CROSSED PRODUCT OF OPERATOR ALGEBRA

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Among various notions of the modern ring theory the idea of the crossed product of an algebra by its group of automorphisms is seemed to be not yet explicitly introduced. J. von Neumann's method of construction of the example of factor shows us the possibility and the way of introducing this notion.

In this paper, we define this notion in a C^*-algebra (§ 1), in a unitary algebra (§ 2), and in a special W^*-algebra (§ 3). We shall mainly concern with the representation of the crossed-product, and finally show that some of the examples of factors by von Neumann can be considered as our just defined crossed product (§4). We only interpretate the von Neumann's example from the view-point of the crossed product, and we don't discuss further problems, for example, "For what kind of W^*-algebra and its group of automorphisms, does the crossed product produce the unknown new factor?" (For some of these problems, cf. N. Suzuki [4]).

1. For a (discrete) group G of the $*$-automorphisms of a $*$-algebra A, we shall consider A-valued functions defined on G, which take 0 except a finite subset of G, and denote any A-valued function which takes $a_i \in A$ at the point $\alpha_i \in G$ for each $i(i=1,2,\ldots,m)$ by $\Sigma_i \alpha_i a_i$. The set $\mathfrak{S}$ of all these functions is clearly a linear space for the usual operations of the addition and the scalar multiplication. Of course the zero element of $\mathfrak{S}$ is the function which takes 0 everywhere.

If we introduce the multiplication-operation and the $*$-operation as follows:

multiplication: $\left(\sum_i \alpha_i a_i\right)\left(\sum_j \beta_j b_j\right) = \sum_{i,j} \alpha_i \beta_j a_i^{\beta_j} b_j$

$*$-operation: $\left(\sum_i \alpha_i a_i\right)^* = \sum_i \alpha_i^{-1} a_i^{*\alpha_i^{-1}}$

where a^α denotes the image of a by the automorphism α, then the set $\mathfrak{S}$ is a $*$-algebra, which we call the *crossed product* of A by G and denote it by (A, G).

As the mapping $a(\in A) \to \varepsilon a(\in (A, G))$ is the $*$-isomorphism from A into (A, G), where ε denotes the unit of G, A is a $*$-subalgebra of (A, G). Furthermore if A has the unit 1 then (A, G) has the unit $\varepsilon 1$.

In the following we shall study the representations of (A, G) for various types of A. We shall begin with the following

DEFINITION 1. When φ is a linear functional on A, we define the linear functional $\tilde{\varphi}$ on (A, G) as follows:

$$\tilde{\varphi}\left(\sum_i \alpha_i a_i\right) = \sum_i \delta_e^{\alpha_i} \varphi(a_i),$$

where δ_α^β denotes the Kronecker symbol $\delta_\alpha^\beta = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta, \end{cases}$ and we call $\tilde{\varphi}$ the *extension* of φ on (A, G).

At first if we assume that A is a C^*-algebra with the unit, then we can prove the following proposition:

PROPOSITION 1. *Let A be a C^*-algebra, and G be a group of $*$-automorphisms of A. If $\sum = \{\sigma\}$ is a complete set*[1] *of positive linear functionals on A, then $\tilde{\sum} = \{\tilde{\sigma}\}$, the set of extensions of elements of $\sum$, is also a complete set of positive linear functionals on (A, G).*

PROOF. Since the additivity and the homogeneity of $\tilde{\sigma} \in \tilde{\sum}$ are obvious, it is sufficient to show the positivity of $\tilde{\sigma}$ and the completeness of the set $\tilde{\sum}$.

Since

$$\begin{aligned} \tilde{\sigma}\left[\left(\sum_i \alpha_i a_i\right)\left(\sum_i \alpha_i a_i\right)^*\right] &= \tilde{\sigma}\left(\sum_{i,j} \alpha_i \alpha_j^{-1} a_i^{\alpha_j^{-1}} a^{*\alpha_j^{-1}}\right) \\ &= \sum_i \sigma\left(a_i^{\alpha_i^{-1}} a_i^{*\alpha_i^{-1}}\right) \geqq 0, \end{aligned}$$

$\tilde{\sigma}$ is positive; moreover, if $\tilde{\sigma}\left[\left(\sum_i \alpha_i a_i\right)\left(\sum_i \alpha_i a_i\right)^*\right] = 0$ for all $\tilde{\sigma} \in \tilde{\sum}$, then from the above computations $\sigma(a_i^{\alpha_i^{-1}} a_i^{*\alpha_i^{-1}}) = 0$ for all $\sigma \in \sum$ and $i = 1, 2, \ldots m$. Therefore by the completeness of $\sum$, we have $a_1 = a_2 = \ldots = a_m = 0$, so that $\sum_i \alpha_i a_i = 0$, which shows the completeness of $\tilde{\sum}$ on (A, G). Q. E. D.

Now, let A, G be a C^*-algebra and a group of $*$-automorphisms of A resp, and let $\sum$, $\tilde{\sum}$ be a complete set of states of A and the set of extensions of elements of $\sum$ resp. We denote by $\mathfrak{P}$ the set of positive type functionals Φ on (A, G) such that

$$\Phi\left(\sum_i \alpha_i a_i\right) = \frac{\tilde{\sigma}\left[\left(\sum_j \beta_j b_j\right)\left(\sum_i \alpha_i a_i\right)\left(\sum_j \beta_j b_j\right)^*\right]}{\tilde{\sigma}\left[\left(\sum_j \beta_j b_j\right)\left(\sum_j \beta_j b_j\right)^*\right]}$$

where $\tilde{\sigma} \in \tilde{\sum}$ and $\sum_j \beta_j b_j$ is an arbitrary element of (A, G). If we introduce a norm in (A, G) by

$$(*) \qquad \left\| \sum_i \alpha_i a_i \right\| = \sup \left[\Phi\left[\left(\sum_i \alpha_i a_i\right)\left(\sum_i \alpha_i a_i\right)^*\right]^{1/2} : \Phi \in \mathfrak{P}\right],$$

then the completion of (A, G) by this norm is a C^*-algebra, which we call

1) Cf. I. E. Segal [3].

a *u-crossed product* of A by G, and we shall denote it by $\mathfrak{A}=C^*(A,G,\sum)$.

Without the G-invariancy of the elements of $\sum$, it is troublesome and fruitless to proceed the study of $C^*(A,G,\sum)$, so that in the rest of this section we shall consider a C^*-algebra A, a group G of $*$-automorphisms of A and a *complete* state σ of A which is *invariant* with respect to G.

To discuss the representation of the C^*-algebra $\mathfrak{A}=C^*(A,G,\sigma)$, we shall have a few preparatory discussions on the representation of C^*-algebra by its state.

Now, since σ is a complete state of a C^*-algebra A, A can be considered as a pre-Hilbert space, by introducing the inner product $(a,b)=\sigma(ab^*)$; we shall denote by H the Hilbert space obtained by the completion of A. Moreover if we define the operator $a^\sharp$ on A by $ba^\sharp=ba$, then $a^\sharp$ is bounded on A, therefore $a^\sharp$ can be extended onto H; and we denote this extension by the same notation $a^\sharp$, then the mapping $a\to a^\sharp$ is a faithful representation of A on H. We say this representation a *canonical representation* of A by the state σ.

In the following we shall study the relation between the canonical representation of A by σ and that of (A,G) by $\tilde{\sigma}$ (the extension of σ). For this purpose we introduce the direct product Hilbert space $G\otimes H$ of G and H following H. Umegaki [5].

Let $F(G)$ be the vector space of all finite-valued numerical functions on G and $F(G)\otimes H$ the algebraic direct product of $F(G)$ and H. Putting f_α the characteristic function of the point α and denoting $f_\alpha\otimes\xi$ as $\alpha\otimes\xi$ conveniently, all elements $\alpha\otimes\xi$, $\alpha\in G$, $\xi\in H$ generate a vector subspace of $F(G)\otimes H$. We shall denote this subspace as $G\odot H$. For elements $\sum\alpha_i\otimes\xi_i$, $\sum\alpha'_j\otimes\xi'_j$ of $G\odot H$, we define the inner product by

$$\left(\sum_i \alpha_i\otimes\xi_i,\ \sum_j\alpha'_j\otimes\xi'_j\right)=\sum_{ij}\delta^{\alpha'_j}_{\alpha_i}(\xi_i,\ \xi'_j),$$

whence $G\odot H$ is a pre-Hilbert space. The completion of this space, we shall call the direct product Hilbert space of G and H and denote it by $G\otimes H$. It is known that $G\otimes H$ is isomorphic to $l^2(G)\otimes H$ in the sense of Murray-von Neumann [2].

Next, we introduce the two kinds of bounded operators on $G\otimes H$:

1°. R_a $(a\in A)$: Define $R_a=1\otimes a^\sharp$, that is,

$$\left(\sum\alpha_i\otimes a_i\right)R_a=\sum\alpha_i\otimes(a_i a^\sharp)$$

on the dense part $G\odot A$ of $G\otimes H$. It is almost obvious that the mapping $a\to R_a$ is a faithful representation of A on $G\otimes H$.

2°. U_α $(\alpha\in G)$: If we define U'_α on $G\odot A$ by the equation $\left(\sum\alpha_i\otimes a_i\right)U'_\alpha=\sum(\alpha_i\alpha)\otimes a_i^\alpha$, then U'_α is bounded, so that U'_α can be extended onto

$G \otimes H$; we denote this extension by U_α. As easily seen, U_α is a unitary operator on $G \otimes H$ and the mapping $\alpha \to U_\alpha$ is a unitary representation of G on $G \otimes H$ such that $U_\alpha^{-1} R_a U_\alpha = R_a{}^\alpha$. Moreover, by the easy computation it follows that

$$(U_\alpha R_a)^* = U_{\alpha^{-1}} R_a{}^{*\alpha^{-1}},$$

and

$$(U_\alpha R_a)(U_\beta R_b) = U_{\alpha\beta} R_a{}^\beta{}_b,$$

therefore the $*$-algebra A which is algebraically generated by $\mathfrak{I} = \{R_a, U_\alpha : a \in A, \alpha \in G\}$ is a set $\left\{\sum U_{\alpha_i} R_{a_i} : a_i \in A, \alpha_i \in G\right\}$. We denote by $\mathbf{A}$ the uniform closure of A.

By means of these notations we have the following

THEOREM 1. *If A is a C^*-algebra, G is a group of $*$-automorphisms of A and σ is a complete state of A which is invariant with respect to G, then the linear mapping*

$$\pi : \sum_i \alpha_i a_i \to \sum_i U_{\alpha_i} R_{a_i}$$

is an isometric representation of the crossed product (A, G) of A by G onto the $*$-algebra $\mathbf{A}$ on the Hilbert space $G \otimes H$.

PROOF. At first we shall show that the above correspondence is one-to-one. Since the implication $\sum \alpha_i a_i = 0 \Rightarrow \sum_i U_{\alpha_i} R_{a_i} = 0$ is clear, we shall show the converse one. If $\sum U_{\alpha_i} R_{a_i} = 0$, then

$$\begin{aligned} 0 &= \left((\varepsilon \otimes 1)\sum_i U_{\alpha_i} R_{a_i},\ (\varepsilon \otimes 1)\sum_i U_{\alpha_i} R_{a_i}\right) \\ &= \sum_{i,j} (\alpha_i \otimes a_i,\ \alpha_j \otimes a_j) \\ &= \sum_{i,j} \sigma(a_i a_i^*). \end{aligned}$$

Since σ is a complete state of A, we have $a_1 = a_2 = \ldots = a_m = 0$, and $\sum_i \alpha_i a_i = 0$.

That the mapping π is a $*$-homomorphism is almost obvious by considering the $*$-operation and the multiplication in each algebra.

The isometric property of π can be seen as follows:

$$\begin{aligned} &\left(\left(\sum \beta_p \otimes b_p\right)\sum U_{\alpha_i} R_{a_i},\ \left(\sum \beta_p \otimes b_p\right)\sum U_{\alpha_i} R_{a_i}\right) \\ &= \sum_{i,j,p,q} (\beta_p \alpha_i \otimes b_p^{\alpha_i} a_i,\ \beta_q \alpha_j \otimes b_q^{\alpha_j} a_j) \\ &= \sum \delta^{\beta_p \alpha_i}_{\beta_q \alpha_j} \sigma(b_p^{\alpha_i} a_i a_j^* b_q^{*\alpha_j}) \\ &= \sum{}' \sigma(b_p^{\alpha_i} a_i a_j^* b_q^{*\alpha_j}), \end{aligned}$$

where $\sum'$ denotes the summation over the indices such that $\beta_p \alpha_i \alpha_j^{-1} \beta_q^{-1} = \varepsilon$.

On the other hand,

$$\tilde{\sigma}\left[\left(\sum\beta_p b_p\right)\left(\sum\alpha_i a_i\right)\left(\sum\alpha_i a_i\right)^*\left(\sum\beta_p b_p\right)^*\right]$$
$$= \tilde{\sigma}\left[\sum_{i,j,p,q}\beta_p\alpha_i\alpha_j^{-1}\beta_q^{-1}\, b_p^{\alpha_i\alpha_j^{-1}\beta_q^{-1}}\, a_i^{\alpha_j-1\beta q-1}\, a_j^{*\alpha_j-1\beta q-1}\, b_q^{*\beta_q^{-1}}\right]$$
$$= \sum{}' \sigma(b_p^{\alpha_i}\, a_i\, a_j^*\, b_q^{*\alpha j}).$$

By the above computations

$$\left(\left(\sum\beta_p\otimes b_p\right)\sum U_{\alpha_i}R_{a_i},\ \left(\sum\beta_p\otimes b_p\right)\sum U_{\alpha_i}R_{a_i}\right)$$
$$= \tilde{\sigma}\left[\left(\sum\beta_p b_p\right)\left(\sum\alpha_i a_i\right)\left(\sum\alpha_i a_i\right)^*\left(\sum\beta_p b_p\right)^*\right],$$

therefore by the definition of norm (*)

$$\left\|\sum\alpha_i a_i\right\| = \left|\sum U_{\alpha_i}R_{a_i}\right| (= \text{operator bound}).$$

REMARK. Since the representation π is isometric, it can be extended as the representation of $\mathfrak{A}$ onto $\mathbf{A}$; we use the same notation to denote this extension.

COROLLARY. *Under the same conditions on A, G and σ as in Theorem* 1, *the canonical representation of $\mathfrak{A} = C^*(A, G, \sigma)$ by the state $\tilde{\sigma}$ is unitarily equivalent to the representation π defined in Theorem* 1.

PROOF. If we define the mapping ϕ from (A, G) onto $G \odot A$:

$$\phi: \sum\alpha_i a_i \to \sum\alpha_i\otimes a_i$$

then ϕ is linear isometric. In fact.

$$\left(\left(\sum_i\alpha_i a_i\right),\ \left(\sum_i\alpha_i a_i\right)\right) = \tilde{\sigma}\left[\left(\sum_i\alpha_i a_i\right)\left(\sum_i\alpha_i a_i\right)^*\right]$$
$$= \sum_{i,j}\tilde{\sigma}\left(\alpha_i\alpha_j^{-1}a_i^{\alpha_j^{-1}}a_j^{*\alpha_j^{-1}}\right)$$
$$= \sum_i\sigma(a_i a_i^*) = \sum_i(a_i,\ a_i)$$
$$= \left(\sum_i\alpha_i\otimes a_i,\ \sum_i\alpha_i\otimes a_i\right).$$

Since (A, G) and $G \odot A$ are dense in $\mathfrak{H}$ and $G \otimes H$ respectively, ϕ induces the unitary operator from $\mathfrak{H}$ onto $G \otimes H$, we use the same notation ϕ to denote it. Whence it is easily verified that

$$\phi^{-1}\left(\sum_i\alpha_i a_i\right)^{\#}\phi = \sum_i U_{\alpha_i}R_{a_i} = \left(\sum_i\alpha_i a_i\right)\pi$$

is valid on $G \odot A \odot$ Since the representation π and the canonical representation are both continuous, the proof is completed.

REMARK. Though in Theorem 1 and its Corollary we assume that the C^*-algebra A has a G-invariant complete state σ, we can show the similar results, by passing the direct sum method, when A has a complete set $\sum = \{\sigma\}$ of G-invariant states; but we don't enter into the detail.

2. In this section we shall develop the similar discussions for a unitary algebra and its group of $*$-automorphisms.

Let A be a unitary algebra[1]; that is, A is an associative $*$-algebra over the complex field and simultaneously A is a pre-Hilbert space with respect to the inner product $(,)$ satisfying the following conditions:

(1) $(a, b) = (b^*, a^*)$ $\quad (a, b \in A)$,

(2) $(ab, c) = (b, a^*c)$ $\quad (a, b, c \in A)$,

(3) for any $a \in A$, the mapping $b \to ba$ is continuous,

(4) $\{ab: a, b \in A\}$ is dense in A.

We denote the Hilbert space obtained by the completion of A by H.

The mapping $b \to ba$ can be extended onto H by the condition (3), we denote this extended operator by $a^\sharp$ and call the right multiplication operator by a. Similarly by the conditions (1), (3), the mapping $b \to ab$ can be extended on H and call this extended operator $a^\flat$ the left multiplication operator. We call the W^*-algebra $\mathfrak{R}(A)$ [resp. $\mathfrak{L}(A)$] which is generated by the right [resp. left] multiplication operators, a right [resp. left] W^*-algebra of A. At last since the mapping $a \to a^*$ is continuous by (1), we can extend this mapping onto H which we denote by j, and call the involution. We can easily show the following relations:

$$(ab)^\sharp = a^\sharp b^\sharp, \quad (ab)^\flat = b^\flat a^\flat, \quad a^\sharp b^\flat = b^\flat a^\sharp$$
$$ja^\sharp j = a^{*\flat}, \quad j^2 = 1$$
$$j\mathfrak{R}(A)j = \mathfrak{L}(A).$$

Now let G be a group of $*$-automorphisms which preserve the inner product: $(a^\alpha, b^\alpha) = (a, b)$ for every $\alpha \in G$, $a, b \in A$. Then since the mapping $a \to a^\alpha$ is continuous on A, it is uniquely extended on H, which we denote by $u(\alpha)$; thus G is a group of unitary operators on H, and satisfy the following relations

$$u(\alpha)j = ju(\alpha), \quad a^{\alpha\sharp} = u(\alpha)^{-1}a^\sharp u(\alpha)$$

which one can find without difficulty.

Under these assumptions, if we define the mapping $\phi: \sum_i \alpha_i a_i \to \sum_i \alpha_i \otimes a_i$ from the crossed product (A, G) onto the pre-Hilbert space $G \otimes A$, ϕ is linear and one-to-one; therefore if we define the inner product in (A, G) by the following way:

$$\left(\sum_i \alpha_i a_i, \sum_j \beta_j b_j\right) = \left(\left(\sum_i \alpha_i a_i\right)\phi, \left(\sum_j \beta_j b_j\right)\phi\right) = \left(\sum_i \alpha_i \otimes a_i, \sum_j \beta_j \otimes b_j\right),$$

then (A, G) becomes a pre-Hilbert space and satisfy the conditions (1)-(4) of a unitary algebra at the beginning of this section. In fact,

Ad (1). $\left(\sum_i \alpha_i a_i, \sum_j \beta_j b_j\right) = \sum_{i,j} \delta^{\gamma j}_{\alpha_i}(a_i, b_j) = \sum \delta^{\alpha i}_{\beta_j \alpha_i^{-1}}(b_j^{*\beta j}, a_i^{*\alpha_i^{-1}})$

1) For a unitary algebra, cf. J. Dixmier: Les algèbres d'opérateurs dans l'espace hilbertien, Paris (1957).

$$= \left(\left(\sum_j \beta_j b_j\right)^*,\ \left(\sum_i \alpha_i a_i\right)^*\right).$$

Ad (2). $\left(\left(\sum \alpha_i a_i\right)\left(\sum \beta_j b_j\right),\ \sum \gamma_k c_k\right) = \sum_{ijk} \delta_{\alpha_i\beta j}^{\gamma k}\left(a_i^{\beta j} b_j,\ c_k\right)$

$$= \sum \delta_{\beta j}^{\alpha_i^{-1}\gamma_k}\left(b_j,\ a_i^{*\alpha_i^{-1}\gamma_k} c_k\right) = \left(\sum \beta_j b_i,\ \left(\sum \alpha_i a_i\right)^*\left(\sum \gamma_k c_k\right)\right).$$

Ad (3). $\left\|\left(\sum \beta_j b_j\right)\left(\sum_{i=1}^m \alpha_i a_i\right)\right\|^2 = \sum \delta_{pja_i}^{\beta_i \alpha p}\left(b_i^{\alpha_i} a_i,\ b_q^{\alpha p} a_q\right)$

$$= \sum_{ij}\left(b_j^{\alpha_i} a_i,\ b_j^{\alpha_i} a_i\right) = \sum_i \left(\sum_j (b_j,\ b_j)\right)|a_i^{\#}|^2$$

$$\leqq M^2 \left\|\sum \beta_j b_j\right\|^2, \qquad \text{where } M = m \max_{1\leqq i\leqq m} |a_i^{\#}|.$$

Ad (4). Since A is a unitary algebra, for any $a \in A$ there exist b, c in A such that the distance between a and bc is arbitrarily small, therefore the distance between αa and αbc in (A, G) is arbitrarily small for all $\alpha \in G$. Since $\alpha bc = (\alpha b)(\varepsilon c)$, the finite linear combinations of the members of $\{(\beta b)(\gamma c) : \beta, \gamma \in G,\ b,\ c \in A\}$ are dense in (A, G), so $\left\{\left(\sum \beta_j b_j\right)\left(\sum \gamma_k c_k\right)\right\}$ is dense in (A, G).

In this way, we see that the crossed product (A, G) is also a unitary algebra.

Now, let $\mathfrak{H}$ be the Hilbert space obtained by the completion of (A, G) and let $\left(\sum \alpha_i a_i\right)^{\#}$ and $\left(\sum \alpha_i a_i\right)^{\flat}$ denote the right and left multiplication operators respectively. We define the following two kinds of operators on $\mathfrak{H}$:

1°. r_a $(a \in A)$: Since the mapping $\sum \alpha_i a_i \to \sum \alpha_i (a_i a^{\#})$ is clearly continuous, it is extendable on $\mathfrak{H}$, we denote this extension by r_a. (Clearly $r_a = (\varepsilon a)^{\#}$, and since A is isomorphic to εA, $a \to r_a$ is a representation of A on $\mathfrak{H}$; furthermore $r_a = \phi R_a \phi^{-1}$, where R_a is the operator defined on $G \otimes H$ in §1.)

2°. u_α $(\alpha \in G)$: If we define u'_α on (A, G) by the equation $\left(\sum \alpha_i a_i\right) u'_\alpha = \sum (\alpha_i \alpha) a_i^\alpha$, then u_α is bounded, so that u' can be extended on $\mathfrak{H}$, we define u_α by this extension. (Clearly $u_\alpha = \phi U_\alpha \phi^{-1}$, where U_α is the operator defined on $G \otimes H$ in §1, so that $\alpha \to u_\alpha$ is a unitary representation of G on $\mathfrak{H}$.)

Then we have the following

THEOREM 2. *Let A be a unitary algebra, and G a group of $*$-automorphisms which preserve the inner product invariantly, then the crossed product (A, G) is also a unitary algebra and the right W^*-algebra is generated by $\{r_a, u_\alpha : a \in A, \alpha \in G\}$.*

PROOF. The first half has been proved before stating the Theorem, and the last half followes from Theorem 1 considering the weak closure instead of uniform closure.

REMARK. It may be interesting to clarify the structure of (A, G) in the

relation to that of A, but we could not obtain the concrete results. Under suitable restrictions on G, we want to determine the structure of the unitary algebra (A, G) elsewhere.

3. In this section we shall discuss a case of a W^*-algebra which has a faithful normal trace as a special case of §2, and define the weakly closed crossed product algebra which is a factor under the suitable restrictions. (Def. 2 and Theorem 3.)

PROPOSITION 2. *Let A be a $*$-algebra which has a faithful trace. If G is a group of $*$-automorphisms of A which preserve the trace $tr(\ )$ invariant, then the extension $\tilde{tr}$ of tr is also a faithful trace of (A, G).*

PROOF. By means of Proposition 1, it is sufficient to prove the equation

$$\tilde{tr}[(\alpha a)(\beta b)] = \tilde{tr}[(\beta b)(\alpha a)].$$

Now, $\tilde{tr}[(\alpha a)(\beta b)] = \tilde{tr}(\alpha\beta\, a^{\beta} b) = \delta_{\epsilon}^{\alpha\beta}\, tr(a^{\beta} b) = \delta_{\epsilon}^{\beta\alpha}\, tr(b a^{\beta}) = \delta_{\epsilon}^{\beta\alpha}\, tr(b^{\alpha} a) = \tilde{tr}[(\beta\alpha)\, b^{\alpha} a] = \tilde{tr}[(\beta b)(\alpha a)]$, which is desired.

In the sequel we use the following notations:

A: A W^*-algebra acting on the Hilbert space H_0 with a faithful normal trace $tr(a) = (\xi_0 a, \xi_0)$, and ξ_0 is a generating and separating vector for A.

G: A group of $*$-automorphism of A, and we shall assume that tr is G-invariant.

As the mapping $a \to \xi_0 a$ is one-to-one, A can be considered as a unitary algebra and G as a group of $*$-automorphisms of a unitary algebra A which preserve the inner product invariant. So we use the same notations as in §2 without to refer.

Then the crossed product (A, G) is also a unitary algebra by Theorem 2.

On the other hand, the unitary algebra (A, G) has a faithful trace $\tilde{tr}$ by Proposition 2, so that the canonical representation of (A, G) by $\tilde{tr}$ can be taken, but as easily seen, the canonical image of $\sum_i \alpha_i a_i \in (A, G)$ coincides with the right multiplication operator $\left(\sum_i \alpha_i a_i\right)^{\#}$ of the unitary algebra (A, G), so we don't distinguish each other in the following.

Since $\varepsilon 1$ is a unit element of (A, G), it is a central element of the unitary algebra (A, G), so $((\varepsilon 1)T, (\varepsilon 1))$ $(T \in \mathfrak{R}(A, G))$ gives a trace of the right W^*-algebra of (A, G), and $((\varepsilon 1)T, (\varepsilon 1)) = tr(T)$, and moreover $\varepsilon 1$ is a generating vector for the right W^*-algebra of (A, G).

DEFINITION 2. For such A, G as above, we call the right W^*-algebra of a unitary algebra (A, G) a *w-crossed product* of A by G and denote it by $W^*(A, G, tr)$.

By Theorems 1, 2, $W^*(A, G, tr)$ is unitarily equivalent to the W^*-algebra $\mathbf{W}$ generated by $\mathbf{I} = \{U_\alpha, R_a : \alpha \in G,\ a \in A\}$ on the Hilbert space $G \otimes H$, where H is the completion of A, and U_α, R_a are the same as in §1. Of course

H is isomorphic to H_0.

In order to analyze $\mathbf{W}$ we introduce the following operators on $G \otimes H$:

J: If we define J' by the equation $\left(\sum \alpha_i \otimes a_i\right) J' = \sum \alpha_i^{-1} \otimes a_i^{*\alpha_i^{-1}}$, then J is continuous on $G \odot A$, so it is extendable onto $G \otimes H$, we shall denote this extension by J; then $J^2 = 1$ holds, and J is unitarily equivalent to the involution of the unitary algebra (A, G).

V_α $(\alpha \in G)$: Define V_α by the equation $JU_\alpha J = V_\alpha$, then V_α acts on $G \otimes A$ as follows:

$$\left(\sum \alpha_i \otimes a_i\right) V_\alpha = \sum (\alpha^{-1} \alpha_i) \otimes a_i.$$

L_a $(a \in A)$: Define L_a by $JR_a{}^*J$, then L_a acts on $G \otimes A$ as follows:

$$\left(\sum \alpha_i \otimes a_i\right) L_a = \sum \alpha_i \otimes (a^{\alpha_i} a_i).$$

The matrix representation of the operator on $G \otimes H$ due to von Neumann [2] is as follows:

$$T = (T^{\alpha,\beta}) \colon T^{\alpha,\beta} \text{ is a bounded operator on } H,$$

and $$< \delta_\alpha^\beta \xi \colon \beta \in G > T = < \xi\, T^{\alpha,\beta} \colon \beta \in G >.$$

Under these definitions and notations we have

$$U_{\alpha_0}^{-1} T U_{\alpha_0} = (u(\alpha_0)^{-1}\, T^{\alpha\alpha_0^{-1},\, \beta\alpha_0^{-1}} u(\alpha_0)), \quad V_{\alpha_0}^{-1} T V_{\alpha_0} = (T^{\alpha_0\alpha,\, \alpha_0\beta}),$$

$$JTJ = (ju(\alpha)^{-1}\, T^{\alpha^{-1},\beta^{-1}} u(\beta)\, j),$$

$$R_a T = (a^{\#} T^{\alpha,\beta}), \quad L_a T = (a^{\alpha b}\, T^{\alpha,\beta}),$$

$$TR_a = (T^{\alpha,\beta}\, a^{\#}), \quad TL_a = (T^{\alpha,\beta}\, a^{\beta b}).$$

Therefore, if we define $\mathbf{I} = \{U_\alpha,\ R_a \colon \alpha \in G,\ a \in A\}$, $\mathbf{J} = \{V_\alpha,\ L_a \colon \alpha \in G,\ a \in A\}$, then $\mathbf{I}'$ and $\mathbf{J}'$ can be characterized as follows:

PROPOSITION 3. *$T = (T^{\alpha,\beta})$ belongs to $\mathbf{I}'$ if and only if there exist $t(\alpha, \beta) \in A$ such that $T^{\alpha,\beta} = jt(\alpha,\beta)^{\#} = u(\alpha_0)^{-1}\, t(\alpha\alpha_0^{-1},\ \beta\alpha_0^{-1})^{\#}\, u(\alpha_0)$ for all $\alpha_0 \in G$; and $S = (S^{\alpha,\beta})$ belongs to $\mathbf{J}'$ if and only if there exist $s(\alpha,\beta) \in A$ such that $S^{\alpha,\beta} = u(\alpha)^{-1}\, s(\alpha^{-1},\ \beta^{-1})^{\#}\, u(\beta)$ and $s(\alpha,\beta)^{\#} = u(\alpha_0)^{-1}\, s(\alpha\alpha_0^{-1},\ \beta\alpha_0^{-1})^{\#}\, u(\alpha_0)$ for all $\alpha_0 \in G$.*

PROOF. If $T = (T^{\alpha,\beta}) \in \mathbf{I}'$, then $U_{\alpha_0}^{-1} T U_{\alpha_0} = T$ for all $\alpha_0 \in G$ and $R_a T = TR_a$ for all $a \in A$, hence $a^{\#} T^{\alpha,\beta} = T^{\alpha,\beta}\, a^{\#}$, i. e., $T^{\alpha,\beta} \in A^{\#\prime} = jA^{\#}j$, therefore there exist $t(\alpha,\beta) \in A$ such that $T^{\alpha,\beta} = jt(\alpha,\beta)^{\#} j$; and $U_{\alpha_0}^{-1} T U_{\alpha_0} = T$ implies $jt(\alpha,\beta)^{\#} j = u(\alpha_0)^{-1}\, jt(\alpha\alpha_0^{-1}, \beta\alpha_0^{-1})^{\#}\, ju(\alpha_0) = ju(\alpha_0)^{-1} t(\alpha\alpha_0^{-1},\ \beta\alpha_0^{-1})^{\#} u(\alpha_0) j$, hence $t(\alpha,\beta)^{\#} = u(\alpha_0)^{-1} t(\alpha\alpha_0^{-1},\ \beta\alpha_0^{-1})^{\#} u(\alpha_0)$ for all $\alpha_0 \in G$.

Conversely, if $T = (T^{\alpha,\beta})$, $T^{\alpha,\beta} = jt(\alpha,\beta)^{\#} j$, and $t(\alpha,\beta)$ satisfies the conditions, then clearly $T \in \mathbf{I}'$.

For the case of $\mathbf{J}'$, the proof is accomplished by considering the involution J.

PROPOSITION 4. $\mathbf{I}' = \mathbf{J}''$, $\mathbf{J}' = \mathbf{I}''\,(= \phi^{-1} \mathfrak{W} \phi)$, $\mathfrak{W} = W^*(A, G, tr)$.

PROOF. This is easily seen from the commutation theorem for the right and the left W^*-algebra of a unitary algebra (A, G), but here we prove it as an application of Proposition 3.

Since $\mathfrak{I} \subset \mathfrak{J}'$ is easily verified, we have $\mathfrak{I}' \supset \mathfrak{J}''$, therefore it is sufficient to show that the converse inclusion is true. Now, if $T = (T^{\alpha,\beta})$ belongs to $\mathfrak{I}'$, and $S = (S^{\alpha,\beta})$ belongs to $\mathfrak{J}'$, then by Proposition 3 we have the following:

$$
\begin{aligned}
(TS)^{\alpha,\gamma} = \sum_\beta T^{\alpha,\beta} S^{\beta,\gamma} &= \sum_\beta jt(\alpha,\beta)^{\#} js(\varepsilon, \gamma^{-1}\beta)^{\#} u(\beta^{-1}\gamma) \\
&= \sum_\beta jt(\alpha, \gamma\beta^{-1})^{\#} js(\varepsilon, \beta^{-1})^{\#} u(\beta) \\
&= \sum_\beta s(\varepsilon, \beta^{-1})^{\#} jt(\alpha, \gamma\beta^{-1})^{\#} ju(\beta) \\
&= \sum_\beta s(\varepsilon, \beta^{-1}\alpha)^{\#} jt(\alpha, \gamma\beta^{-1}\alpha)^{\#} ju(\alpha^{-1}\beta) \\
&= \sum_\beta s(\varepsilon, \beta^{-1}\alpha)^{\#} ju(\alpha^{-1})\, t(\varepsilon, \gamma\beta^{-1})^{\#} u(\alpha)\, ju(\alpha^{-1}\beta) \\
&= \sum_\beta s(\varepsilon, \beta^{-1}\alpha)^{\#} ju(\alpha^{-1})\, u(\beta) t(\beta, \gamma)^{\#} u(\beta^{-1}) u(\alpha) u(\alpha^{-1}) u(\beta) j \\
&= \sum_\beta s(\varepsilon, \beta^{-1}\alpha)^{\#} u(\alpha^{-1}\beta)\, jt(\beta, \gamma)^{\#} j \\
&= \sum_\beta S^{\alpha,\beta} T^{\beta,\gamma} = (ST)^{\alpha,\gamma}.
\end{aligned}
$$

This completes the proof.

Now, following J. Dixmier [1], we shall have the following

DEFINITION. A group G of the $*$-automorphisms of a W^*-algebra A is called *ergodic* if and only if the operator of the center of A which is invariant by G is only the scalar multiple of the unit.

If A is abelian, then the $*$-automorphism can be considered as the homeomorphism on the character space of A; in this case, the above ergodicity is equivalent to the usual one on the character space of A. (Cf. J. Dixmier [1])

THEOREM 3. *Let A be a W^*-algebra with a faithful normal trace tr such that $tr(a) = (\xi_0 a, \xi_0)$ and ξ_0 being a generating and separating vector for A. Moreover, let G be a group of $*$-automorphisms of A which preserve tr invariant. If G is ergodic, and satisfies the condition*

$$(\S) \qquad a^b = u(\alpha) b^{\#},\ \alpha \neq \varepsilon \Rightarrow a = b = 0 \quad (a,\ b \in A)$$

then the w-crossed product $\mathfrak{W} = W^*(A, G, tr)$ is a factor.

PROOF. By Proposition 4, $\mathfrak{W}$ is unitarily equivalent to the W^*-algebra $\mathfrak{I}'' = \mathfrak{J}'$, so that it is sufficient to prove that the W^*-algebra $\mathfrak{I}'' = \mathfrak{J}'$ is a factor.

If T belongs to the center of $\mathfrak{I}'' = \mathfrak{J}'$, then T belongs to $\mathfrak{I}' \cap \mathfrak{J}'$; whence by Proposition 3, we have $T = (T^{\alpha,\beta})$, $T^{\alpha,\beta} = jt(\alpha,\beta)^{\#} j$, $t(\alpha,\beta)^{\#} = u(\alpha_0^{-1})\, t(\alpha\alpha_0^{-1}, \beta\alpha_0^{-1})^{\#} u(\alpha_0)$ for all $\alpha_0 \in G$ and $T^{\alpha,\beta} = u(\alpha^{-1}\beta)\, s(\alpha^{-1}\beta, \varepsilon)^{\#}$ for some $t(\alpha,\beta)$ and $s(\alpha,\beta)$ of A. If $\alpha = \beta$, we have $jt(\alpha, \alpha)^{\#} j = s(\varepsilon, \varepsilon)^{\#}$: that is $t(\alpha,\alpha)^* = s(\varepsilon,\varepsilon)$

for all $\alpha \in G$ and $t(\alpha,\alpha)^* = s(\varepsilon,\varepsilon)$ belongs to the center of A. Moreover, $s(\varepsilon,\varepsilon)^\# = u(\alpha^{-1})\, s(\varepsilon,\varepsilon)^\# u(\alpha)$, that is, $s(\varepsilon,\varepsilon)^\#$ is permutable for all $u(\alpha)$. Since $\{u(\alpha): \alpha \in G\}' \cap A^{\#\dagger} = \{\lambda 1\}$ by the ergodicity of the group G, we have $s(\varepsilon,\varepsilon) = \lambda 1$ (λ: scalar), i.e., $T_{\alpha,\alpha} = \lambda 1$ for all $\alpha \in G$.

Next if $\alpha \neq \beta$, from $jt(\alpha,\beta)^\# j = u(\alpha^{-1}\beta)\, s(\alpha^{-1}\beta,\varepsilon)^\#$ and the condition (§) we have $t(\alpha,\beta) = 0$, i.e., $T^{\alpha\beta} = 0$. Consequently we have $T = \lambda 1$, this completes the proof. Q.E.D.

4. In this section we shall interprete an example of the factor due to von Neumann from the view-point of our just defined crossed product.

Let $\mathfrak{G}$ be an ergodic m-group in the measure space (S,μ) in the sense of von Neumann [2; p. 195, Definition 12.1.5] and moreover we assume that $\mu(\{s\}) = 0$ for all $s \in S$ and $\mu(S) = 1$. Let A be a multiplication algebra of the measure space (S,μ), then for $f(s) \in A$, the functional $tr_\mu(f) = (1f^\#, 1) = \int f(s)\, d\mu$ is a normal trace and 1 is a generating and separating vector for A, where $f^\#$ denotes the multiplication operator by the function $f(s)$. Define the automorphism of A as follows:

$$\text{For } \alpha \in \mathfrak{G}\ f \in A,\ f(s) \to f^\alpha(s) : f^\alpha(s) = f(s\alpha^{-1}).$$

Whence, by a slight modification such that $\overline{U}_\alpha$, $\overline{V}_\alpha$ in von Neumann's notation mean U_α^{-1}, V_α^{-1} respectively in this paper, we can see that $G = \{\alpha\}$ is ergodic in our sense by [2; p. 196, Lemma 12.2.4] and satisfies the condition (§) by [2; p. 197, Lemma 12.2.3], so that the w-crossed product $W^*(A,G,tr_\mu)$ is a factor, this is the case discussed in [2; p. 200, Lemma 12.3.4].

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