [4]). *Let $\mathbf{M}_i$ be factors on Hilbert spaces $\mathfrak{H}_i$ for all $i \in \mathbf{I}$. If a certain factor $\mathbf{M}_{i_0}$ is not normal, then $\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i$ is not normal for any equivalence-class $\mathfrak{C}$.*

PROOF. As $\mathbf{M}_{i_0}$ is not normal, then there exists a W^*-subalgebra $\mathbf{N}_{i_0}$ of $\mathbf{M}_{i_0}$ such that

$$(\mathbf{N}_{i_0}' \cap \mathbf{M}_{i_0}) \cap \mathbf{M}_{i_0} \supsetneqq \mathbf{N}_{i_0}.$$

Now $\overline{\mathbf{N}}_{i_0} \subset \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i$ on $\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathfrak{H}_i$ for any $\mathfrak{C}$ and

$$[\overline{\mathbf{N}}_{i_0}' \cap \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i]' \cap \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i = \mathbf{R}(\overline{\mathbf{N}}_{i_0}, (\prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i)') \cap \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathbf{M}_i$$

$$\supset \mathbf{R}(\overline{\mathbf{N}}_{i_0}, \overline{\mathbf{M}}_{i_0}') \cap \overline{\mathbf{M}}_{i_0} = \overline{\mathbf{R}(\mathbf{N}_{i_0}, \mathbf{M}_{i_0}')} \cap \overline{\mathbf{M}}_{i_0}$$

$$\supset \overline{\mathbf{R}(\mathbf{N}_{i_0}, \mathbf{M}_{i_0}') \cap \mathbf{M}_{i_0}} = \overline{(\mathbf{N}_{i_0}' \cap \mathbf{M}_{i_0})' \cap \mathbf{M}_{i_0}} \supsetneqq \overline{\mathbf{N}}_{i_0} \text{ in } \prod \bigotimes_{i\in I}^{\mathfrak{C}} \mathfrak{H}_i$$

for any $\mathfrak{C}$, which proves the theorem.

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