

# ON THE ABEL SUMMABILITY OF MULTIPLE FOURIER SERIES BY SPHERICAL MEANS

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**1. Introduction.** Let  $x = (x_1, x_2, \dots, x_k)$  be a point in the  $k$  dimensional Euclidian space and  $f(x) = f(x_1, x_2, \dots, x_k)$  be a function of the Lebesgue class  $L$  having the period  $2\pi$  in each variables; let

$$(1.1) \quad f(x) \sim \sum_{n_1, n_2, \dots, n_k} a_{n_1, n_2, \dots, n_k} e^{i(n_1 x_1 + n_2 x_2 + \dots + n_k x_k)}$$

be its Fourier series, that is

$$(1.2) \quad a_{n_1, n_2, \dots, n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} f(x_1, x_2, \dots, x_k) e^{-i(n_1 x_1 + \dots + n_k x_k)} dx_1 \dots dx_k.$$

We shall consider the spherical means of the series (1.1). This method was inaugurated by Prof. Bochner [1] and developed by other writers.

By the spherical sum of (1.1) we mean

$$(1.3) \quad S_n(x) = \sum_{\nu < n} a_{n_1, \dots, n_k} e^{i(n_1 x_1 + \dots + n_k x_k)} \quad n_1^2 + n_2^2 + \dots + n_k^2 = \nu$$

and write the spherical mean of the function  $f(x)$  at a point  $x$  by

$$(1.4) \quad f_x(t) = \frac{\Gamma(k/2)}{2(\pi)^{k/2}} \int_{\sigma} f(x_1 + t\xi_1, x_2 + t\xi_2, \dots, x_k + t\xi_k) d\sigma_{\xi},$$

where  $\sigma$  denotes the unite sphere  $\xi_1^2 + \xi_2^2 + \dots + \xi_k^2 = 1$ , and  $d\sigma_{\xi}$  its  $k-1$  dimensional volume element.

The general Abel mean of the series (1.3) is given by

$$(1.5)' \quad \frac{1}{\varepsilon} \int_0^{\infty} \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt.$$

If we put  $m = 0$ , (1.5)' becomes ordinary Abel mean, which is established by Bochner [1]. The formula (1.5)' is reduced to

$$(1.5) \quad \frac{1}{\varepsilon} \int_0^1 \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt + o(1), \quad \text{as } \varepsilon \rightarrow 0,$$

by the localization theorem, where  $m > -\frac{1}{k+1}$ , since

$$\left(\frac{1}{\varepsilon}\right) \int_1^{\infty} \frac{(t/\varepsilon)^{k-1}}{[(t/\varepsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt$$

$$\begin{aligned}
 &= \left| \left( \frac{1}{\varepsilon} \right) \varepsilon^{(1+m)(k+1)} \sum_{j=1}^{\infty} \int_j^{j+1} \frac{1}{[t^{2(1+m)} + \varepsilon^{2(1+m)}]^{(k+1)/2}} t^{k-1} f_x(t) dt \right| \\
 &\leq \varepsilon^{mk+m+1} \sum_{j=0}^{\infty} \int_j^{j+1} \frac{1}{t^{2(1+m)(k+1)}} |t^{k-1} f_x(t)| dt \\
 &\leq \varepsilon^{mk+m+1} \sum_{j=1}^{\infty} j^{(1+m)(k+1)} \cdot O(j^{k-1}) \cdot \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} |f(t_1, \dots, t_k)| dt_1 \dots dt_k \\
 &= \varepsilon^{mk+m+1} \cdot O\left( \sum_{j=1}^{\infty} \frac{1}{j^{m(k+m+1)}} \right) = o(1).
 \end{aligned}$$

The general Abel mean of the function (1.4) is given by

$$(1.6) \quad \varepsilon \int_0^1 \left( \frac{\varepsilon}{t} \right)^\alpha e^{-\left(\frac{\varepsilon}{t}\right)^{1+n}} \left( \frac{t}{\varepsilon} \right)^{k-1} f_x(t) \frac{dt}{t^2}$$

where  $n > -1$ .

The formulas (1.5) and (1.6) were given by N. Levinson [3] for the case of one variable.

The object of this paper is to establish the relations between (1.5) and (1.6).

**2. Tauberian lemmas.** The following lemmas are essentially due to N. Levinson [3]. and N. Wiener [7].

LEMMA 1. *Let*

$$(2.0) \quad \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon} \right)^k \int_0^1 N_1 \left( \frac{t}{\varepsilon} \right) f(t) dt = 0 \quad \text{boundedly, } (k \geq 1),$$

where

$$(2.1) \quad \int_0^1 |f(t)| dt < \infty, \quad f(t) = t^{k-1} f_x(t)$$

and

$$(2.2) \quad |N_1(t)| < A.$$

Let  $R(t)$  be a function such that

$$(2.3) \quad \int_0^\infty t^{k-1} |R(t)| dt < \infty$$

and

$$(2.4) \quad \int_0^\infty |R(t)| \frac{dt}{t} < \infty.$$

Then, if

$$(2.5) \quad N_2(t) = \int_0^\infty R(y)N_1\left(\frac{t}{y}\right)\frac{dy}{y}$$

then

$$(2.6) \quad \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon}\right)^k \int_0^1 N_2\left(\frac{t}{\epsilon}\right)f(t) dt = 0.$$

PROOF. Since the following integral is absolutely convergent, we have

$$(2.7) \quad \begin{aligned} &\left(\frac{1}{\epsilon}\right)^k \int_0^\infty R\left(\frac{y}{\epsilon}\right)\frac{dy}{y} \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \\ &= \left(\frac{1}{\epsilon}\right)^k \int_0^1 f(t) dt \int_0^\infty R\left(\frac{y}{\epsilon}\right)N_1\left(\frac{t}{y}\right)\frac{dy}{y} = \left(\frac{1}{\epsilon}\right)^k \int_0^1 N_2\left(\frac{t}{\epsilon}\right)f(t) dt. \end{aligned}$$

(2.0), (2.1) and (2.2) yields

$$\left| \left(\frac{1}{y}\right)^k \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right| < M,$$

and we get

$$\begin{aligned} &\left| \left(\frac{1}{\epsilon}\right)^k \int_0^\infty R\left(\frac{y}{\epsilon}\right)\frac{dy}{y} \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right| = \left| \left(\frac{1}{\epsilon}\right)^k \int_0^\infty y^{k-1}R\left(\frac{y}{\epsilon}\right)\frac{dy}{y^k} \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right| \\ &\leq \left| \left(\frac{1}{\epsilon}\right)^k \left\{ \int_0^\delta + \int_\delta^\infty \right\} R\left(\frac{y}{\epsilon}\right)\frac{dy}{y} \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right| \\ &\leq \left| \left(\frac{1}{\epsilon}\right)^k \int_0^\delta y^{k-1}R\left(\frac{y}{\epsilon}\right) dy \left\{ \frac{1}{y^k} \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right\} \right| + M \left| \left(\frac{1}{\epsilon}\right)^k \int_\delta^\infty R\left(\frac{y}{\epsilon}\right)y^{k-1} dy \right|. \end{aligned}$$

Let we take small  $\delta > 0$ , we have  $\left| \left(\frac{1}{y}\right)^k \int_0^1 N_1\left(\frac{t}{y}\right)f(t) dt \right| < \eta$  for any small

$\eta > 0$ , by (2.0). Therefore the last term is less than

$$\eta \left| \int_0^\infty R(u)u^{k-1} du \right| + M \int_{\delta/\epsilon}^\infty |u^{k-1}R(u)| du.$$

Therefore for sufficiently small  $\eta$  the first term is arbitrarily small independent of  $\epsilon$ , and for small  $\epsilon$  and fixed  $\delta$  the second term is arbitrarily small. Combined this fact with (2.7) we have

$$\lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon}\right)^k \int_0^1 N_2\left(\frac{t}{\epsilon}\right)f(t) dt = 0,$$

which proves the lemma.

LEMMA 2. *Lemma 1 remains valid if (2.4) is replaced by*

$$|R(t)| < A, \quad \text{for } t < \frac{1}{2},$$

and

$$\int_0^\infty |N_1(t)| \frac{dt}{t} < \infty.$$

PROOF. The proof is the same as that of N. Levinson [3].

LEMMA 3. *Let for some fixed  $b$ ,  $t^{-b}N_1(t)$  and  $t^{-b}N_2(t)$  belong to  $L(0, \infty)$ , and let*

$$k_1(w) = \int_0^\infty N_1(t)t^{-w}dt,$$

$$k_2(w) = \int_0^\infty N_2(t)t^{-w}dt,$$

and

$$\gamma(w) = \frac{k_2(w)}{k_1(w)}.$$

If  $\gamma(w)$  is analytic, and

$$\int_{-\infty}^\infty |\gamma(u + iv)|^2 dv < M < \infty$$

in the strip,  $b - \delta \leq u \leq b + \delta$ , for some fixed  $\delta > 0$ , then

$$(2.8) \quad R(t) = \text{l. i. m.}_{\epsilon \rightarrow \infty} \frac{1}{2\pi i} \int_{-1.1+\epsilon}^{1.1+\epsilon} \gamma(w)t^{w-1}dw, \quad (b - \delta \leq u \leq b + \delta),$$

is a solution of the equation (2.5) and

$$(2.9) \quad t^{-b}R(t)$$

belongs to the class  $L(0, \infty)$ .

PROOF. The proof is analogous to that of N. Levinson [3].

**3. Summability theorems.** Let  $P(m)$  represent

$$(3.1) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^1 \frac{(t/\epsilon)^{k-1}}{[(t/\epsilon)^{2(1+m)} + 1]^{(k+1)/2}} f_x(t) dt = 0, \quad m > -\frac{1}{k+1},$$

and  $E(n, \alpha)$  represent

$$(3.2) \quad \lim_{\epsilon \rightarrow 0} \epsilon \int_0^1 \left(\frac{\epsilon}{t}\right)^\alpha e^{-\left(\frac{\epsilon}{t}\right)^{1+n}} \left(\frac{t}{\epsilon}\right)^{k-1} f_x(t) \frac{dt}{t^2} = 0, \quad n > -1, \quad \alpha \geq k-1.$$

THEOREM 1.  $E(n, \alpha)$  for  $n > m$ , implies  $P(m)$ ; while  $P(m)$  for  $m > n$ ,

implies  $E(n, \alpha)$ .

PROOF. Let

$$N_1(t) = \frac{1}{[t^{2(1+m)} + 1]^{(k+1)/2}} \quad \text{and} \quad N_2(t) = t^{-\alpha-2} e^{-(\frac{1}{t})^{1+n}},$$

then

$$(3.3) \quad P(m) \quad \text{means} \quad \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon}\right)^k \int_0^1 N_1\left(\frac{t}{\epsilon}\right) t^{k-1} f_x(t) dt = 0,$$

and

$$(3.4) \quad E(n, \alpha) \quad \text{means} \quad \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon}\right)^k \int_0^1 N_2\left(\frac{t}{\epsilon}\right) t^{k-1} f_x(t) dt = 0.$$

In the sequel we shall use the notations of Lemmas 1,2 and 3. Then we have

$$(2.2) \quad |N_1(t)| < A \quad \text{and} \quad |N_2(t)| < A$$

for  $m > -\frac{1}{k+1}$ ,  $n > -1$  and  $t \in (0, \infty)$ . Also, we have

$$\int_0^\infty |N_2(t)| \frac{dt}{t} < \infty.$$

By the Mellin transforms, we have

$$(3.5) \quad k_1(w) = k_1(u + iv) = \frac{1}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)} \Gamma\left(\frac{1-w}{2(1+m)}\right) \Gamma\left(\frac{k+1}{2} - \frac{1-w}{2(1+m)}\right)$$

for  $1 > u > -m(k+1) - k$ , and

$$(3.6) \quad k_2(w) = k_2(u + iv) = \frac{1}{1+n} \Gamma\left(\frac{1+w+\alpha}{1+n}\right)$$

for  $u > -(\alpha+1)$ , (c. f. Titchmarsh [4] p.192), where  $\Gamma(x)$  denotes Gamma function.

First we prove that  $P(m)$  implies  $E(n, \alpha)$  if  $m > n$  and  $\alpha \geq k-1$ . We use the Lemma 1 and Lemma 3. Since  $t^{(k-1)}N_1(t)$  and  $t^{(k-1)}N_2(t)$  belong to the class  $L(0, \infty)$  under the hypothesis of the theorem, we have

$$(3.7) \quad \gamma(w) = \gamma(u + iv) = \frac{k_2(w)}{k_1(w)} = \frac{2(1+m)\Gamma\left(\frac{k+1}{2}\right)}{1+n} \cdot \frac{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)}{\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2} - \frac{1-w}{2(1+m)}\right)}.$$

The well known formula of  $\Gamma$ -function

$$(3.8) \quad |\Gamma(u + iv)| \sim \sqrt{2\pi} |v|^{u-\frac{1}{2}} \exp\left\{-\frac{\pi}{2}|v|\right\}, \quad \text{as } |v| \rightarrow \infty,$$

implies that

$$(3.9) \quad |\gamma(u + iv)| \sim C|v|^{\frac{1+u+\alpha}{1+n} - \frac{k}{2}} \exp\left\{-\frac{\pi}{2}\left[\left|\frac{v}{1+n}\right| - \left|\frac{v}{1+m}\right|\right]\right\}, \text{ as } |v| \rightarrow \infty,$$

in the strip  $-(k-1) - \delta \leq u \leq -(k-1) + \delta$  for some fixed  $\delta > 0$ , and some constant  $C$ . Since  $\gamma(u + iv)$  is analytic in that strip, we have

$$(3.10) \quad \int_{-\infty}^{\infty} |\gamma(u + iv)|^2 dv < M, \text{ in } -(k-1) - \delta \leq u \leq -(k-1) + \delta.$$

By Lemma 3, if we put  $b = -(k-1)$ , there exists a function  $R(t)$  such that (2.3) and (2.5) of Lemma 1 are satisfied. Since  $1/\Gamma(w)$  is an entire function, the regular property of the function  $\gamma(u + iv)$  depends only on the behavior of numerator of the function. Consider the same way as above in the strip  $-\delta \leq u \leq 1 + \delta$ , we have (2.9) in Lemma 3 for  $b = -1$ . (The details of this statement, see N. Levinson [3].) Thus we have (2.4).

Therefore all the conditions of Lemma 1 are satisfied, we have the theorem in this case.

Finally, we shall prove that  $E(n, \alpha)$  implies  $P(m)$  for  $n > m$  and  $\alpha \geq k-1$ . In Lemmas 1, 2 and 3, interchange  $N_1, N_2, k_1$  and  $k_2$  with  $N_2, N_1, k_2$  and  $k_1$  respectively. We have

$$(3.12) \quad \gamma(w) = \gamma(u + iv) = \frac{k_1(w)}{k_2(w)} = \frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)} \frac{\Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2} - \frac{1-w}{2(1+m)}\right)}{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)},$$

$$(3.13) \quad |\gamma(u + iv)| \sim C|v|^{\frac{k}{2} - \frac{1+u+\alpha}{1+n}} \exp\left\{-\frac{\pi}{2}\left[\left|\frac{v}{1+m}\right| - \left|\frac{v}{1+n}\right|\right]\right\}, \text{ as } |v| \rightarrow \infty,$$

in the strip  $-(k-1) - \delta \leq u \leq -(k-1) + \delta$  for some fixed  $\delta > 0$ , and some constant  $C$ .  $\gamma(w)$  is analytic in the strip  $1 > u > -m(k+1) - k$  and by (3.13), it belongs to  $L^2$  on any ordinate of this strip. Thus as before, the conditions of Lemma 1 are satisfied except for (2.4). Again by Lemma 3, if we put  $b = 0$ , we have an absolutely integrable

$$(3.14) \quad R(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \gamma(w)t^{w-1}dw.$$

The integrand of (3.14) has poles at  $w = 1$  and  $w = 2m + 3$ , but it has no pole in the strip  $1 < u < 2m + 3$ . We displace the path of integration to the right of  $w = 1$  and observe that  $w = 1$  is a pole, we have

$$(3.15) \quad R(t) = \frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1+1+\alpha}{1+n}\right)} + \frac{1+n}{2(1+m)\Gamma\left(\frac{k+1}{2}\right)} \frac{1}{2\pi i} \int_{-i\infty+2+2m}^{i\infty+2+2m} \Gamma\left(\frac{1-w}{2(1+m)}\right)\Gamma\left(\frac{k+1}{2} - \frac{1-w}{2(1+m)}\right) t^{w-1}dw,$$

$$\frac{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)}{\Gamma\left(\frac{1+w+\alpha}{1+n}\right)}$$

so that

$$(3.16) \quad |R(t)| \leq A + A't^{1+2m}$$

where  $A$  and  $A'$  are some positive constants. Since  $m > -\frac{1}{2}$ ,  $R(t)$  is bounded for finite  $t$  and the conditions of Lemma 2 are fulfilled. This proves the theorem of this case.

REMARK. The essential parts of our theorem are as follows;  $E(n, \alpha)$  for  $n > 0$ ,  $\alpha \geq k - 1$  implies  $P(0)$ , while  $P(0)$  implies  $E(n, \alpha)$  for  $0 > n > -1$  and  $\alpha \geq k - 1$ .

**4. Absolute summability theorems.** The Tauberian treatment of the absolute summability theorems was inaugurated by Prof. G. Sunouchi [4]. In his method, we get the following lemma.

LEMMA 4. *Under the hypothesis of Lemma 1 or Lemma 2*

$$(4.0) \quad \int_0^\infty \left| d_\epsilon \left( \frac{1}{\epsilon} \right) \int_0^1 N_1 \left( \frac{t}{\epsilon} \right) f(t) dt \right| < \infty$$

implies

$$(4.1) \quad \int_0^\infty d_\epsilon \left( \frac{1}{\epsilon} \right)^k \int_0^1 N_2 \left( \frac{t}{\epsilon} \right) f(t) dt < \infty.$$

PROOF. Put

$$(4.2) \quad S(x) = \int_0^x t^{k-1} R(t) dt,$$

$$(4.3) \quad F_1(y) = \left( \frac{1}{y} \right)^k \int_0^1 N_1 \left( \frac{t}{y} \right) f(t) dt,$$

and

$$(4.4.) \quad F_2(y) = \left( \frac{1}{y} \right)^k \int_0^1 N_2 \left( \frac{t}{y} \right) f(t) dt$$

then by (2.3),  $S(0)$  and  $S(\infty)$  exist. By (2.7), we get

$$(4.5) \quad \begin{aligned} F_2(\epsilon) &= \left( \frac{1}{\epsilon} \right)^k \int_0^\infty R \left( \frac{y}{\epsilon} \right) \frac{dy}{y} \int_0^1 N_1 \left( \frac{t}{y} \right) f(t) dt \\ &= \left( \frac{1}{\epsilon} \right)^k \int_0^\infty y^{k-1} R \left( \frac{y}{\epsilon} \right) \frac{dy}{y^k} \int_0^1 N_1 \left( \frac{t}{y} \right) f(t) dt \\ &= \left( \frac{1}{\epsilon} \right)^k \int_0^\infty y^{k-1} R \left( \frac{y}{\epsilon} \right) F_1(y) dy \end{aligned}$$

$$= \frac{1}{\varepsilon} \int_0^{\infty} \left(\frac{y}{\varepsilon}\right)^{k-1} R\left(\frac{y}{\varepsilon}\right) F_1(y) dy.$$

Integrating by parts, the last term is

$$(4.6) \quad \left[ S\left(\frac{y}{\varepsilon}\right) F_1(y) \right]_0^{\infty} - \int_0^{\infty} S\left(\frac{y}{\varepsilon}\right) dF_1(y).$$

Since  $S(0)$ ,  $S(\infty)$ ,  $F_1(0)$  and  $F_2(\infty)$  exist by (2.3) and (4.0), we have

$$(4.7) \quad \int_0^{\infty} \left| d_{\varepsilon} \int_0^{\infty} S\left(\frac{y}{\varepsilon}\right) dF_1(y) \right| = \int_0^{\infty} |dF_2(\varepsilon)|.$$

Then, by (4.2) and Cameron-Martin's unsymmetric Fubini theorem [2],

$$(4.8) \quad \begin{aligned} \int_0^{\infty} \left| d_{\varepsilon} \int_0^{\infty} S\left(\frac{y}{\varepsilon}\right) dF_1(y) \right| &\leq \int_0^{\infty} |dF_1(y)| \int_0^{\infty} \left| d_{\varepsilon} S\left(\frac{y}{\varepsilon}\right) \right| \\ &\leq \int_0^{\infty} |dF_1(y)| \int_0^{\infty} \left| \frac{y}{\varepsilon^2} \left(\frac{y}{\varepsilon}\right)^{k-1} R\left(\frac{y}{\varepsilon}\right) \right| d\varepsilon \\ &\leq \int_0^{\infty} |dF_1(y)| \int_0^{\infty} |u^{k-1} R(u)| du < \infty. \end{aligned}$$

This proves the Lemma.

If we denote by  $|P(m)|$  the fact that

$$(4.9) \quad \frac{1}{\varepsilon} \int_0^{\infty} \frac{\left(\frac{t}{\varepsilon}\right)^{k-1}}{\left[\left(\frac{t}{\varepsilon}\right)^{2(1+m)} + 1\right]^{\frac{k+1}{2}}} f_x(t) dt, \quad m \geq 0,$$

is of bounded variation in  $(0, \infty)$ , and by  $|E(n, \alpha)|$  the fact that

$$(4.10) \quad \varepsilon \int_0^1 \left(\frac{\varepsilon}{t}\right)^{\alpha} e^{-\left(\frac{\varepsilon}{t}\right)^{1+n}} f_x(t) \frac{dt}{t^2}, \quad n > -1, \alpha \geq k-1,$$

is of bounded variation in  $(0, \infty)$ . Then we have

**THEOREM 2.**  $|E(n, \alpha)|$  for  $n > m$  and  $\alpha \geq k-1$  implies  $|P(m)|$ , while  $|P(m)|$  implies  $|E(n, \alpha)|$  for  $m > n$ ,  $\alpha \geq k-1$ .

**PROOF.** To prove the localization property of absolute summability, we need to show that

$$(4.11) \quad \int_0^{\infty} \left| d_{\varepsilon} \frac{1}{\varepsilon} \int_1^{\infty} \frac{\left(\frac{t}{\varepsilon}\right)^{k-1}}{\left[\left(\frac{t}{\varepsilon}\right)^{2(1+m)} + 1\right]^{\frac{1}{2}(k+1)}} f_x(t) dt \right| < \infty, \quad m \geq 0.$$

Since

$$\begin{aligned}
 (4.12) \quad & \frac{d}{d\varepsilon} \left\{ \left( \frac{1}{\varepsilon} \right) \int_1^\infty \frac{\left( \frac{t}{\varepsilon} \right)^{k-1}}{\left[ \left( \frac{t}{\varepsilon} \right)^{2(1+m)} + 1 \right]^{\frac{1}{2}(k+1)}} f_\alpha(t) dt \right\} \\
 & \leq A \sum_{j=1}^\infty \frac{1}{\left[ \left( \frac{j}{\varepsilon} \right)^{2(1+m)} + 1 \right]^{\frac{1}{2}(k+1)}} \left[ \left( \frac{1}{\varepsilon} \right)^{k+1} + \left( \frac{1}{\varepsilon} \right)^{k+1} \frac{\left( \frac{j}{\varepsilon} \right)^{2(1+m)}}{\left( \frac{j}{\varepsilon} \right)^{(1+m)} + 1} \right] \\
 & \quad \cdot \int_j^{j+1} |t^{k-1} f_\alpha(t)| dt \\
 & = \begin{cases} O(\varepsilon^{m(k+1)}) & \text{as } \varepsilon \rightarrow 0, \\ O(\varepsilon^{-(k+1)}) & \text{as } \varepsilon \rightarrow \infty, \end{cases}
 \end{aligned}$$

where  $A$  is a constant, we have (4.11).

The existence of the solution  $R(y)$  of (2.5) is the same as that of Theorem 1. Using Lemma 4 instead of Lemmas 1 and 2, we can show Theorem 2.

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