ON THE ERGODIC THEOREMS CONCERNING MARKOV PROCESSES

SHIGERU TSURUMI

(Received February 24, 1958)

1. Introduction. Concerning a Markov process with an invariant measure the mean ergodic theorem for all the integrable functions was proved by K. Yosida [11]. On the other hand, the individual ergodic theorem for the p-th integrable functions ($p > 1$) was proved by S. Kakutani [5] and J. L. Doob [1]. At length in 1954 E. Hopf [4] has established the individual ergodic theorem for all the integrable functions by means of remarkably powerful arguments. In the present paper we shall concern with general Markov processes and study the necessary and sufficient conditions for the validities of the individual and the mean ergodic theorems. Concerning the measurable point transformations such studies were already taken by N. Dunford-D. S. Miller [2], F. Riesz [7], C. Ryll-Nardzewski [8] and the author [9]. The corresponding results will be obtained in this paper.

2. Notations and preliminaries. Let X be a fixed abstract space, $\mathfrak{F}$ a fixed σ-field of subsets of X and μ a fixed finite measure defined on $\mathfrak{F}$. It is supposed that $X \in \mathfrak{F}$.

A function $P(x, A)$ of two variables $x \in X$, $A \in \mathfrak{F}$ is called a *transition probability of Markov process* if it satisfies

(i) for each fixed $x \in X$, $P(x, A)$ is a probability measure on $\mathfrak{F}$ as a set function of a variable $A \in \mathfrak{F}$;

(ii) for each fixed $A \in \mathfrak{F}$, $P(x, A)$ is a $\mathfrak{F}$-measurable function of a variable $x \in X$.

We consider in the sequel a fixed transition probability $P(x, A)$. If we define

$$P^{(1)}(x, A) = P(x, A),$$

$$P^{(n)}(x, A) = \int P^{(n-1)}(y, A) P^{(1)}(x, dy)^{1)} \qquad (n = 2, 3, \ldots\ldots\ldots),$$

then $P^{(n)}(x, A)$'s satisfy

$$P^{(m+n)}(x, A) = \int P^{(m)}(y, A) P^{(n)}(x, dy) \qquad (m, n = 1, 2, \ldots\ldots\ldots).$$

In the sequel every function (every set, every measure) under consideration will be a real-valued $\mathfrak{F}$-measurable function (a set in $\mathfrak{F}$, a measure defined

1) $\int$ means the integral over the whole space X, $\int_X$.

on $\mathfrak{F}$).

We consider an operator S which is defined over the finite measures and maps to the same sort as follows:

$$\varphi \rightarrow S\varphi = \psi: \quad \psi(A) = \int P(x, A)\varphi(dx).$$

Then $S\varphi(X) = \varphi(X)$ and

$$S^j\varphi(A) = \int P^{(j)}(x, A)\varphi(dx) \qquad (j = 1, 2, \ldots\ldots\ldots).$$

A finite measure φ is called an *invariant measure* if $S\varphi(A) = \varphi(A)$ for every A. For a measure φ, the transition probability $P(x, A)$ is called *φ-measurable* if $\varphi(A) = 0$ implies $S\varphi(A) = 0$ or, equivalently, if $\varphi(A) = 0$ implies that $P(x, A) = 0$ for φ-a.a. x.[2] If $P(x, A)$ is φ-measurable and $\varphi(A) = 0$, then $P^{(j)}(x, A) = 0$ for φ-a.a. x $(j = 1, 2, \ldots\ldots\ldots)$. For instance, we shall prove the case of $j = 2$. Let $B = \{y; P(y, A) > 0\}$, then $\varphi(B) = 0$. Further, let $C = \{x; P(x, B) = 0\}$, then $\varphi(C) = 1$. Hence we have

$$P^{(2)}(x, A) = \int_B P(y, A)P(x, dy) = 0 \qquad \text{for all } x \in C,$$

as was to be shown. If φ is an invariant measure, $P(x, A)$ is clearly φ-measurable.

Now let φ be a finite invariant measure. Then for any $f \in L_1(\varphi)$ the integral $\int f(y)P(x, dy)$ exists for φ-a.a. x and

$$\iint f(y)P(x, dy)\varphi(dx) = \int f(x)\varphi(dx).$$

Hence we can define an operator T of $L_p(\varphi)$ $(p \geqq 1)$ into itself as follows:

$$f \rightarrow Tf = g: \quad g(x) = \int f(y)P(x, dy).$$

The operator T is a linear bounded operator of $L_p(\varphi)$ into itself and $\|T\|_p = 1$. Further T is a positive operator, that is, if $f(x) \geqq 0$ for φ-a.a. x, so is $Tf(x)$.

Now we recall the ergodic theorems concerning a Markov process $P(x, A)$ with a finite invariant measure φ.

Individual Ergodic Theorem. *Let φ be a finite invariant measure. Then for every $f \in L_p(\varphi)$ $(p \geqq 1)$ there exists a function $\tilde{f} \in L_p(\varphi)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \tilde{f}(x) \qquad \varphi\text{-a.e.,} \tag{2.1}$$

2) "for φ-a.a. x" and "φ-a.e." mean "for almost all x in X with respect to φ" and "almost everywhere in X with respect to φ", respectively.

$$\int \tilde{f}(x)\varphi(dx) = \int f(x)\varphi(dx), \tag{2.2}$$

$$T\tilde{f}(x) = \tilde{f}(x) \qquad \varphi\text{-a. e.}. \tag{2.3}$$

MEAN ERGODIC THEOREM. *Let φ be a finite invariant measure. Then for every $f \in L_p(\varphi)$ $(p \geqq 1)$ there exists a function $\tilde{f} \in L_p(\varphi)$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j f - \tilde{f} \right\|_p = 0, \tag{2.4}$$

$$T\tilde{f}(x) = \tilde{f}(x) \qquad \varphi\text{-a. e.}. \tag{2.5}$$

Let $P(x, A)$ be φ-measurable. Then, note that for $f \in L_p(\varphi)$, $Tf(x)$ may not be defined. However for every $f \in L_\infty(\varphi)$, $Tf(x)$ is defined for φ-a. a. x, and $T1 = 1$, $\|Tf\|_\infty \leqq \|f\|_\infty$. Further we note that, for every $f \in L_\infty(\varphi)$

$$T^j f(x) = \int f(y) P^{(j)}(x, dy) \qquad \varphi\text{-a. e.},$$

$$\int T^j f(x)\varphi(dx) = \int f(x) S^j \varphi(dx).$$

A set A is called a *compressible* (φ) *set* if $P(x, A) \leqq e_A(x)$[3] for φ-a. a. x and $P(x, A) < e_A(x)$ for x in a set of positive φ-measure. A set A is called an *incompressible* (φ) *set* if $P(x, A) \leqq e_A(x)$ for φ-a. a. x and if A includes no compressible (φ) set. If A is an incompressible (φ) set, then from the definition it follows necessarily that $P(x, A) = e_A(x)$ for φ-a. a. x. Note that if φ is an invariant measure there exists no compressible (φ) set and so X itself is an incompressible (φ) set.

In the sequel it is not supposed that μ is an invariant measure, but it is supposed that $P(x, A)$ is μ-measurable.

LEMMA 1. *The union of two incompressible (μ) sets is an incompressible (μ) set.*

PROOF. We note first that, for two sets E and F, if $P(x, E) \leqq e_E(x)$, $P(x, F) \leqq e_F(x)$ for μ-a. a. x then $P(x, E \cup F) \leqq e_{E\cup F}(x)$, $P(x, E \cap F) \leqq e_{E\cap F}(x)$ for μ-a. a. x.

Suppose that both A and B are the incompressible (μ) sets. Then by the remark we have

$$P(x, A \cup B) \leqq e_{A\cup B}(x) \qquad \mu\text{-a. e.}$$

For any set $C \subset A \cup B$ such that $P(x, C) \leqq e_C(x)$ μ-a. e., it is sufficient to show that $P(x, C) = e_C(x)$ μ-a. e.. By the remark,

$$P(x, C \cap A) \leqq e_{C\cap A}(x), \quad P(x, C \cap B) \leqq e_{C\cap B}(x),$$
$$P(x, C \cap A \cap B) \leqq e_{C\cap A\cap B}(x) \qquad \mu\text{-a. e.}$$

3) e_A denotes the indicator of a set A.

Since A and B include no compressible (μ) set,

$$P(x, C \cap A) = e_{C\cap A}(x), \qquad P(x, C \cap B) = e_{C\cap B}(x),$$
$$P(x, C \cap A \cap B) = e_{C\cap A\cap B}(x) \qquad \mu\text{-a. e.},$$

so that

$$P(x, C) = P(x, C \cap A) + P(x, C \cap B) - P(x, C \cap A \cap B)$$
$$= e_{C\cap A}(x) + e_{C\cap B}(x) - e_{C\cap A\cap B}(x) = e_C(x) \qquad \mu\text{-a. e.},$$

as was to be shown.

THEOREM 1.[4] *The space X splits into two disjoint sets Y and Z such that*

(2.6) *Y is an incompressible (μ) set;*

(2.7) *Z includes no incompressible (μ) set of positive μ-measure.*

The sets Y and Z are called the *incompressible* (μ) and the *dissipative* (μ) *part* of X.

PROOF. Let a denote the supremum of μ-measures of the incompressible (μ) sets in X. Then, by virtue of Lemma 1, there exists a sequence of the incompressible (μ) sets Y_n $(n = 1, 2, \ldots)$ such that

$$Y_1 \subset Y_2 \subset \ldots, \qquad \lim_{n\to\infty} \mu(Y_n) = a.$$

We set

$$Y = \bigcup_{n=1}^{\infty} Y_n, \qquad Z = X - Y,$$

then Y and Z are mutually disjoint and satisfy (2.6) and (2.7) In fact, suppose that $A \subset Y$ and $P(x, A) \leqq e_A(x)$, μ-a. e. Since each Y_n is an incompressible (μ) set, $P(x, A \cap Y_n) \leqq e_{A\cap Y_n}(x)$ μ-a. e. and hence $P(x, A \cap Y_n) = e_{A\cap Y_n}(x)$ μ-a. e., so that $P(x, A) = e_A(x)$ μ-a. e. This concludes (2.6). Next, suppose that $A \subset Z$ and A is an incompressible (μ) set of positive μ-measure. Then, by Lemma 1, $Y \cup A$ is an incompressible (μ) set and the μ-measure of $Y \cup A$ is greater than a. This contradicts the definition of a, so that (2.7) must be true.

3. Individual ergodic theorem. We shall first state several propositions.

(I. 1) *For every $f \in L_p(\mu)$ $(p \geqq 1)$, $T^j f(x)$ $(j = 1, 2, \ldots\ldots)$ are defined for μ-a. a. x in the incompressible (μ) part Y and there exists a function $\widetilde{f} \in L_p(Y, \mu)$[5] such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \widetilde{f}(x) \qquad \mu\text{-a. e. in } Y.$$

4) Either Y or Z may be of μ-measure zero. The present decomposition theorem is different from the decomposition theorem [4] with respect to the ajoint operator T^* of T, and corresponds to the decomposition theorem [3] with respect to a measurable point transformation.

5) $L_p(Y, \mu)$ denotes the family of all the functions which are p-th integrable (with respect to μ) over Y.

Moreover, for every $f \in L_\infty(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \tilde{f}(x) \qquad \mu\text{-}a.e.$$

(I. 2) *For every* $f \in L_p(\mu)$ $(p \geqq 1)$, $T^j f(x)$ $(j = 1, 2, \ldots)$ *are defined for* μ-*a. a.* x *and there exists a function* $\tilde{f} \in L_p(\mu)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \tilde{f}(x) \qquad \mu\text{-}a.e.$$

(I. 3) *For every* $f \in L_p(\mu)$ $(p \geqq 1)$, $Tf(x)$ $(j = 1, 2, \ldots\ldots)$ *are defined for* μ-*a. a.* x *and there exists a function* $\tilde{f} \in L_p(\mu)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} Tf(x) = \tilde{f}(x) \qquad \mu\text{-}a.e.$$

Moreover, if f *is a non-negative function and positive in a set of positive* μ-*measure, so is* $\tilde{f}$.

(C. 1) *There exists a positive constant* K *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \leqq K\cdot\mu(A)$$

for all A.

(C. 2)[6] *There exists a positive constant* K *such that*

$$0 < \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \leqq K\cdot\mu(A)$$

for any set A *of positive* μ-*measure.*

In this section we shall prove the following

THEOREM 2. (C. 1) $\rightarrow$ (I. 1)[7], (I. 2) $\rightarrow$ (C. 1).

THEOREM 3. (C. 2) $\leftrightarrows$ (I. 3). *If they hold, then* X *is an incompressible* (μ) *set.*

THEOREM 4. *If* X *is an incompressible* (μ) *set, then* (C. 1) $\leftrightarrows$ (C. 2) $\leftrightarrows$ (I. 1) $\rightleftarrows$ (I. 2) $\leftrightarrows$ (I. 3).

Before the proofs we shall prepare several lemmas.

For a set A, let $(A \mid A_k) = (A \mid A_k,\ k = 1, 2, \ldots\ldots\ldots m)$ denote the partition of A such that

$$A = \bigcup_{k=1}^{m} A_k, \qquad A_i \cap A_j = 0 \ (i \neq j).$$

6) Since $P(x, A)$ is μ-measurable, $\mu(A) = 0$ implies $S^j\mu(A) = 0$ for $j = 1, 2, \ldots\ldots$, so that the proposition (C. 2) implies the proposition (C. 1).

7) With respect to the propositions P and Q "$P \rightarrow Q$" means that P implies Q.

A set function α defined on $\mathfrak{F}$ is called *subadditive* if $\alpha(A \cup B) \leqq \alpha(A) + \alpha(B)$ for any sets A and B.

LEMMA 2. *Let α be a non-negative subadditive set function on $\mathfrak{F}$ and have a dominant finite measure φ, that is, $\alpha(A) \leqq \varphi(A)$ for all A. For every set A, define*

$$\beta(A) = \sup_{(A|A_k)} \sum_{(A|A_k)} \alpha(A_k),$$

where $\sum_{(A|A_k)}$ means a summation with respect to the sets A_k's of a partition $(A|A_k)$ and $\sup_{(A|A_k)}$ denotes the supremum for all the partitions $(A|A_k)$'s of A. Then β is a finite measure with the dominant φ.

PROOF. It is clear that $0 \leqq \beta(A) \leqq \varphi(A)$ for all A. In order to prove that β is a finite measure, it is sufficient to show that β is finitely additive, since β has a dominant finite measure φ.

Take a finite number of disjoint sets $A^1, \ldots\ldots, A^N$ and set $A = \bigcup_{i=1}^{N} A^i$. Let ε be any positive number. Then, for each A^i, there exists a partition of A^i, $(A^i|A^i_k;\ k = 1, \ldots\ldots, m_i)$, such that

$$\beta(A^i) - \frac{\varepsilon}{N} < \sum_{(A^i|A^i_k)} \alpha(A^i_k).$$

Here note that the collection of sets $A^i_k\ (k = 1, \ldots\ldots, m_i;\ i = 1, \ldots\ldots, N)$ gives a partition of A, $(A|A^i_k)$. Hence we obtain

$$\sum_{i=1}^{N} \beta(A^i) - \varepsilon < \sum_{i=1}^{N} \sum_{(A^i|A^i_k)} \alpha(A^i_k) = \sum_{(A|A^i_k)} \alpha(A^i_k) \leqq \beta(A). \tag{3.1}$$

On the other hand, there exists a partition of A, $(A|A_k;\ k = 1, \ldots\ldots, m)$, such that

$$\beta(A) - \varepsilon < \sum_{(A|A_k)} \alpha(A_k).$$

We set $A^i_k = A^i \cap A_k\ (i = 1, \ldots\ldots, N;\ k = 1, \ldots\ldots, m)$. Then, for each i, the collection of sets $A^i_k\ (k = 1, \ldots\ldots, m)$ gives a partition of A^i, $(A^i|A^i_k)$. Since α is subadditive, we obtain

$$\sum_{(A|A_k)} \alpha(A_k) \leqq \sum_{i=1}^{N} \sum_{(A^i|A^i_k)} \alpha(A^i_k) \leqq \sum_{i=1}^{N} \beta(A^i).$$

Hence

$$\beta(A) - \varepsilon < \sum_{i=1}^{N} \beta(A_i). \tag{3.2}$$

Since ε in (3.1) and (3.2) is any positive number,

$$\sum_{i=1}^{N} \beta(A^i) = \beta(A).$$

This shows that β is finitely additive.

LEMMA 3. *For a set A, define*

$$B = A \cup \bigcup_{j=1}^{\infty} \{x;\ P^{(j)}(x, A) > 0\}.$$

Then

$$P(x, B) = Te_B(x) \leqq e_B(x) \qquad \mu\text{-a. e.}$$

PROOF. We note first that, for any $f, g \in L_\infty(\mu)$,

(3.3) $$(Tf)^+(x) \leqq Tf^+(x)^{8)} \qquad \mu\text{-a. e.,}$$

(3.4) if $f(x) \geqq g(x)$ μ-a. e., then

$$nf(x) - [nf-1]^+(x) \geqq ng(x) - [ng-1]^+(x) \qquad \mu\text{-a. e.}$$

Now we set

$$f_m(x) = e_A(x) + \sum_{j=1}^{m} P^{(j)}(x, A),\ \ B_m = \{x;\ f_m(x) > 0\}$$

$$(m = 1, 2, \ldots\ldots).$$

Then

$$Tf_m(x) = f_{m+1}(x) - e_A(x) \leqq f_{m+1}(x) \quad \mu\text{-a. e.} \quad (m = 1, 2, \ldots\ldots).$$

Further, $nf_m(x) - [nf_m - 1]^+(x)$ increases and converges to $e_{B_m}(x)$ as $n \to \infty$, and $e_{B_m}(x)$ increases and converges to $e_B(x)$ as $m \to \infty$. Hence, by the convergence theorem, (3.3) and (3.4), we have

$$\begin{aligned} P(x, B) = Te_B(x) &= T(\lim_{m\to\infty}\lim_{n\to\infty}\{nf_m(x) - [nf_m - 1]^+(x)\}) \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\{Tnf_m(x) - T[nf_m - 1]^+(x)\} \\ &\leqq \lim_{m\to\infty}\lim_{n\to\infty}\{nTf_m(x) - [nTf_m - 1]^+(x)\} \\ &\leqq \lim_{m\to\infty}\lim_{n\to\infty}\{nf_{m+1}(x) - [nf_{m+1} - 1]^+(x)\} = e_B(x) \ \ \mu\text{-a. e.,} \end{aligned}$$

as was to be proved.

LEMMA 4. *If* (C.1) *holds, there exists a finite invariant measure* λ *such that*

(3.5) $$\lambda(A) \leqq K^2 \cdot \mu(A) \qquad \textit{for all } A,$$

(3.6) $$\lambda(A) \geqq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \qquad \textit{for all } A,$$

(3.7) $$\textit{if } \lambda(A) = 0 \textit{ then } \mu(A \cap Y) = 0.$$

8) For a function f, $f^+(x)$ denotes $\max(f(x), 0)$

PROOF. If we set

$$\alpha(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \qquad \text{for all } A,$$

then α is a non-negative subadditive set function on $\mathfrak{F}$ and has a dominant finite measure $K\cdot\mu$ by virtue of (C. 1). Hence, if we define

$$\beta(A) = \sup_{(A|A_k)} \sum_{(A|A_k)} \alpha(A_k) \qquad \text{for all } A,$$

then, by Lemma 2, β is a finite measure such that

(3. 8) $$\beta(A) \leqq K\cdot\mu(A) \qquad \text{for all } A,$$

(3. 9) $$\beta(A) \geqq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \qquad \text{for all } A,$$

(3. 10) $$\beta(A) \leqq S\beta(A) \qquad \text{for all } A.$$

Both (3. 8) and (3. 9) are clear. We shall prove (3. 10). For any fixed set A, set

$$X_{mk} = \left\{x;\ \frac{k-1}{m} < P(x, A) \leqq \frac{k}{m}\right\}$$
$$(k = 1, \ldots\ldots, m;\ m = 1, 2, \ldots\ldots).$$

Then we note that, for any finite measure φ,

$$S\varphi(A) = \inf_{m\geqq 1} \sum_{k=1}^{m} \frac{k}{m}\, \varphi(X_{mk}).$$

Let ε be any positive number. Then there exist a positive integer m such that

$$S\beta(A) + \varepsilon > \sum_{k=1}^{m} \frac{k}{m} \beta(X_{mk}).$$

By (3. 9) there exists a positive integer n_0 such that

$$\beta(X_{mk}) + \frac{\varepsilon}{m} > \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(X_{mk})$$
$$(k = 1, \ldots\ldots\ m;\ n = n_0, n_0 + 1, \ldots\ldots).$$

Hence, for any $n \geqq n_0$, we have

$$S\beta(A) + \varepsilon > \sum_{k=1}^{m} \frac{k}{m} \left(\frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(X_{mk}) - \frac{\varepsilon}{m} \right)$$
$$\geqq \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=1}^{m} \frac{k}{m} S^j\mu(X_{mk}) - \varepsilon \geqq \frac{1}{n} \sum_{j=0}^{n-1} S^{j+1}\mu(A) - \varepsilon,$$

so that

$$S\beta(A) + 2\varepsilon \geqq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^{j+1}\mu(A) = \alpha(A).$$

Since ε is any positive number,

$$S\beta(A) \geqq \alpha(A).$$

Hence, for any set A and for any partition $(A|A_k)$ of A,

$$S\beta(A) = \sum_{(A|A_k)} S\beta(A_k) \geqq \sum_{(A|A_k)} \alpha(A_k),$$

so that by the definition of β we have (3.10).

By (3.10),

(3.11) $$\beta(A) \leqq S\beta(A) \leqq S^2\beta(A) \leqq \cdots\cdots \qquad \text{for all } A.$$

Now we define

$$\lambda(A) = \lim_{n\to\infty} S^n\beta(A) \qquad \text{for every } A.$$

Then we shall prove that λ is the desired finite invariant measure. It is clear that λ is a non-negative, finitely additive set function on $\mathfrak{F}$. By (3.8) and (C.1), it follows that

$$\begin{aligned}\lambda(A) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\beta(A) \\ &\leqq K\cdot\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(A) \leqq K^2\cdot\mu(A) \qquad \text{for all } A,\end{aligned}$$

which is just (3.5). Since λ is a non-negative, finitely additive set function on $\mathfrak{F}$ and has a domiant finite measure $K^2\cdot\mu$ by (3.5), λ is a finite measure. (3.6) follows directly from the definition of λ, (3.11) and (3.9). Since $S^n\beta(A)$ converges to $\lambda(A)$ for every set A as $n\to\infty$ and is dominated by $\lambda(A)$,

$$\begin{aligned}S\lambda(A) &= \int P(x,A)\lambda(dx) = \lim_{n\to\infty}\int P(x,A)S^n\beta(dx) \\ &= \lim_{n\to\infty} S^{n+1}\beta(A) = \lambda(A) \qquad \text{for every } A,\end{aligned}$$

which shows that λ is an invariant measure.

Finally we shall prove (3.7). Suppose $\lambda(A) = 0$. Since λ is an invariant measure, $S^j\lambda(A) = 0$ $(j = 1, 2, \cdots\cdots)$ and so

$$e_A(x) = 0, \quad P^{(j)}(x, A) = 0 \qquad \lambda\text{-a. e.} \quad (j = 1, 2 \cdots\cdots,).$$

Hence, if we set

$$B = A \cup \bigcup_{j=1}^{\infty} \{x;\ P^{(j)}(x,A) > 0\},$$

then $\lambda(B) = 0$.

On the other hand, by virtue of Lemma 3,

$$P(x,B) \leqq e_B(x) \qquad \mu\text{-a. e.}$$

Since $P(x,Y) = e_Y(x)$ μ-a. e.,

$$P(x, B\cap Y) \leqq e_{B\cap Y}(x) \qquad \mu\text{-a. e.}$$

Since Y includes no compressible (μ) set,

$$P(x, B\cap Y) = e_{B\cap Y}(x) \qquad \mu\text{-a. e.}$$

and so

$$P^{(j)}(x, B \cap Y) = e_{B\cap Y}(x) \qquad \mu\text{-a. e.} \qquad (j = 1, 2, \ldots\ldots\ldots).$$

Hence

$$S^j\mu(B \cap Y) = \mu(B \cap Y) \qquad (j = 1, 2, \ldots\ldots\ldots),$$

so that, by (3.6),

$$\lambda(B \cap Y) \geqq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j\mu(B \cap Y) = \mu(B \cap Y).$$

Since $A \subset B$ and $\lambda(B) = 0$,

$$\mu(A \cap Y) = 0,$$

as was to be shown. Thus the proof of Lemma 4 is terminated.

For a measure φ, a function f is called an *invariant* (φ) *function* if $Tf(x) = f(x)$ for φ-a. a. x.

LEMMA 5. *Let λ be the finite invariant measure defined in Lemma 4 and let $f \in L_1(\lambda)$ be an invariant (λ) function. Then, for any real numbers $\alpha, \beta (\alpha < \beta)$,*

$$P(x, A(\alpha, \beta)) = e_{A(\alpha,\beta)}(x) \qquad \lambda\text{-a. e.},$$

where

$$A(\alpha, \beta) = \{x;\ \alpha < f(x) \leqq \beta\}.$$

PROOF. We note first that if $g \in L_1(\lambda)$ is an invariant (λ) function, so is g^+. In fact,

$$g^+(x) = (Tg)^+(x) \leqq Tg^+(x) \qquad \lambda\text{-a. e.}$$

Since λ is an invariant measure,

$$\int Tg^+(x)\lambda(dx) = \int g^+(x)S\lambda(dx) = \int g^+(x)\lambda(dx).$$

Hence

$$Tg^+(x) = g^+(x) \qquad \lambda\text{-a. e.}$$

For any real number α, set

$$A(\alpha) = \{x; f(x) > \alpha\}.$$

Then $[n(f-\alpha)]^+(x) - [n(f-\alpha)-1]^+(x)$ increases and converges to $e_{A(\alpha)}(x)$ as $n \to \infty$. Since f is an invariant (λ) function, $[n(f-\alpha)]^+$ and $[n(f-\alpha)-1]^+$ are the invariant (λ) functions. Hence

$$\begin{aligned} P(x, A(\alpha)) &= Te_{A(\alpha)}(x) \\ &= T(\lim_{n\to\infty}\{[n(f-\alpha)]^+(x) - [n(f-\alpha)-1]^+(x)\}) \\ &= \lim_{n\to\infty}\{T[n(f-\alpha)]^+(x) - T[n(f-\alpha)-1]^+(x)\} \\ &= \lim_{n\to\infty}\{[n(f-\alpha)]^+(x) - [n(f-\alpha)-1]^+(x)\} \\ &= e_{A(\alpha)}(x) \qquad \lambda\text{-a. e.} \end{aligned}$$

Thus, for the set $A(\alpha,\beta)$ in the lemma,

$$P(x, A(\alpha,\beta)) = P(x, A(\alpha)) - P(x, A(\beta))$$
$$= e_{A(\alpha)}(x) - e_{A(\beta)}(x) = e_{A(\alpha,\beta)}(x) \qquad \lambda\text{-a. e.},$$

as was to be proved.

LEMMA 6. *Let* λ *be the finite invariant measure defined in Lemma* 4. *Then, if* $f \in L_p(\lambda)$ $(p \geqq 1)$ *is an invariant* (λ) *function*, $f \in L_p(Y,\mu)$.

PROOF. Suppose that $f \in L_p(\lambda)$ is an invariant (λ) function. Then f^+ and $|f| \left(= \frac{1}{2}(f + f^+)\right)$ are the invariant (λ) functions as was noted in the proof of Lemma 5. Now, set

$$A_{nk} = \left\{ x; \left(\frac{k-1}{n}\right)^{1/p} < |f(x)| \leqq \left(\frac{k}{n}\right)^{1/p} \right\}$$
$$= \left\{ x; \frac{k-1}{n} < |f(x)|^p \leqq \frac{k}{n} \right\} \qquad (n, k = 1, 2, \ldots\ldots\ldots).$$

Then, by Lemma 5, each A_{nk} satisfies

$$P(x, A_{nk}) = e_{A_{nk}}(x) \qquad \lambda\text{-a. e.}$$

Since $P(x, Y) = e_Y(x)$ μ-a. e., it follows by (3.5) that $P(x, Y) = e_Y(x)$ λ-a. e. Hence

$$P(x, A_{nk} \cap Y) \leqq e_{A_{nk} \cap Y}(x) \qquad \lambda\text{-a. e.},$$

so that, by (3.7)

$$P(x, A_{nk} \cap Y) \leqq e_{A_{nk} \cap Y}(x) \qquad \mu\text{-a. e. in } Y.$$

Since $P(x, A_{nk} \cap Y)$ and $e_{A_{nk} \cap Y}(x)$ are majorized by $P(x, Y)$ and $e_Y(x)$, respectively, and $P(x, Y)$ $(= e_Y(x))$ vanishes for μ-a. a. x outside of Y,

$$P(x, A_{nk} \cap Y) \leqq e_{A_{nk} \cap Y}(x) \qquad \mu\text{-a. e.}$$

Since Y includes no compressible (μ) set,

$$P(x, A_{nk} \cap Y) = e_{A_{nk} \cap Y}(x) \qquad \mu\text{-a. e.}$$

and so

$$P^{(j)}(x, A_{nk} \cap Y) = e_{A_{nk} \cap Y}(x) \ \mu\text{-a. e.} \qquad (j = 1, 2, \ldots\ldots\ldots).$$

Hence

$$S^j\mu(A_{nk} \cap Y) = \mu(A_{nk} \cap Y)) \qquad (j = 1, 2, \ldots\ldots\ldots),$$

so that, by (3.6),

$$\lambda(A_{nk} \cap Y) \geqq \mu(A_{nk} \cap Y).$$

Therefore

$$\int_Y |f(x)|^p \mu(dx) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{k}{n} \mu(A_{nk} \cap Y)$$
$$\leqq \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{k}{n} \lambda(A_{nk}) = \int |f(x)|^p \lambda(dx) < \infty,$$

which shows $f \in L_p(Y, \mu)$.

LEMMA 7. *If $\widetilde{T}$ is a linear positive operator of $L_1(\mu)$ into itself, then $\widetilde{T}$ is a bounded operator.*

For the proof, see [8].

PROOF OF THEOREM 2. Proof of "(C. 1) → (I, 1)": Let λ be the finite invariant measure defined in Lemma 4. Take any $f \in L_p(\mu)$ ($p \geqq 1$). Then, by (3.5), $f \in L_p(\lambda)$. Hence, by the individual ergodic theorem concerning Markov process $P(x, A)$ with a finite invariant measure λ, there exists an invariant (λ) function $\widetilde{f} \in L_p(\lambda)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \widetilde{f}(x) \qquad \lambda\text{-a. e.}$$

Hence, by Lemma 6, $\widetilde{f} \in L_p(Y, \mu)$. Further, by (3.7), $T^j f(x)$ ($j = 1, 2, \ldots\ldots..$) are defined for μ-a. a. x in Y and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \widetilde{f}(x) \qquad \mu\text{-a. e. in } Y.$$

Thus the first half of (I. 1) is proved.

Next we shall prove the second half of (I. 1). Suppose $f \in L_\infty(\mu)$. For the proof it is no loss of generality to assume that $0 \leqq f(x) \leqq 1$ μ-a. e. Now we set

$$g(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) - \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x).$$

Then it is sufficient to show $\int g(x)\mu(dx) = 0$. Since $0 \leqq \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) \leqq 1$

μ-a. e. ($n = 1, 2, \ldots\ldots..$), it follows, by the convergence theorem, that

$$Tg(x) \geqq g(x) \qquad \mu\text{-a. e.,}$$

and so

$$T^n g(x) \geqq g(x) \qquad \mu\text{-a. e.} \qquad (n = 1, 2, \ldots\ldots..). \tag{3.12}$$

Set

$$X_{mk} = \left\{ x ; \frac{k-1}{m} < g(x) \leqq \frac{k}{m} \right\}$$
$$(k = 1, \ldots\ldots.., m ; m = 1, 2, \ldots\ldots..).$$

Let ε be any positive number. Then there exists a positive integer m such that

$$\int g(x)\lambda(dx) + \varepsilon > \sum_{k=1}^{m} \frac{k}{m} \lambda(X_{mk}).$$

By (3.6) there exists a positive integer n such that

$$\lambda(X_{mk}) + \frac{\varepsilon}{m} > \frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(X_{mk}) \qquad (k = 1, \ldots\ldots, m).$$

Hence

$$\int g(x)\lambda(dx) + \varepsilon > \sum_{k=1}^{m} \frac{k}{m}\left(\frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(X_{mk}) - \frac{\varepsilon}{m}\right) \tag{3.13}$$

$$\geqq \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1}^{m} \frac{k}{m} S^j\mu(X_{mk}) - \varepsilon$$

$$\geqq \frac{1}{n}\sum_{j=0}^{n-1}\int g(x)S^j\mu(dx) - \varepsilon.$$

Here, by (3.12),

$$\int g(x)S^j\mu(dx) = \int T^j g(x)\mu(dx)$$

$$\geqq \int g(x)\mu(dx) \qquad (j = 0, \ldots\ldots, n-1).$$

Thus we obtain, from (3.13),

$$\int g(x)\lambda(dx) + \varepsilon > \int g(x)\mu(dx) - \varepsilon.$$

Since ε is any positive number,

$$\int g(x)\lambda(dx) \geqq \int g(x)\mu(dx). \tag{3.14}$$

On the other hand, $\frac{1}{n}\sum_{j=0}^{n-1} T^j f(x)$ converges λ-a. e. as $n \to \infty$ as was shown in the proof of the first half of (I. 1), so that $g(x) = 0$ λ-a. e. and so

$$\int g(x)\lambda(dx) = 0. \tag{3.15}$$

Consequently, by (3.14) and (3.15),

$$\int g(x)\mu(dx) = 0,$$

as was to be shown.

Proof of "(I. 2) → (C. 1)": We shall use the case of $p = 1$ of (I. 2). Define an operator $\widetilde{T}$ of $L_1(\mu)$ into itself as follows:

$$f \to \widetilde{T}f = \widetilde{f}: \quad \widetilde{f}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f(x).$$

Then $\widetilde{T}$ is a linear positive operator of $L_1(\mu)$ into itself, so that, by Lemma 7, $\widetilde{T}$ is a bounded operator. Let K denote the norm of $\widetilde{T}$, $\|\widetilde{T}\|_1$. Since, for every set A, $\frac{1}{n}\sum_{j=0}^{n-1} T^j e_A(x)$ converges boundedly to $\widetilde{T}e_A(x)$ μ-a. e. as $n\to\infty$, it holds that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(A) = \lim_{n\to\infty}\int \frac{1}{n}\sum_{j=0}^{n-1} T^j e_A(x)\mu(dx)$$
$$= \int \widetilde{T}e_A(x)\mu(dx) \leqq K\cdot \int e_A(x)\mu(dx) = K\cdot\mu(\mathrm{A}).$$

PROOF OF THEOREM 3. Proof of "(I. 3) → (C. 2)": It is clear that (I. 3) → (I. 2), and, by Theorem 2, (I. 2) → (C. 1). Hence it is sufficient to prove that if $\mu(A) > 0$ then $\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(A) > 0$. Suppose $\mu(A) > 0$. Then, by (I. 3), $\widetilde{e}_A(x)$ is positive in a set of positive μ-measure, so that

$$0 < \int \widetilde{e}_A(x)\mu(dx) = \lim_{n\to\infty}\int \frac{1}{n}\sum_{j=0}^{n-1} T^j e_A(x)\mu(dx)$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} S^j(A),$$

as was to be proved.

Proof of "(C. 2)→(I. 3)": We shall first prove that X is an incompressible (μ) set. Since (C. 2) → (C. 1), we can consider the finite invariant measure λ defined in Lemma 4. By (3. 6) and (C. 2),

$$\text{if } \mu(A) > 0 \text{ then } \lambda(A) > 0 \tag{3.16}$$

or, equivalently,

$$\text{if } \lambda(A) = 0 \text{ then } \mu(A) = 0.$$

Now we shall show that no compressible (μ) set exists. Suppose

$$P(x, A) \leqq e_A(x) \qquad \mu\text{-a. e.}$$

Then, by (3. 5),

$$P(x, A) \leqq e_A(x) \qquad \lambda\text{-a. e.}$$

Since λ is an invariant measure, it follows that

$$P(x, A) = e_A(x) \qquad \lambda\text{-a. e,}$$

Hence, by (3. 16),

$$P(x, A) = e_A(x) \qquad \mu\text{-a. e.,}$$

which shows that A is not a compressible (μ) set. Since no compressible (μ) set exists, X must be an incompressible (μ) set.

Hence, in the present case, (I. 1) coincides (I. 2), and by Theorem 2, (C. 1) → (I. 1). Thus for the proof of (I. 3) it is sufficient to show that if $f \in L_p(\mu)$ is a non-negative function and positive in a set of positive μ-measure, so is $\widetilde{f}$. Suppose that $f \in L_p(\mu)$ is a non-negative function and $\mu\{x; f(x) > 0\} > 0$. Then, by (3. 5), $f \in L_p(\lambda)$ and, by (3. 16), $\lambda\{x;\ f(x) > 0\} > 0$. Hence, by the ergodic theorem concerning Markov process $P(x, A)$ with a finite invariant measure λ,

$$\int \tilde{f}(x)\lambda(dx) = \int f(x)\lambda(dx) > 0,$$

so that $\lambda\{x; \tilde{f}(x) > 0\} > 0$. Hence, by (3.5), $\mu\{x; \tilde{f}(x) > 0\} > 0$, as was to be proved.

PROOF OF THEOREME 4. We shall prove the implications "(C.1) $\to$ (I.1) $\to$ (I.2) $\to$ (I.3) $\to$ (C.2) $\to$ (C.1)". The implications "(C.1) $\to$ (I.1')' and "(I.3) $\to$ (C.2)" follow from Theorems 2 and 3, respectively. The implication "(C.2) $\to$ (C.1)" is trivial. Since X is an incompressible (μ) set, (I.1) coincides (I.2). Thus it remains to prove the implication "(I.2) $\to$ (I.3)". Let λ be the finite invariant measure defined in Lemma 4. Since X is an incompressible (μ) set, (3.7) coincides with:

(3.16') $\quad$ if $\lambda(A) = 0$ then $\mu(A) = 0$

or, equivalently, with:

$$\text{if } \mu(A) > 0 \text{ then } \lambda(A) > 0.$$

Then the proof of "(I.2) $\to$ (I.3)" is the same to that of the latter half of "(C.2) $\to$ (I.3)" of Theorem 3. Thus the proof of Theorem 4 is terminated.

4. Mean ergodic theorem. We shall first state three propositions.

(M.1) *The operator T is a bounded operator of $L_p(\mu)$ $(p \geqq 1)$ into itself and, for every $f \in L_p(\mu)$, there exists a function $\tilde{f} \in L_p(\mu)$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j f - \tilde{f} \right\|_p = 0.$$

(M.2) *The operator T is a bounded operator of $L_p(\mu)$ $(p \geqq 1)$ into itself and, for every $f \in L_p(\mu)$, there exists a function $\tilde{f} \in L_p(\mu)$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j f - \tilde{f} \right\|_p = 0.$$

Moreover, if f is a non-negative function and positive in a set of positive μ-measure, so is $\tilde{f}$.

(C.3) *There exists a positive constant K such that*

$$\frac{1}{n} \sum_{j=0}^{n-1} S^j \mu(A) \leqq K \cdot \mu(A) \qquad (n = 1, 2, \ldots\ldots)$$

for all A.

Now we shall prove the following

THEOREM 5. (C.3) $\rightleftarrows$ (M.1).

THEOREM 6. (M.1) $\to$ (I.1).

THEOREM 7. (M. 2) $\rightarrow$ (I. 3).

THEOREM 8. *If X is an incompressible* (μ) *set,* (M. 1) $\rightarrow$ (M. 2).

Theorem 5 (the case of $p=1$) has been proved by I. Miyadera [6]. The proof depends on the method of that of the Riesz, Yosida and Kakutani ergodic theorem in Banach space (see, for example, [10]). In the present paper, Theorems 5, 7 and 8 will be proved on using the results in §3. Theorem 6 follows from Theorems 5 and 2, since (C. 3) $\rightarrow$ (C. 1) trivially.

PROOF OF THEOREM 5. Proof of "(C. 3) $\rightarrow$ (M. 1)": For every $f \in L_p(\mu)$ ($p \geqq 1$),

$$\begin{aligned} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j f \right\|_p &= \left\{ \int \left| \int f(y) \frac{1}{n} \sum_{j=0}^{n-1} P^{(j)}(x, dy) \right|^p \mu(dx) \right\}^{1/p} \text{ [9]} \\ &\leqq \left\{ \int\int |f(y)|^p \frac{1}{n} \sum_{j=0}^{n-1} P^{(j)}(x, dy) \mu(dx) \right\}^{1/p} \\ &= \left\{ \int |f(y)|^p \frac{1}{n} \sum_{j=0}^{n-1} S^j \mu(dy) \right\}^{1/p} \\ &\leqq \left\{ K \cdot \int |f(y)|^p \mu(dy) \right\}^{1/p} \\ &= K^{1/p} \cdot \|f\|_p \leqq K \cdot \|f\|_p \qquad (n = 1, 2, \ldots\ldots\ldots). \end{aligned}$$

Hence

$$(4.1) \qquad \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j \right\|_p \leqq K \qquad (n = 1, 2, \ldots\ldots\ldots).$$

On the other hand, the implication "(C. 3) $\rightarrow$ (C. 1)" is trivial and, by Theorem 2, (C. 1) $\rightarrow$ (I. 1). Hence, for every $f \in L_\infty(\mu)$ there exists a function $\tilde{f} \in L_\infty(\mu)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = \tilde{f}(x) \qquad \mu\text{-a. e.}$$

Since $\left| \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) \right| \leqq \|f\|_\infty$ μ-a. e. ($n = 1, 2, \ldots\ldots\ldots$), by the convergence theorem it follows that, for every $p \geqq 1$,

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j f - \tilde{f} \right\|_p = 0.$$

Further, for every $p \geqq 1$, $L_\infty(\mu)$ is dense in $L_p(\mu)$. Hence, by (4.1), it holds that, for every $f \in L_p(\mu)$, $\frac{1}{n} \sum_{j=0}^{n-1} T^j f$ converges to a function $\tilde{f} \in L_p(\mu)$ in the sense of $L_p(\mu)$-norm as $n \to \infty$.

9) Here $P^{(0)}(x, A)$ denotes $e_A(x)$.

Proof of "(M.1) → (C.3)": We shall use the case of $p = 1$ of (M.1). Since, for every $f \in L_1(\mu)$, $\frac{1}{n}\sum_{j=0}^{n-1} T^j f$ converges in a sense of $L_1(\mu)$-norm as $n \to \infty$, by the Banach theorem there exists a positive constant K such that

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} T^j \right\|_1 \leqq K \qquad (n = 1, 2, \ldots\ldots).$$

Hence, for every set A,

$$\frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(A) = \int \frac{1}{n}\sum_{j=0}^{n-1} T^j e_A(x)\mu(dx)$$
$$\leqq \left\| \frac{1}{n}\sum_{j=0}^{n-1} T^j \right\|_1 \|e_A\|_1 \leqq K\cdot\mu(A) \qquad (n = 1, 2, \ldots\ldots).$$

Proof of Theorem 7. By the trivial implications and Theorem 5 it is clear that (M.1) → (C.1). If we prove the implication "(C.1) → (C.2)" under the assumption (M.2), then we obtain (I.3) by virtue of Theorem 3. For the proof it is sufficient to show that if $\mu(A) > 0$ then $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(A) > 0$. Suppose $\mu(A) > 0$. Then, by (M.2), $\tilde{e}_A$ is positive in a set of positive μ-measure. Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} S^j\mu(A) = \lim_{n\to\infty}\int \frac{1}{n}\sum_{j=0}^{n-1} T^j e_A(x)\mu(dx)$$
$$= \int \tilde{e}_A(x)\mu(dx) > 0.$$

Proof of Theorem 8. By Theorem 6, (M.1) → (I.1). Since X is an incompressible (μ) set, by Theorem 4 it holds that (I.1) → (I.3). The propositions (I.3) and (M.1) together imply (M.2).

Finally we shall give one of the applications of our theorems.

Corollary. *Let the transition probability $P(x, A)$ be generated by a bounded function $p(x,y)$ of two variables x, $y \in X$ as follows:*

$$P(x, A) = \int_A p(x,y)\mu(dy).$$

Then (I.2) *and* (M.1) *hold.*

Proof. Let $p(x,y) \leqq K$. Then, for each A,

$$P^{(1)}(x, A) = \int_A p(x,y)\mu(dy) \leqq K\cdot\mu(A),$$

$$P^{(2)}(x, A) = \int P(y, A)P(x, dy) \leqq \int K\cdot\mu(A)P(x, dy) = K\cdot\mu(A),,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

$$P^{(n)}(x, A) = \int P^{(n-1)}(y, A) P(x, dy) \leqq \int K\cdot\mu(A) P(x, dy) = K\cdot\mu(A),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Hence

$$S^n\mu(A) = \int P^{(n)}(x, A)\mu(dx) \leqq K\cdot\mu(A) \qquad (n = 1, 2, \ldots\ldots\ldots),$$

so that, by Theorems 5 and 6, both (M.1) and (I.1) hold.

Next we shall prove (I.2). Consider any $f \in L_p(\mu)$ ($p \geqq 1$). Let ε be any positive number. Then there exists a function $g \in L_\infty(\mu)$ such that

$$\int |f(x) - g(x)|\mu(dx) < \frac{\varepsilon}{2K}.$$

By virtue of (I.1), $\frac{1}{n}\sum_{j=0}^{n-1} T^j g(x)$ converges to a finite limit for μ-a.a. x as $n \to \infty$. Thus

$$\begin{aligned} &\limsup_{m,n\to\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1} T^j f(x) - \frac{1}{n}\sum_{j=0}^{n-1} T^j f(x)\right| \\ &\leqq \limsup_{m\to\infty}\int |f(y) - g(y)|\,\frac{1}{m}\sum_{j=0}^{m-1} P^{(j)}(x, dy) \\ &\quad + \limsup_{m,n\to\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1} T^j g(x) - \frac{1}{n}\sum_{j=0}^{n-1} T^j g(x)\right| \\ &\quad + \limsup_{n\to\infty}\int |g(y) - f(y)|\,\frac{1}{n}\sum_{j=0}^{n-1} P^{(j)}(x, dy) \\ &\leqq 2K\cdot\int |f(y) - g(y)|\mu(dy) < \varepsilon \qquad \mu\text{-a.e.} \end{aligned}$$

Since ε is any positive number, $\frac{1}{n}\sum_{j=0}^{n-1} T f(x)$ converges to a finite limit for μ-a.a. x as $n \to \infty$ and, by virtue of (M.1), the limit function must belong to $L_p(\mu)$. Hence (I.2) holds.

REFERENCES

[1] J. L. DOOB, Asymptotic properties of Markoff transition probabilities, Trans. Amer. Math. Soc. 63 (1948), 393-421.

[2] N. DUNFORD and D. S. MILLER, On the ergodic theorem, Trans. Amer. Math. Soc. 60 (1946), 538-549.

[3] E. HOPF, Ergodentheorie, Berlin (1937).

[4] ———, The general temporally discrete Markoff process, Jour. of Rat. Mech. and Analys. 3 (1954), 13-45.

[5] S. KAKUTANI, Ergodic theorems and the Markoff process with a stable distribu-

tion, Proc. Imp. Acad. Tokyo, 16(1940), 49-54.

[6] I. MIYADERA, On the mean ergodic theorem concerning a Markov process (in Japanese), The Monthly of Real Analysis, 7(1953), 63-66.

[7] F. RIESZ, On a recent generalisation of G. D. Birkhoff's ergodic theorem, Acta de Szeged, 11 (1946-1948), 193-200.

[8] C. RYLL-NARDZEWSKI, On the ergodic theorems (I) (Generalized ergodic theorems), Studia Math. 12(1951), 65-73.

[9] S. TSURUMI, On the ergodic theorem, Tôhoku Math. Jour. 6(1954) 53-68.

[10] K. YOSIDA, Mean ergodic theorem in Banach spaces, Proc. Imp. Acad. Tokyo, 14(1938), 292-294.

[11] ———, The Markoff process with a stable distribution, Proc. Imp. Acad. Tokyo, 16(1940), 43-48.

MATHEMATICAL INSTITUTE, TOKYO METROPOLITAN UNIVERSITY.