ON THE PROJECTION OF NORM ONE IN W^*-ALGEBRAS, III

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This paper is a continuation of the author's preceding papers [8], [9], in which we discuss certain existence-problems of σ-weakly continuous projections of norm one of different types of W^*-algebras.

By a projection of norm one we mean a projection mapping from a Banach space onto its subspace whose norm is one. In the following we concern with the projection of norm one in a W^*-algebra $\mathbf{M}$. We denote by $\mathbf{M}_*$ the space of all σ-weakly continuous linear functionals on $\mathbf{M}$. On the other hand $\mathbf{M}^*$ means the conjugate space of $\mathbf{M}$ and the second conjugate space of $\mathbf{M}$ is written by $\mathbf{M}^{**}$ usually. However, in case $\mathbf{M}$ is a W^*-algebra $\mathbf{M}^{**}$ is the W^*-algebra that plays a special rôle for $\mathbf{M}$ (cf. [3], [7]) so that we denote especially by $\widetilde{\mathbf{M}}$. A positive linear functional φ on a W^*-algebra is called singular if there exists no non-zero positive σ-weakly continuous functional such as $\psi \leqq \varphi$. The closed subspace of $\mathbf{M}^*$ generated by all singular linear functionals is denoted by $\mathbf{M}_*^{\perp}$. Then we get $\mathbf{M}^* = \mathbf{M}_* \bigoplus \mathbf{M}_*^{\perp}$: the sum is l^1-direct sum. A uniformly continuous linear mapping π from a W^*-algebra $\mathbf{M}$ to another W^*-algebra $\mathbf{N}$ is called singular if ${}^t\pi(\mathbf{N}_*) \subset \mathbf{M}_*^{\perp}$ where ${}^t\pi$ means the transpose of π.

All other notations and definitions are referred to [7] and [8]. Before going to discussions, the author expresses his hearty thanks to Mr. M. Takesaki for his valuable suggestions and co-operations.

1. General decomposition theorem.

THEOREM 1. *Let $\mathbf{M}$, $\mathbf{N}$ be W^*-algebras, then any uniformly continuous linear mapping from $\mathbf{M}$ to $\mathbf{N}$ is uniquely decomposed into the σ-weakly continuous part and the singular part.*

PROOF. Let π be a uniformly coninuous linear mapping from $\mathbf{M}$ into $\mathbf{N}$, then ${}^t\pi$ is the mapping from $\mathbf{N}^*$ to $\mathbf{M}^*$. Consider the restriction of ${}^t\pi$ to $\mathbf{N}_*$. The transpose of this restriction is a σ-weakly continuous linear mapping $\widetilde{\pi}$ from $\widetilde{\mathbf{M}}$ to $\mathbf{N}$ and clearly $\widetilde{\pi}$ is a σ-weakly continuous extension of π to $\widetilde{\mathbf{M}}$. Denote by $\mathbf{M}_*^0$ the polar of $\mathbf{M}_*$ in $\widetilde{\mathbf{M}}$, then we get a central projection z in $\widetilde{\mathbf{M}}$ such as $\mathbf{M}_*^0 = \widetilde{\mathbf{M}}(1-z)$.

Put $\pi_1(a) = \widetilde{\pi}(az)$, $\pi_2(a) = \widetilde{\pi}(a(1-z))$ for each $a \in \mathbf{M}$. We have, clearly, $\pi = \pi_1 + \pi_2$. Moreover we get

$$< a,\ {}^t\pi_1(\varphi) > \;=\; < \tilde{\pi}(az),\ \varphi > \;=\; < az,\ {}^t\tilde{\pi}(\varphi) > \;=\; < a,\ R_z{}^t\pi(\varphi) >$$

for all $a \in \mathbf{M}$ and $\varphi \in \mathbf{N}_*$. Hence ${}^t\pi_1(\mathbf{N}_*) \subset \mathbf{M}_*$ i. e. π_1 is σ-weakly continuous. Similarly, we get $< a,\ {}^t\pi_2(\varphi) > \;=\; < a, R_{(1-z)}{}^t\pi(\varphi) >$ for all $a \in \mathbf{M}$ and $\varphi \in \mathbf{N}_*$ so that ${}^t\pi_2(\mathbf{N}_*) \subset \mathbf{M}_*^{\perp}$. On the other hand the unicity is clear. This completes the proof.

COROLLARY 1. 1. *Let* $\mathbf{M}$ *be a* W^*-algebra, $\mathbf{N}$ *a* W^**-representable* **-subalgebra of* $\mathbf{M}$ *and* π *a projection of norm one from* $\mathbf{M}$ *to* $\mathbf{N}$. *Then* π *is uniquely decomposed into* π_1, *and* π_2 *where* π_1 *is a positive normal* $\mathbf{N}$*-module homomorphism from* $\mathbf{M}$ *to* $\mathbf{N}$ *and* π_2 *a positive singular* $\mathbf{N}$*-module homomorphism from* $\mathbf{M}$ *to* $\mathbf{N}$.

PROOF. In this case, the σ-weakly continuous extension of π to $\widetilde{\mathbf{M}}$, $\tilde{\pi}$, is also a projection of norm one from $\widetilde{\mathbf{M}}$ to $\mathbf{N}$. Therefore the above decomposition shows that all postulates on π_1 and π_2 are satisfied.

THEOREM 2. *Let* π *be a positive linear mapping from a* W^**-algebra* $\mathbf{M}$ *to a* W^**-algebra* $\mathbf{N}$, *then* π *is singular if and only if there exists no non-zero positive normal linear mapping* π' *such as* $\pi'(a) \leqq \pi(a)$ *for every positive element* $a \in \mathbf{M}$.

PROOF. Suppose π is singular and π' a positive normal linear mapping from $\mathbf{M}$ to $\mathbf{N}$ such as $\pi'(a) \leqq \pi(a)$ for positive $a \in \mathbf{M}$. Take a positive normal linear functional φ of $\mathbf{N}$. We have, from the assumption, $0 \leqq {}^t\pi'(\varphi) \leqq {}^t\pi(\varphi)$. Since ${}^t\pi(\varphi)$ is a singular linear functional, ${}^t\pi'(\varphi) = 0$ so that we get ${}^t\pi'(\mathbf{N}^*) = 0$. Therefore $\pi' = 0$.

Now suppose π has the property stated above. By Theorem 1, we have $\pi = \pi_1 + \pi_2$ where π_1 is σ- weakly continuous and π_2 singular. Moreover π_1 and π_2 are positive in this case. Therefore $\pi_1(a) \leqq \pi(a)$ for positive $a \in \mathbf{M}$ which imples $\pi_1 = 0$. Hence $\pi = \pi_2$ i. e. π is singular.

2. Existence-problems of the σ-weaky continuous proiection of norm one on different types of W^*-algebras. If $\mathbf{N}$ is a semi-finite W^*-algebra and $\mathbf{M}$ a purely infinite W^*-algebra, then thier direct product $\mathbf{N} \otimes \mathbf{M}$ is purely infinite by [4] and there exists a σ-weakly continuous projection of norm one from $\mathbf{N} \otimes \mathbf{M}$ to $\mathbf{N}$. Similarly, we can see that there may also exist a σ-weakly continuous projection of norm one from a W^*-algebra of type **II** to its W^*-subalgebra of type **I**. Now in the following we study on the converse existence-problems of these facts. The next theorem is essentially due to Sakai [4].

THEOREM 3. *If there exists a projection of norm one* π *from a semi-finite* W^**algebra* $\mathbf{M}$ *to its purely infinite* W^**-subalgebra* $\mathbf{N}$, *then* π *is always* *singular.*

To prove this theorem we need the following

LEMMA 3. 1. *Let π be a projection of norm one from a W^*-algebra $\mathbf{M}$ to its W^*-subalgebra $\mathbf{N}$ and $\pi = \pi_1 + \pi_2$ the decomposition in Corollary 1. 1., then π_1 is strongly continuous on the unit sphere of $\mathbf{M}$.*

PROOF. By the relation $\pi_1((a - \pi_1(a))^*(a - \pi_1(a)))$ for each $a \in \mathbf{M}$, one can get $\pi_1(a^*a) \geqq \pi_1(a)^*\pi_1(a)\,(2I - \tilde{\pi}(z))$ where $\tilde{\pi}$ is the σ-weakly continuous extension of π to $\tilde{\mathbf{M}}$ and z a cental projection of $\tilde{\mathbf{M}}$ in the proof of Theorem 1. By [8], we see easily, $\tilde{\pi}(z) \in \mathbf{N}^{\natural}$. Hence $\pi_1(a)^*\,\pi_1(a)\,(2I - \tilde{\pi}(z)) \geqq \pi_1(a)^*\,\pi_1(a)$. Therefore π_1 is strongly continuous on the unit sphere of $\mathbf{M}$(cf. [2: Chap. 1 §4]).

PROOF OF THEOREM 3. Let π_1 be the σ-weakly continuous part of π. By the above lemma and the property of π_1, we can proceed the same argument as in [4: Proof of Theorem 2] concerning π_1 and get $\pi_1 = 0$.

PROPOSITION. *Let $\mathbf{M}$ be a semi-finite W^*-algebra and $\mathbf{N}$ its finite W^*-subalgebra. We denote by $\mathfrak{M}$ for the definition ideal of a faithful normal semi-finite trace τ_0 on $\mathbf{M}$ (cf. [2]). If there exists a non-trivial σ-weakly continuous $\mathbf{N}$-module homomorphism π from $\mathbf{M}$ to $\mathbf{N}$, we have $\mathbf{N}' \cap \mathfrak{M} \neq \{0\}$.*

PROOF. Without loss of generality we may assume that $\mathbf{N}$ is countably decomposable. Take a faithful normal finite trace τ on $\mathbf{N}$. We have $< xy, {}^t\pi(\tau) > = < \pi(xy), \tau > = < \pi(x)y, \tau > = < y\pi(x), \tau > = < \pi(yx), \tau > = < yx, {}^t\pi(\tau) >$ for every $x \in \mathbf{M}$ and $y \in \mathbf{N}$.

Now $< x, {}^t\pi(\tau) > = < x\cdot a, \tau_0 >$ for all $x \in \mathbf{M}$ where a is an operator belonging to $L^1(\mathbf{M}, \tau_0)$ and $x\cdot a$ the strong product defined in [5] and $< x\cdot a, \tau_0 >$ means the extended value of τ_0 (cf. also [5]). We get $< xy\cdot a, \tau_0 > = < xy, {}^t\pi(\tau) > = < y\cdot x\cdot a, \tau_0 > = < x\cdot a\cdot y, \tau_0 >$ for every $x \in \mathbf{M}$ and $y \in \mathbf{N}$, which implies $a \eta \mathbf{N}'$. Here we may assume, without loss of generality, that a is self-adjoint. By [5: Corollary 12. 6] we can find a non-zero spectral projection e of a such as $e \in \mathfrak{M}$. Since it is clear that $e \in \mathbf{N}'$, we have $\mathbf{N}' \cap \mathfrak{M} \neq \{0\}$.

THEOREM 4. *If there exists a projection of norm one from a W^*-algebra $\mathbf{M}$ of type $\mathbf{I}$ to its W^*-subalgebra $\mathbf{N}$ of type $\mathbf{II}$, then π is always singular.*

PROOF. It is sufficient to prove this theorem in case that $\mathbf{N}$ is of type $\mathbf{II}_1$, because if there exists a projection of norm one which is not singular from $\mathbf{M}$ to its W^*-subalgebra of type $\mathbf{II}_\infty$ we see, by [4 : Proposition 3], that there also exists a projection of norm one which is not singular from $\mathbf{M}$ to its certain W^*-subalgebra of type $\mathbf{II}_1$.

If $\pi_1 \neq 0$ in Corollary 1. 1., we can choose a non-zero projection $e \in \mathfrak{M} \cap \mathbf{N}'$ by the above proposition ($\mathfrak{M}$ being taken as in the preceding proposition) and e is easily seen to be a finite projection of $\mathbf{M}$. We have $e\mathbf{M}e \supset e\mathbf{N}$.

Now $e\mathbf{N}$ is isomorphic to a W^*-algebra $z(e)\mathbf{N}$ which is of type $\mathbf{II}_1$ where $z(e)$ denotes the central envelope of e in $\mathbf{N}'$. On the other hand $e\mathbf{M}e$ is a finite W^*-algebra of type $\mathbf{I}$. Therefore $e\mathbf{M}e \supset e\mathbf{N}$ is impossible, which is a contradiction. Hence $\pi_1 = 0$ i. e. π is singular.

REMARK. As Theorem 3 concerns with the direct product of a semi-finite W^*-algebra and a purely infinite W^*-algebra (cf. [4]) this theorem is also closedly related to the direct product of a W^*-algebra of type $\mathbf{I}$ and that of type $\mathbf{II}$.

LEMMA 5. 1. *If φ is a positive singular linear functional on a W^*-subalgebra $\mathbf{N}$ of a W^*-algebra $\mathbf{M}$, any positive extension of φ to $\mathbf{M}$ is singular.*

PROOF. Denote by ψ the positive extension of φ to $\mathbf{M}$ and let $\psi = \psi_1 + \psi_2$ be the decomposition into its normal and singular parts. Since ψ_1 is normal on $\mathbf{M}$, it is also normal on $\mathbf{N}$. But, as $\varphi = \psi \geqq \psi_1$ on $\mathbf{N}$, we get $\psi_1 = 0$ on $\mathbf{N}$. Therefore $\psi_1 = 0$ on $\mathbf{M}$, whence $\psi = \psi_2$ is singular.

LEMMA 5. 2. *Let $\mathbf{M}$ be a W^*-algebra, $\mathbf{N}$ a W^*-subalgebra and π a projection of norm one from $\mathbf{M}$ to $\mathbf{N}$, then ${}^t\pi(\mathbf{N}_*^\perp) \subset \mathbf{M}_*^\perp$.*

PROOF. Take a singular positive linear functional φ of $\mathbf{N}$. Clearly ${}^t\pi(\varphi)$ is positive and ${}^t\pi(\varphi) = \varphi$ on $\mathbf{N}$. Hence ${}^t\pi(\varphi)$ is singular by the above lemma. Since $\mathbf{N}_*^\perp$ is generated by its positive elements,we get ${}^t\pi(\mathbf{N}_*^\perp) \subset \mathbf{M}_*^\perp$.

By help of these lemmas we can generalize the result of M. Takesaki (cf. [6]).

THEOREM 5. *Let $\mathbf{M}$ be a W^*-factor of type $\mathbf{I}$ and $\mathbf{N}$ its W^*-subalgebra. In order that there exists a σ-weakly continuous projection of norm one from $\mathbf{M}$ to $\mathbf{N}$ it is necessary and sufficient that $\mathbf{N}$ is the second dual of a two-sided ideal.*

PROOF. Without loss of generality, we may assume that $\mathbf{M}$ is the full operator algebra on some Hilbert space $\mathbf{H}$. Suppose π is a σ-weakly continuous projection of norm one from $\mathbf{M}$ to $\mathbf{N}$. $\mathbf{M} = C^{**}$ where C is the ideal of all completely continuous operators on $\mathbf{H}$. By [1] one easily verifies $C^0 = \mathbf{M}_*^\perp$ where C^0 means the polar of C in $\mathbf{M}^*$. Consider $\pi(C)$, the image of

C by π, then $\pi(C)$ is a σ-weakly dense ideal of $\mathbf{N}$ by [8].

Take $\varphi \in \pi(C)^*$ and denote by ψ the extension of φ to $\mathbf{N}$. We can say that the σ-weakly continuous part of ψ is independent from ψ. In fact, if ψ and ψ' are extensions of φ and $\psi = \psi_1 + \psi_2$, $\psi = \psi'_1 + \psi'_2$ are decompositions into their σ-weakly continuous parts and singular parts, we have, by Lemma 5. 2., ${}^t\pi(\psi_2)$, ${}^t\pi(\psi'_2) \in C^0$. Hence $< \pi(a), \varphi > = < \pi(a), \psi > = < a, {}^t\pi(\psi) > = < a, {}^t\pi(\psi_1) + {}^t\pi(\psi_2) > = < a, {}^t\pi(\psi_1) > = <\pi(a), \psi_1 > = < \pi(a), \psi_1 >$ for each $a \in C$. Therefore $\psi_1 = \varphi = \psi'_1$ on $\pi(C)$ which implies $\psi_1 = \psi'_1$.

Now, correspond φ to the unique σ-weakly continuous part ψ_1 of the extension ψ of φ. One easily verifies that this mapping is linear and one-to-one from $\pi(C)^*$ onto $\mathbf{N}_*$. On the other hand the unit sphere of C is σ-weakly dense in that of $\mathbf{M}$ as $\mathbf{M}$ is the second dual of C, so that the unit sphere of $\pi(C)$ is σ-weakly dense in that of $\mathbf{N}$. Therefore the above correspondence is isometric. This completes the necessity of the theorem.

Next, let $\mathbf{N} = C_1^{**}$ for an ideal C_1 of $\mathbf{N}$. For any central projection z of $\mathbf{N}$, zC_1 is a subspace of C_1. Hence $\mathbf{N}z = (C_1 z)^{**}$, which implies $\mathbf{N}z$ must contain a direct summand of a factor of type **I**. Therefore $\mathbf{N} = \Sigma_\alpha \mathbf{N}_\alpha$, central direct sum of those factors $\{\mathbf{N}_\alpha\}$ of type **I**.

Now one verifies easily that there exists a σ-weakly continuous projection π_α of norm one from $\mathbf{M}$ to each $\mathbf{N}_\alpha$. Put $\pi(a) = \Sigma_\alpha \pi_\alpha(a)$ for each $a \in \mathbf{M}$; π is clearly a σ-weakly continuous projection of norm one from $\mathbf{M}$ to $\mathbf{N}$. Thus the proof is completed.

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