CROSSED PRODUCTS OF RINGS OF OPERATORS

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Introduction. The purpose of the present paper is to introduce the notion of the crossed product to certain operator rings on a Hilbert space, the so-called factors.[1] This notion played the important rôle in the theory of the classical algebra may be brought in the modern theory of the operator rings. It seems that it will play as well the affirmative and active role for operator rings. Indeed, our notion is already found in the construction of factors due to Murray and von Neumann [4] [6] for a certain maximal abelian algebra and a group of its automorphisms. In view of this point, the extension of factors in this manner will be apparently expected to get factors of different algebraical types from the original one by varying the groups of automorphisms. On the other hand, it invites the algebraic decomposition of factors by its subfactors. Although it has, of course, the innate meaning as in the classical algebra, we have begun this study with the possibility as above. Therefore, the present paper is the first step in our program, and we shall only give a way for the extension of operator rings.

We first define the crossed product of a finite factor with the invariant $=1$ by a group of its automorphisms,[2] and show some basic properties of it. Then, all elements in the crossed product are determined uniquely by the original factor and a group of automorphisms. The question naturally arises whether the crossed product is also a factor or not, for a given group of automorphisms. We shall give the negative answer for this.

Among all factors, our main object to study is those of the finite continuous case i. e. $(\mathbf{II}_1)$ case. What we first ought to do is to see when the crossed products are factors. Indeed, let $\mathbf{M}$ be a $\mathbf{II}_1$-factor and let G a group of outer automorphisms (i. e. a group of automorphisms in which all but the unit are outer.), then the crossed product of $\mathbf{M}$ by G is shown to be a factor of type $\mathbf{II}_1$. At the final section, we shall find a necessary and sufficient condition that a W^*-algebra be the crossed product of the subfactor.

1. The notion of the crossed product. We shall begin with the unitary

1) A W^*-algebra means a weakly closed self-ajoint operator algebra with the identity on a Hilbert space, and in particular, a factor fostered by Murray and von Neumann means a W^*-algebra whose center consists of scalar multiples of the identity. cf. [1] [4].

2) By an automorphism of a factor, we always understand a $*$-automorphism.

representation of a group of automorphisms. The elements of a group of automorphisms are denoted by $\alpha, \beta, \ldots, \sigma, \tau, \ldots$ and its unit by e, and the image of an element a in a factor by its automorphim α is expressed by a^{α}. Let $\mathbf{M}$ be a finite factor with the invariant $C = 1$ on a Hilbert space $\mathbf{H}$ and let G a group of its automorphisms. Then G is represented to a unitary group on $\mathbf{H}$. That is:

LEMMA 1. *The group G of automorphisms admits a faithful unitary representation $\sigma \in G \rightarrow u_{\alpha}$ on $\mathbf{H}$ such that $u_{\sigma}^{*} a u_{\sigma} = a^{\sigma}$ for all $a \in \mathbf{M}$.*

PROOF. Let φ be a separating and generating trace vector of $\mathbf{M}$, i. e. $< a^{*} a \varphi, \varphi > = < a a^{*} \varphi, \varphi >$ and $[\mathbf{M} \varphi] = [\mathbf{M} \varphi] = \mathbf{H}$. Define the operator u_{σ} as follows:

$$u_{\sigma} a \varphi = a^{\sigma^{-1}} \varphi \qquad \text{for all } a \in \mathbf{M}.$$

Then, u_{σ} is uniquely extended to a bounded operator on $\mathbf{H}$ and $\sigma \rightarrow u_{\sigma}$ is the unitary representation of G as desired. In fact, since the trace $< (,) \varphi, \varphi >$ is invariant by G, i. e. $< a^{\sigma} \varphi, \varphi > = < a \varphi, \varphi >$ $(a \in \mathbf{M}, \sigma \in G)$,

$$\|u_{\sigma} a \varphi\|^{2} = \|a^{\sigma^{-1}} \varphi\|^{2} = < (a^{*} a)^{\sigma^{-1}} \varphi, \varphi > = < a^{*} a \varphi, \varphi > = \|a \varphi\|^{2}.$$

Thus, u_{σ} is unitary, and obviously $u_{\sigma\tau} = u_{\sigma} u_{\tau}$. Further, since φ is separating for $\mathbf{M}$, this correspondence is one-to-one. Finally, $u_{\sigma}^{*} a u_{\sigma} b \varphi = a^{\sigma} b \varphi$ for all $a, b \in \mathbf{M}$ yield $u_{\sigma}^{*} a u_{\sigma} \psi = a^{\tau} \psi$ for all vectors $\psi \in \mathbf{H}$.

Henceforward, for the sake of convenience, a unitary representation of a group of automorphisms on $\mathbf{H}$ means any unitary representation which satisfies the property in Lemma 1.

Next, we shall consider a unitary representation of G on the direct product $\mathfrak{H} = \mathbf{H} \otimes l_{2}(G)$ of $\mathbf{H}$ and $l_{2}(G)$. Denoting by $\{\varepsilon_{\alpha}\}_{\alpha \in G}$ a complete orthonormal system of $l_{2}(G)$, each vector of $\mathbf{H} \otimes l_{2}(G)$ is expressed in the form

$$\sum_{\alpha \in G} \varphi_{\alpha} \otimes \varepsilon_{\alpha}$$

where φ_{α} are vectors of $\mathbf{H}$ such that $\sum_{\alpha \in G} \|\varphi_{\alpha}\|^{2}$ is finite. The operator $a \otimes I$ $(a \in \mathbf{M})$ means the operator on $\mathfrak{H}$ defined by $(a \otimes I) \left(\sum_{\alpha \in G} \varphi_{\alpha} \otimes \varepsilon_{\alpha} \right) = \sum_{\alpha \in G} a \varphi_{\alpha} \otimes \varepsilon_{\alpha}$. Then, $a \rightarrow a \otimes I$ is a $*$-isomorphism of $\mathbf{M}$ into the full operator ring on $\mathfrak{H}$ and the set of operators $a \otimes I$ is a W^{*}-algebra on $\mathfrak{H}$, denoted by $\mathbf{M} \otimes \mathbf{I}$. Each $\sigma \in G$ induces an automorphism of $\mathbf{M} \otimes \mathbf{I}$ by $a \otimes I \rightarrow a^{\sigma} \otimes I$ $(a \in \mathbf{M})$ and so G induces a group of automorphisms $\mathbf{M} \otimes \mathbf{I}$, which is denoted by the same notation G, without confusions.

LEMMA 2. *The group G of automorphisms admits a faithful unitary representation $\sigma \rightarrow U_{\sigma}$ on $\mathbf{H} \otimes l_{2}(G)$ such that*

(1) *$U_{\sigma}^{*} A U_{\sigma} = A^{\sigma}$ for each $A \in \mathbf{M} \otimes \mathbf{I}$.*

(2) $\{(\mathbf{M}\otimes\mathbf{I})U_\sigma(\varphi\otimes\varepsilon_\alpha)\}_{\sigma\in G}$ *are mutually orthogonal for each vector* $\varphi\otimes\varepsilon_\alpha$ *(a fixed* $\alpha\in G$*).*

PROOF. Using the representation in Lemma 1, we define the operators U_σ on $\mathbf{H}\otimes l_2(G)$ for each $\sigma\in G$ as follows:

$$U_\sigma\left(\sum_\alpha\varphi_\alpha\otimes\varepsilon_\alpha\right)=\sum_\alpha u_\sigma\varphi_\alpha\otimes\varepsilon_{\sigma\alpha}.$$

Then, it is immediately verified that $\sigma\to U_\sigma$ is a faithful unitary representation of G on $\mathbf{H}\otimes l_2(G)$, satisfying the property (1). It is left only to prove the property (2). If $\sigma\neq\tau$, for $a,b\in\mathbf{M}$,

$$<(a\otimes I)U_\sigma(\varphi\otimes\varepsilon_\alpha),\ (b\otimes I)U_\tau(\varphi\otimes\varepsilon_\alpha)>\ =\ <au_\sigma\varphi\otimes\varepsilon_{\sigma\alpha},\ bu_\tau\varphi\otimes\varepsilon_{\tau\alpha}>$$
$$=\ <au_\sigma\varphi, bu_\tau\varphi><\varepsilon_{\sigma\alpha},\ \varepsilon_{\tau\alpha}>\ =0.$$

Hence the property (2) holds.

REMARK 1. It shoud be noted that each U_σ is determined by the matrix $(u_{\alpha,\beta})_{\alpha\ \beta\in G}$ where

$$u_{\alpha,\beta}=\begin{cases}u_\sigma & \text{if } \alpha\beta^{-1}=\sigma\\ 0 & \text{if } \alpha\beta^{-1}\neq\sigma\end{cases}.$$

In fact, denoting by J_α the linear isometry $\varphi\to\varphi\otimes\varepsilon_\alpha$ of $\mathbf{H}$ onto the subspace $\mathfrak{H}_\alpha$ in $\mathbf{H}\otimes l_2(G)$ and setting $J_\alpha^*=J_\alpha^{-1}$ on $\mathfrak{H}_\alpha$, $=0$ on $\mathfrak{H}_\alpha^\perp$,

$$J_\alpha^*U_\sigma J_\beta\varphi=J_\alpha^*U_\sigma(\varphi\otimes\varepsilon_\beta)=J_\alpha^*(u_\sigma\varphi\otimes\varepsilon_{\sigma\beta})$$
$$=\begin{cases}u_\sigma\varphi & \text{if } \alpha\beta^{-1}=\sigma\\ 0 & \text{if } \alpha\beta^{-1}\neq\sigma\end{cases}$$

for each vector $\varphi\in\mathbf{H}$.

At present, we shall define the crossed product of a factor by a group of its automorphisms. The concept of the crossed product we are going to give concerns with finite factors with the invariant $C=1$. Let $\mathbf{M}$ be a finite factor with $C=1$ on a Hilbert space $\mathbf{H}$ and let G a group of its automorphisms. Passing the unitary repersentation of G on $\mathbf{H}\otimes l_2(G)$ in Lemma 2, obtained from a unitary representation $\mathfrak{U}$ of G on $\mathbf{H}$, with the same notation, we consider a system $\mathfrak{S}$ of all linear froms

$$\sum_{\alpha\in G}A_\alpha U_\alpha$$

where A_α are elements of $\mathbf{M}\otimes\mathbf{I}$ and all but a finite number of them are zero. Then, Since $(AU_\alpha)^*=U_\alpha^*A^*=A^{*\alpha}U_{\alpha^{-1}}$ and $(AU_\alpha)(BU_\alpha)=AB^{\alpha^{-1}}U_{\alpha\beta}(A, B\in\mathbf{M}\otimes\mathbf{I})$ the system $\mathfrak{S}$ is a $*$-algebra. Now we shall give the definition of the crossed product.

DEFINITION. The W^*-algebra on $\mathbf{H}\otimes l_2(G)$ generated by the system $\mathfrak{S}$ is said to be *the crossed product of* $\mathbf{M}$ *by the group* G *of automorphisms* and denoted by $(\mathbf{M}, G, \mathfrak{U})$.

The crossed product defined above seems to depend on the choice of the representation $\mathfrak{U}$ of G on $\mathbf{H}$, but it will be shown that the crossed product is uniquely determined $\mathbf{M}$ and G within unitary equivalence. The original form of our notion introduced on operator rings is found in the so-called factor construction due to Murray and von Neumann [4] [6], where this notion concerns with a measure space (Ω, ν) and a group of homeomorphisms on Ω, called an ergodic m-group. Recently, this fact has been explained largely by T. Turumaru [8].

2. The general properties of the crossed product. In this section, we discuss the general properties of the crossed product defined in the preceding section. We shall show that all elements in the crossed product $(\mathbf{M}, G, \mathfrak{U})$ are uniquely determined by a family of elements in $\mathbf{M}\otimes\mathbf{I}$ and $\{U_\alpha\}_{\alpha\in G}$.

LEMMA 3. *For each element A in the crossed product* $(\mathbf{M}, G, \mathfrak{U})$, *there exists a unique family* $\{A_\alpha\}_{\alpha\in G}$ *in* $\mathbf{M}\otimes\mathbf{I}$ *such that*

$$A(\varphi\otimes\varepsilon_e)=\sum\nolimits_{\alpha\in G}A_\alpha U_\alpha(\varphi\otimes\varepsilon_e)$$

for all vectors $\varphi\otimes\varepsilon_e(\varphi\in\mathbf{H})$.

PROOF. Let A be an element in the unit sphere of $(\mathbf{M}, G, \mathfrak{U})$, by Kaplansky's density theorem [3], there exists a directed family $A_\lambda\in\mathfrak{S}$ in the unit sphere which converges strongly to A. Put $A_\lambda=\sum_{\alpha\in G}(a_\alpha^{(\lambda)}\otimes I)\ U_\alpha$ (all but a finite number of $a_\alpha^{(\lambda)}$ in $\mathbf{M}$ are zero),

$$\begin{aligned}\|\varphi\|^2 &\geqq \Big\|\sum\nolimits_\alpha(a_\alpha^{(\lambda)}\otimes I)\,U_\alpha(\varphi\otimes\varepsilon_e)\Big\|^2=\Big\|\sum\nolimits_\alpha a_\alpha^{(\lambda)}u_\alpha\varphi\otimes\varepsilon_\alpha\Big\|^2\\ &=\sum\nolimits_\alpha\|a_\alpha^{(\lambda)}u_\alpha\varphi\otimes\varepsilon_\alpha\|^2=\sum\nolimits_\alpha\|a_\alpha^{(\lambda)}u_\alpha\varphi\|^2\end{aligned}$$

for all $\varphi\in\mathbf{H}$. Thus $\|a_\alpha^{(\lambda)}u_\alpha\varphi\|\quad\|\varphi\|$ and so all elements $a_\alpha^{(\lambda)}$ belong to the unit sphere of $\mathbf{M}$.

Now we show that a directed family $a_\alpha^{(\lambda)}$ for each $\alpha\in G$ is cauchy in the strong topology on $\mathbf{M}$. Indeed, by the property (2) in Lemma 2 we have

$$\begin{aligned}\Big\|\sum\nolimits_\alpha(a_\alpha^{(\lambda)}\otimes I-a_\alpha^{(\mu)}\otimes I)\,U_\alpha(\varphi\otimes\varepsilon_e)\Big\|^2&=\Big\|\sum\nolimits_\alpha(a_\alpha^{(\lambda)}-a_\alpha^{(\mu)})u_\alpha\varphi\otimes\varepsilon_\alpha\Big\|^2\\ =\sum\nolimits_\alpha\|(a_\alpha^{(\lambda)}-a_\alpha^{(\mu)})u_\alpha\varphi\otimes\varepsilon_\alpha\|^2&=\sum\nolimits_\alpha\|(a_\alpha^{(\lambda)}-a_\alpha^{(\mu)})u_\alpha\varphi\|^2.\end{aligned}$$

Since the left side converges to 0, each $\|(a_\alpha^{(\lambda)}-a_\alpha^{(\mu)})u_\alpha\varphi\|\to 0$, or $\|(a_\alpha^{(\lambda)}-a_\alpha^{(\mu)})\varphi\|\to 0$ for all $\varphi\in\mathbf{H}$. Observing that the unit sphere of $\mathbf{M}$ is complete in the strong topology, each directed famly $a_\alpha^{(\lambda)}$ converges strongly to a_α in $\mathbf{M}$, therefore each $a_\alpha^{(\lambda)} \otimes I$ converges strongly to $a \otimes I$, because the isomorphism $\mathbf{M} \to \mathbf{M} \otimes \mathbf{I}$ is strongly continuous in the unit sphere.

Setting $A_\alpha = a_\alpha \otimes I$ and $A_\alpha^{(\lambda)} = a_\alpha^{(\lambda)} \otimes I$, it must be shown that $\sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e)$ converges and $A(\varphi \otimes \varepsilon_e) = \sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e)$. For each $\varepsilon > 0$, there exists λ_0 such that

$$\left\| \left(A - \sum_{\alpha \in G} A_\alpha^{(\lambda)} U_\alpha\right)(\varphi \otimes \varepsilon_e) \right\| < \varepsilon/3 \quad \text{for } \lambda \geqq \lambda_0 \qquad (1),$$

and so

$$\left\| \left(\sum_{\alpha \in G} A_\alpha^{(\lambda)} U_\alpha - \sum_{\alpha \in G} A_\alpha^{(\lambda_0)} U_\alpha\right)(\varphi \otimes \varepsilon_e) \right\| < 2\,\varepsilon/3.$$

Put $J_0 =$ the finite set $\{\alpha \in G;\ A_\alpha^{(\lambda_0)} \neq 0\}$, then, by the property (2) in Lemma 2,

$$\left\| \left(\sum_{\alpha \in J} A_\alpha^{(\lambda)} U_\alpha - \sum_{\alpha \in J} A_\alpha^{(\lambda_0)} U_\alpha\right)(\varphi \otimes \varepsilon_e)) \right\| < 2\,\varepsilon/3$$

for all finite sets $J \supset J_0$ in G. Hence

$$\left\| \left(\sum_{\alpha \in J} A_\alpha^{(\lambda_0)} U_\alpha - \sum_{\alpha \in J} A_\alpha U_\alpha\right)(\varphi \otimes \varepsilon_e) \right\| \leqq 2\varepsilon/3 \qquad (2)$$

Combining (1) and (2), we conclude that for all finite sets $J \supset J_0$ in G,

$$\left\| \left(A - \sum_{\alpha \in J} A_\alpha U_\alpha\right)(\varphi \otimes \varepsilon_e) \right\| < \varepsilon.$$

Finally, it left only to prove that such expression is unique. In fact, if $\sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e) = 0$, $0 = \left\| \sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e) \right\|^2 = \sum_{\alpha \in G} \| a_\alpha u_\alpha \varphi \|^2$ for all $\varphi \in \mathbf{H}$. Thus $a_\alpha u_\alpha = 0$ for each $\alpha \in G$ and so $a_\alpha = 0$, or $A_\alpha = 0$. Therefore, our statement holds for any element in the crossed product $(\mathbf{M}, G, \mathfrak{U})$.

REMARK 2. As easily seen, for $A \in (\mathbf{M}, G, \mathfrak{U})$, the family $\{A_\alpha\}_{\alpha \in G}$ in Lemma 3 is uniquely determind as follows:

$$A(\varphi \otimes \varepsilon_\sigma) = \sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_\sigma)$$

for all $\sigma \in \mathbf{G}$, $\varphi \in \mathbf{H}$. Hence $A = 0$ if and only if $A_\alpha = 0$ for all $\alpha \in G$.

LEMMA 4. *The crossed product* $(\mathbf{M}, G, \mathfrak{U})$ *has a separating and generating vector.*

PROOF. Since the invariant of $\mathbf{M}$ equals to one, it is known that there exists a separating and generating vector φ of $\mathbf{M}$. Then each $u_\alpha \varphi$ is also separating and generating for $\mathbf{M}$. In fact, if $a u_\alpha \varphi = 0\,(a \in \mathbf{M})$, $0 = \langle a u_\alpha \varphi, a u_\alpha \varphi \rangle = \langle u_\alpha^* a^* a u_\alpha \varphi, \varphi \rangle = \langle (a^* a)^\alpha \varphi, \varphi \rangle = \| a_\alpha \varphi \|^2$ and hence $a_\alpha \varphi = 0$, $a_\alpha = 0$ since φ is separting for $\mathbf{M}$, and so $a = 0$. On the other hand, $u_\alpha^*[\mathbf{M} u_\alpha \varphi] = [\mathbf{M}\varphi] = \mathbf{H}$, thus $\mathbf{H} = u_\alpha \mathbf{H} = [\mathbf{M} u_\alpha \varphi]$. Now, $\varphi \otimes \varepsilon_e$ is acceptable as a separating and generating vector of $(\mathbf{M}, G, \mathcal{U})$. Indeed, using Lemma 3, $A(\varphi \otimes \varepsilon_e) = \sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e) = \sum_{\alpha \in G} a_\alpha u_\alpha \varphi \otimes \varepsilon_\alpha = 0 (A \in (\mathbf{M}, G, \mathcal{U}))$yields $a_\alpha u_\alpha \varphi = 0$ for all $\alpha \in G$. But, since $u_\alpha \varphi$ are separating for $\mathbf{M}$, $a_\alpha = 0$ for all $\alpha \in G$ and so $A = 0$. Moreover, $[(\mathbf{M}, G, \mathcal{U})(\varphi \otimes \varepsilon_e)] = [\sum_{\alpha \in G} A_\alpha U_\alpha(\varphi \otimes \varepsilon_e)]$ $= [\sum_{\alpha \in G} a_\alpha u_\alpha \varphi \otimes \varepsilon_\alpha] = \mathbf{H} \otimes l_2(G)$ since $u_\alpha \varphi$ are generating for $\mathbf{M}$.

The crossed product of $\mathbf{M}$ by G defined in the section 1 depends on the choice of the unitary representation $\mathcal{U}$ of G. But that it is independent of the unitary representation is desirous.

LEMMA 5. *The crossed product is uniquely determined by* $\mathbf{M}$ *and* G, *i.e. let* $\{u_\alpha\}$, $\{v_\alpha\}$ $(\alpha \in G)$ *be two unitary representation of* G *on* $\mathbf{H}$, *then the crossed product* $(\mathbf{M}, G, \mathcal{U})$ *is spatially isomorphic to* $(\mathbf{M}, G, \mathcal{V})$.

PROOF. Let $\{U_\alpha\}_{\alpha \in G}$, $\{V_\alpha\}_{\alpha \in G}$ be two unitary representations of G on $\mathbf{H} \otimes l_2(G)$ in Lemma 2 corresponding to $\{u_\alpha\}_{\alpha \in G}$, $\{v_\alpha\}_{\alpha \in G}$ respectively and let $\mathfrak{S}, \mathfrak{S}$ be the sets $\{\sum_{\alpha \in g} A_\alpha U_\alpha\}$, $\sum_{\alpha \in g} A_\alpha V_\alpha\}$ (where $A_\alpha \in \mathbf{M} \otimes \mathbf{I}$ and g runs over finite subsets of G) respectively. We shall prove that the mapping $\psi : \sum_{\alpha \in g} A_\alpha U_\alpha \rightarrow \sum_{\alpha \in g} A_\alpha V_\alpha$ of $\mathfrak{S}$ onto $\mathfrak{S}'$ is a spatial isomorphism. Since $\mathfrak{S}$, $\mathfrak{S}$ is dense in $(\mathbf{M}, G, \mathcal{U})$, $(\mathbf{M}, G, \mathcal{V})$ respectively, It is assured that $(\mathbf{M}, G, \mathcal{U})$ is spatially isomorphic to $(\mathbf{M}, G, \mathcal{V})$.

Let φ be a separating and generating vector of $\mathbf{M}$, then $\varphi \otimes \varepsilon_e$ is a separating and generating vector of $(\mathbf{M}, G, \mathcal{U})$ and $(\mathbf{M}, G, \mathcal{V})$ by Lemma 4. Since there exist unitary operators $W'_\alpha \in (\mathbf{M} \otimes \mathbf{I})$ such that $U_\alpha = W'_\alpha V_\alpha$, it holds from the property (2) in Lemma 2 that

$$\left\| \left(\sum_{\alpha \in g} A_\alpha U_\alpha \right) (\varphi \otimes \varepsilon_e) \right\|^2 = \sum_{\alpha \in g} \| A_\alpha U_\alpha(\varphi \otimes \varepsilon_e) \|^2 = \sum_{\alpha \in g} \| A_\alpha W'_\alpha V_\alpha(\varphi \otimes \varepsilon_e) \|^2$$
$$= \sum_{\alpha \in g} \| W'_\alpha A_\alpha V_\alpha(\varphi \otimes \varepsilon_e) \|^2 = \sum_{\alpha \in g} \| A_\alpha V_\alpha(\varphi \otimes \varepsilon_e) \|^2 = \left\| \sum_{\alpha \in g} A_\alpha V_\alpha(\varphi \otimes \varepsilon_e) \right\|^2.$$

Therefore, we can find a unitary operator W on $\mathbf{H} \otimes l_2(G)$ such that

$$W\left(\left(\sum_{\alpha \in g} A_\alpha U_\alpha\right) (\varphi \otimes \varepsilon_e)\right) = \left(\sum_{\alpha \in g} A_\alpha V_\alpha\right) (\varphi \otimes \varepsilon_e),$$

because $\mathfrak{S}(\varphi \otimes \varepsilon_e)$, $\mathfrak{S}(\varphi \otimes \varepsilon_e)$ are dense in $\mathbf{H} \otimes l_2(G)$. Then, it must be shown that $W\left(\sum_{\alpha \in g} A_\alpha U_\alpha\right) W^{-1} = \sum_{\alpha \in g} A_\alpha V_\alpha$. Indeed, for each vector $\left(\sum_{\beta \in h} A_\beta V_\beta\right)(\varphi \otimes \varepsilon_e)$, ($h$ runs over finite subsets of G), $W\left(\sum_{\alpha \in g} A_\alpha U_\alpha\right) W^{-1} \left(\sum_{\beta \in h} A_\beta V_\beta\right)(\varphi \otimes \varepsilon_e) = W\left(\sum_{\alpha \in g} A_\alpha U_\alpha\right)\left(\sum_{\beta \in h} A_\beta U_\beta\right)(\varphi \otimes \varepsilon_e) = W\left(\sum_{\alpha, \beta} A_\alpha A_\beta^{\alpha-1} U_{\alpha\beta}\right)$ $(\varphi \otimes \varepsilon_e) = \sum_{\alpha, \beta} A_\alpha A_\beta^{\alpha-1} V_{\alpha\beta}(\varphi \otimes \varepsilon_e) = \left(\sum_{\alpha \in g} A_\alpha V_\alpha\right)\left(\sum_{\beta \in h} A_\beta V_\beta\right)(\varphi \otimes \varepsilon_e)$. Thus, $W\left(\sum_{\alpha \in g} A_\alpha U_\alpha\right) W^{-1}\psi = \left(\sum_{\alpha \in g} A_\alpha V_\alpha\right)\psi$ for all vectors $\psi \in \mathbf{H} \otimes l_2(G)$. Therefore, The mapping $W(\ ,\ )W^{-1}$ induces a spatial isomorphism of $(\mathbf{M}, G, \mathfrak{U})$ onto $(\mathbf{M}, G, \mathfrak{V})$.

Now it is allowed to express the crossed product $(\mathbf{M}, G, \mathfrak{U})$ by the notation $(\mathbf{M}, G)$, in what follows. We shall state the main result in this section by using the lemmas obtained up to now.

THEOREM 1. *Let* $\mathbf{M}$ *be a finite factor with the invariant* $C=1$ *and let* G *a group of its automorphisms. Then, the crossed product* $(\mathbf{M}, G)$ *is a finite* W^**-algebra with the invariant* $C=1$, *and for each element* A *of* $(\mathbf{M}, G)$, *there exists a unique family* $\{A_\alpha\}_{\alpha\in G}$ *in* $\mathbf{M}\otimes\mathbf{I}$ *such that*

$$A=\sum\nolimits'_{\alpha\in G} A_\alpha U_\alpha$$

where Σ' *is taken in the sense of the metrical convergence*[4].

PROOF. We first prove that $(\mathbf{M}, G)$ is finite. To do it, we must show that there exists a faithful normal trace of $(\mathbf{M}, G)$. Let φ be a separating trace vector of $\mathbf{M}$, then $\varphi\otimes\varepsilon_e$ is a separating trace vector of $(\mathbf{M}, G)$. Indeed, since $\varphi\otimes\varepsilon_e$ is separating for $(\mathbf{M}, G)$ by Lemma 4, it is sufficient to prove that $\|A(\varphi\otimes\varepsilon_e)\|^2=\|A^*(\varphi\otimes\varepsilon_e)\|^2$ for all $A\in(\mathbf{M}, G)$. By Lemma 3, there exists a family $\{a_\alpha\otimes I\}_{\alpha\in G}$ in $\mathbf{M}\otimes\mathbf{I}$ such that

$$A(\varphi\otimes\varepsilon_e)=\sum\nolimits_{\alpha\in G}(a_\alpha\otimes I)U_\alpha(\varphi\otimes\varepsilon_e),$$

and then $\|A(\varphi\otimes\varepsilon_e)\|^2=\sum_\alpha\|(a_\alpha\otimes I)\,U_\alpha(\varphi\otimes\varepsilon_e)\|^2=\sum_\alpha\|a_\alpha u_\alpha\varphi\otimes\varepsilon_\alpha\|^2=\sum_\alpha\|a_\alpha u_\alpha\varphi\|^2=\sum_\alpha\|a_\alpha\varphi\|^2=\sum_\alpha\|a_\alpha^*\varphi\|^2=\sum_\alpha\|u_\alpha^*a_\alpha^*\varphi\|^2=\sum_\alpha\|u_\alpha^*a_\alpha^*\varphi\otimes\varepsilon_{\alpha^{-1}}\|^2=\sum_\alpha\|U_\alpha^*(a_\alpha^*\otimes I)\,(\varphi\otimes\varepsilon_e)\|^2$. Thus $\sum_{\alpha\in G}U_\alpha^*(a_\alpha^*\otimes I)(\varphi\otimes\varepsilon_e)$ converges and the desired identity holds because of $A^*(\varphi\otimes\varepsilon_e)=\sum_{\alpha\in G}U_\alpha^*(a_\alpha^*\otimes I)(\varphi\otimes\varepsilon_e)$[5]. In addition,the invariant C equals to one by Lemma 4.

Now, applying Lemma 3 to the above fact, we assure that for each $A\in(\mathbf{M}, G)$, there exists a family $\{A_\alpha\}_{\alpha\in G}$ in $\mathbf{M}\otimes\mathbf{I}$ such that $A=\sum'_{\alpha\in G}A_\alpha U_\alpha$. Indeed, this family $\{A_\alpha\}_{\alpha\in G}$ is unique, because $\sum'_\alpha A_\alpha U_\alpha=0$ yields $\sum_\alpha A_\alpha U_\alpha(\varphi\otimes\varepsilon_e)=0$, and so $A_\alpha U_\alpha(\varphi\otimes\varepsilon_e)=0$ for all $\alpha\in G$ as we have seen, or $A_\alpha=0$ since $\varphi\otimes\varepsilon_e$ is separating for $(\mathbf{M}, G)$.

In connection with this theorem, it is convenient to introduce the follow-

4) Let $\mathbf{M}$ be a finite W^*-algebra with a separating and generating trace vector φ. Then, $\mathbf{M}$ becomes a topological space in a new way with the metric $[[a]]=\|a\varphi\|$. A directed family $\{a_i\}_{i\in I}$ in $\mathbf{M}$ is said to be metrically convergent to a in $\mathbf{M}$ if $[[a_i-a]]\to 0$. For this metric $[[\ \ ]]$, cf. [5 : Chap. 1] [6 : Chap. 1].

5) Putting $A_\alpha=a_\alpha\otimes 1$, then $<\psi\otimes\varepsilon_\sigma,\ \sum_\alpha U_\alpha^*A_\alpha^*(\varphi\otimes\varepsilon_e)>=\sum_\alpha<\psi\otimes\varepsilon_\sigma,\ U_\alpha^*A_\alpha^*(\varphi\otimes\varepsilon_e)>=\sum_\alpha<A_\alpha U_\alpha(\psi\otimes\varepsilon_\sigma),\ \varphi\otimes\varepsilon_e>=<A(\psi\otimes\varepsilon_\sigma),\ \varphi\otimes\varepsilon_e>=<\psi\otimes\varepsilon_\sigma,\ A^*(\varphi\otimes\varepsilon_e)>$ for all $\psi\in\mathbf{H}$, $\sigma\in G$.

ing phrase : For each element $A \in (\mathbf{M}, G)$, the elements A_α in $\mathbf{M} \otimes \mathbf{I}$ in Theorem 1 are called *the α-component* of A. Further, we shall often make use of the following relations: If $A = \sum'_{\alpha \in G} A_\alpha U_\alpha$(in Theorem 1),

$$U_\sigma A = \sum'_{\alpha \in G} U_\sigma A_\alpha U_\alpha, \ AU_\alpha = \sum'_{\alpha \in G} A_\alpha U_\alpha U_\sigma.$$

and for $B \in \mathbf{M} \otimes \mathbf{I}$,

$$BA = \sum'_{\sigma \, G} BA_\alpha U_\alpha, \ AB = \sum'_{\alpha \in G} A_\alpha U_\alpha B.$$

These facts follow immediately from the property of the metric [[ ]].

An important question now arising is whether the crossed product $(\mathbf{M}, G)$ is a factor or not for any group G of automorphisms. The answer for this question is generally negative. Let $\mathbf{M}$ be a factor of type $\mathbf{II}_1$ on a Hilbert space $\mathbf{H}$, then one can find easily a unitary operator in $\mathbf{M}$ such that $u^2 = I$. Denoting by G a group of automorphisms of $\mathbf{M}$ induced by u and I, the crossed product $(\mathbf{M}, G)$ is not a factor. Indeed, put

$$P = \begin{pmatrix} \frac{1}{2} I & \frac{1}{2} I \\ \frac{1}{2} I & \frac{1}{2} I \end{pmatrix},$$

it is immediate to see that P is a projection $\neq 0, I$ in $\mathbf{H} \otimes l_2(G)$ and is expressed in the form:

$$P = \begin{pmatrix} \frac{1}{2} I & 0 \\ 0 & \frac{1}{2} I \end{pmatrix} + \begin{pmatrix} \frac{1}{2} u & 0 \\ 0 & \frac{1}{2} u \end{pmatrix} \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}.$$

Recall that $U = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$ is the representation of the automorphism induced by $u^*(\ , \ ) \, u$ in Lemma 2, the direct computation shows that P is an element of the center of $(\mathbf{M}, G)$. That is, $(\mathbf{M}, G)$ is not a factor.

This fact tells us that a group of inner automorphisms is, in general, not appropriate for the purpose of the so-called factor construction as mentioned in the introduction. In connection with this example, we shall find the condition under which $(\mathbf{M}, G)$ is a factor, for an abelian group G.

LEMMA 6. *If G is an abelian group of automorphisms of* $\mathbf{M}$ *by which only the center is elementwise invariant, then* $(\mathbf{M}, G)$ *is a factor.*

PROOF. Suppose that A is an element in the center of $(\mathbf{M}, G)$ and $\{A_\alpha\}_{\alpha \in G}$ a family of α-components of A. Then for each U_σ, $\sum'_\alpha U_\sigma A_\alpha U_\alpha =$

$\sum_{\alpha}' A_\alpha U_\alpha U_\sigma$ and so $\sum_{\alpha}' A_\alpha^{\sigma^{-1}} U_{\sigma\alpha} = \sum_{\alpha}' A_\alpha U_{\alpha\sigma} = \sum_{\alpha}' A_\alpha U_{\sigma\alpha}$. Thus we obtain by the uniqueness of a family $\{A_\alpha\}_{\alpha\in G}$ that

$$A_\alpha^{\sigma^{-1}} = A_\alpha \qquad \text{for each } \alpha \in G.$$

Then the assumption shows that $A_\alpha = \lambda_\alpha I_{\mathfrak{H}}$ for some scalars λ_α where $I_{\mathfrak{H}}$ is the identity on $\mathfrak{H} = \mathbf{H} \otimes l_2(G)$. On the other hand, for each $B \in \mathbf{M} \otimes \mathbf{I}$, $\sum_{\alpha}' BA_\alpha U_\alpha = \sum_{\alpha}' A_\alpha U_\alpha B$, or $\sum_{\alpha}' BA_\alpha U_\alpha = \sum_{\alpha}' A_\alpha B^{\alpha^{-1}} U_\alpha$. Using again the uniqueness of $\{A_\alpha\}_{\alpha\in G}$, we get

$$BA_\alpha = A_\alpha B^{\alpha^{-1}} \qquad \text{for each } \alpha \in G.$$

Now, if $A \neq 0$ for some $\alpha \neq e$, $B = B^{\alpha^{-1}}$ for all $B \in \mathbf{M} \otimes \mathbf{I}$ since $A_\alpha = \lambda_\alpha I_{\mathfrak{H}} \neq 0$, which contradicts to $\alpha \neq e$; hence $A_\alpha = 0$ for every $\alpha \neq e$.

3. The crossed product of the factor of type $\mathbf{II}_1$. In this section, we concern only with $\mathbf{II}_1$-factors. Indeed, in our theory, we take an interest in the factors of this type alone. First we wish to see the existence of a group of outer automorphisms mentioned in the introduction. Already, it was known that there exist $\mathbf{II}_1$-factors, having an outer automorphism. In particular, an approximately finite factor on a separable Hilbert space has always such automorphism, as shown in [2], i. e. an automorphism of the algebraic extension K of a finite field induces an outer automorphism of it. In this place, replacing an automorphism of K by a group of automorphisme of K, we can obtain the desired group of automorphisms. Recently the author has shown that an arbitrary countable group is isomorphic to a group of outer automorphisms of the approximately finite factor on a separable Hilbert space [7]. That is to say, since this kind of factors are all isomorphic each other [5], we have the following

THEOREM 2. *The approximately finite factor on a separable Hilbert space has a group of outer automorphisms isomorphic to an arbitrary countable group.*

Next, we shall investigate the crossed product of a $\mathbf{II}_1$-factor by a group of outer automorphisms. First we must ask whether the crossed product obtained in this case is a factor or not.

THEOREM 3. *Let* $\mathbf{M}$ *be a* $\mathbf{II}_1$*-factor with invariant* $C = 1$ *and let* G *a group of outer automorphisms of* $\mathbf{M}$. *Then the commutant of* $\mathbf{M} \otimes \mathbf{I}$ *in the crossed product* $(\mathbf{M}, G)$ *coincides with the center of* $\mathbf{M} \otimes \mathbf{I}$. *That is, the crossed product* $(\mathbf{M}, G)$ *is a factor.*

PROOF. Let A be an element of the commutant of $\mathbf{M} \otimes \mathbf{I}$ in $(\mathbf{M}, G)$, then we must show that it is scalar multiples of the identity I on $\mathbf{H} \otimes$

$l_2(G)$. By Theorem 1,

$$A = \sum'_{\alpha\in G} A_\alpha U_\alpha \ (A_\alpha \in \mathbf{M} \otimes \mathbf{I}).$$

In this case, we obtain $\sum'_\alpha XA_\alpha U_\alpha = \sum'_\alpha A_\alpha U_\alpha X = \sum'_\alpha A_\alpha X^{\alpha^{-1}} U_\alpha$ for all $X \in \mathbf{M} \otimes \mathbf{I}$, and hence, by the uniqueness of a family $\{A_\alpha\}_{\alpha\in G}$,

$$XA_\alpha U_\alpha = A_\alpha X^{\alpha^{-1}} U_\alpha = A_\alpha U_\alpha X. \qquad (1)$$

for each $\alpha \in G$. Thus A_e is scalar multiples of the identity I. Now, suppose that A_α is non-zero for some $\alpha \neq e$, then $A_\alpha U_\alpha \in (\mathbf{M} \otimes \mathbf{I})$ and $U_\alpha^* A_\alpha^* \in (\mathbf{M} \otimes \mathbf{I})$ yield $U_\alpha^* A_\alpha^* A_\alpha U_\alpha \in (\mathbf{M} \otimes \mathbf{I})'$, but it belongs to $\mathbf{M} \otimes \mathbf{I}$, so that $A_\alpha^* A_\alpha = \lambda_\alpha I$ (λ_α : a non-zero positive number). On the other hand, $A_\alpha A_\alpha^* = A_\alpha U_\alpha U_\alpha^* A_\alpha^* \in (\mathbf{M} \otimes \mathbf{I})'$ and so $A_\alpha A_\alpha^* = \lambda_\alpha I$ (for $A_\alpha^* A_\alpha$ and $A_\alpha A_\alpha^*$ have the same spectrum). Therefore, passing the polar decomposition, $A_\alpha = \lambda_\alpha^{1/2} W_\alpha$ where W_α is a partial isometry of $\mathbf{M} \otimes \mathbf{I}$, so that $\lambda_\alpha I = A_\alpha^* A_\alpha = \lambda_\alpha W_\alpha^* W_\alpha$ and $\lambda_\alpha I = A_\alpha A_\alpha^* = \lambda_\alpha W_\alpha W_\alpha^*$. Thus it follows that $W_\alpha^* W_\alpha = W_\alpha W_\alpha^* = I$. Hence we obtain

$$A_\alpha = \lambda_\alpha^{1/2} W_\alpha \qquad (2)$$

for the unitary operator W_α of $\mathbf{M} \otimes \mathbf{I}$. Combinig (1) and (2), $XW_\alpha = W_\alpha X^{\alpha^{-1}}$ for all $X \in \mathbf{M} \otimes \mathbf{I}$ and so $X^{\alpha^{-1}} = W_\alpha^* X W_\alpha$. This contradicts to the fact that $\alpha \neq e$ are outer. Thus $A_\alpha = 0$ for every $\alpha \neq e$ in G. This completes the proof.

In succession, we shall determine the type of our crossed product being deduced easily from Theorem 1.

THEOREM 4. *Let* $\mathbf{M}$ *be a* $\mathbf{II}_1$*-factor with the invariant* $C = 1$ *and let* G *a group of automorphisms of* $\mathbf{M}$*, then the crossed product* $(\mathbf{M}, G)$ *is of type* $\mathbf{II}_1$. *In particular, if* G *is a group of outer automorphisms, it is a factor of type* $\mathbf{II}_1$.

PROOF. By Theorem 1, assuming that $(\mathbf{M}, G)$ is of type $\mathbf{I}$, we may show that this assumption yields the contradiction. Since $\mathbf{M}$ is of type $\mathbf{II}_1$, we can choose a strictly monotone decreasing infinite directed set of projections $\{e_i\}_{i\in I}$ in $\mathbf{M}$. Then $\{e_i \otimes I\}_{i\in I}$ is also a strictly monotone decreasing infinite directed set of projections in $(\mathbf{M}, G)$. But since $(\mathbf{M}, G)$ is considered to be the ring of all bounded operators on a convenable finite dimensional Hilbert space, the directed set $\{e_i \otimes I\}_{i\in I}$ is impossible to be infinite, strictly decreasing.

With respect to factors of type $\mathbf{II}_1$, the difficult and significant problem is to construct factors of the different algebraic type from the approximately finite factor in this manner. We wish to discuss fully this problem elsewhere.

4. The subfactor of the crossed product. A group of unitary operators $\{U_\alpha\}$ on a Hilbert space $\mathcal{H}$ is said to conserve a factor $\mathcal{A}$ on $\mathcal{H}$ if it leaves $\mathcal{A}$ invariably (i. e. $U_\alpha^* A\, U_\alpha \subseteqq \mathcal{A}$) and all U_α don't belong to $\mathcal{A}$ except the unit. The crossed product $(\mathbf{M}, G)$ is, as we have seen, generated by the subfactor $\mathbf{M} \otimes \mathbf{I}$ and a unitary group $\{U_\alpha\}_{\alpha\in G}$ which conserves $\mathbf{M}\otimes\mathbf{I}$, and in which all but the unit are orthogonal to $\mathbf{M}\otimes\mathbf{I}$ (in the sense of the structure of the prehilbert space $\mathbf{M}$ defined by the trace). Now, we are going to consider the converse of this fact.

THEOREM 5. *Let $\mathcal{A}$ be a countably decomposable, finite W^*-algebra with the invariant $C = 1$ on $\mathcal{H}$ and let $\mathcal{B}$ a subfactor of $\mathcal{A}$. If there exists a unitary group $\mathcal{G} = \{U_\alpha\}$ in $\mathcal{A}$ conserving $\mathcal{B}$, in which all but the unit are orthogonal to $\mathcal{B}$, and $\mathcal{A}$ is generated by $\mathcal{B}$ and $\mathcal{G}$, then $\mathcal{A}$ is spatially isomorphic to the crossed product $(\mathbf{B}, G)$ where $\mathbf{B}$ is a factor with the invariant $C = 1$ and isomorphic to $\mathcal{B}$ and G is a group of automorphisms of $\mathbf{B}$ isomorphic to $\mathcal{G}$.*

PROOF. Let φ be a normalized, separating and generating trace vector of $\mathcal{A}$. Consider the isometry Φ of the prehilbert space $\mathcal{A}$ (induced by the trace $<(\,,\,)\,\varphi, \varphi>$) onto the dense set $\mathcal{A}\varphi$ in $\mathcal{H}$ as follows

$$\Phi: \qquad A \in \mathcal{A} \to A\varphi \in \mathcal{H}.$$

Then, $P_{[\mathcal{B}U_\alpha\varphi]}$ are mutually orthogonal and $I = \sum_\alpha P_{[\mathcal{B}U_\alpha\varphi]}$. In fact, for $A, B \in \mathcal{B}$, $<AU_\alpha\varphi, BU_\beta\varphi> = <U_\beta^* B^* A U_\alpha\varphi, \varphi> = <(U_\beta^* B^* A U_\beta) U_\beta^* U_\alpha\varphi, \varphi> = 0$ if $\alpha \neq \beta$. Further, passing the isometry Φ, the fact that $\mathcal{A}$ is generated by $\{\mathcal{B}U_\alpha\}$ yields easily $I = \sum_\alpha P_{[\mathcal{B}U_\alpha\varphi]}$.

Putting $\mathbf{B} = \mathcal{B}_{[\mathcal{B}\varphi]}$, $\mathbf{B}$ is a factor with the invariant $C = 1$ and isomorphic to $\mathcal{B}$ since φ is separating for $\mathcal{B}$. Now, denote by G the group of automorphisms α of $\mathbf{B}$ induced by U_α, then $\{U_\alpha\}$ is considered to be a unitary representation of G on $\mathcal{H}$ (recall that $\{U_\alpha\}$ defines a group of automorphisms of $\mathcal{B}$ by $U_\alpha^*(\,,\,)U_\alpha$).

We shall show that $\mathbf{R}(\mathcal{B}, U_\alpha\,;\, \alpha\in G) = \mathcal{A}$ is spatially isomorphic to the crossed product $(\mathbf{B}, G)$ of $\mathbf{B}$ by G. Since $\|AU_\alpha\varphi\|^2 = \|U_\alpha^* A U_\alpha\varphi\|^2 = \|A\varphi\|^2$ for $A \in \mathcal{B}$, we obtain partial isometries W_α on $\mathcal{H}$ which maps $[\mathcal{B}\varphi]$ on $[\mathcal{B}U_\alpha\varphi]$. Setting $\mathbf{H} = [\mathcal{B}\varphi]$, we denote by S_α the isometries of $\mathbf{H}$ onto the subspaces $\mathbf{H}_\alpha$ in $\mathbf{H}\otimes l_2(G)$, carring $A\varphi (A \in \mathcal{B})$ on $A\varphi\otimes\varepsilon_\alpha$. Then, it is immediately verified that the isometry $\sum_{\alpha\in G} S_\alpha W_\alpha^*$ of $\mathcal{H}$ onto $\mathbf{H}\otimes l_2(G)$ carries $\mathcal{B}$ on $\mathbf{B}\otimes\mathbf{I}$ and the inverse of $\sum_{\alpha\in G} S_\alpha W_\alpha^*$ is the mapping $\sum_{\alpha\in G} W_\alpha S_\alpha^*$, where $S_\alpha^* = S_\alpha^{-1}$ on $\mathbf{H}_\alpha$, $= 0$ on $\mathbf{H}_\alpha^\perp$.

Now, define unitary operators v_α on $\mathbf{H}$ by

$$v_\alpha A\varphi = U_\alpha A U_\alpha^* \varphi \qquad \text{for all } A \in \mathcal{B}.$$

Then, by Lemma 1, $\{v_\alpha\}$ $(\alpha \in G)$ is a unitary representation of G on $\mathbf{H}$ (for $A\varphi = A_{[\mathcal{B}\varphi]}\varphi (A_{[\mathcal{B}\varphi]} \in \mathbf{B})$ and $U_\alpha A U_\alpha^* \varphi = (A_{[\mathcal{B}\varphi]})^{\alpha^{-1}}\varphi)$. We shall complete the proof by showing that, for each $\sigma \in G$

$$\left(\sum_\alpha S_\alpha W_\alpha^*\right) U_\sigma \left(\sum_\alpha S_\alpha W_\alpha^*\right)^{-1} = V_\sigma$$

where V_σ is a unitary representation of σ in Lemma 2 obtained from v_σ. For each vector $\sum_\alpha A_\alpha\varphi \otimes \varepsilon_\alpha$ $(A_\alpha \in \mathcal{B})$ in $\mathbf{H} \otimes l_2(G)$, we have.

$$\left(\sum_\alpha S_\alpha W_\alpha^*\right) U_\sigma \left(\sum_\alpha S_\alpha W_\alpha^*\right)^{-1} \left(\sum_\alpha A_\alpha \varphi \otimes \varepsilon_\alpha\right) = \left(\sum_\alpha S_\alpha W_\alpha^*\right) U_\sigma \left(\sum_\alpha A_\alpha U_\alpha \varphi\right)$$

$$= \left(\sum_\alpha S_\alpha W_\alpha^*\right) \left(\sum_\alpha (U_\sigma A_\alpha U_\sigma^*) U_{\sigma\alpha}\varphi\right) = \sum_\alpha (U_o A_\alpha U_\sigma^*)\varphi \otimes \varepsilon_{\sigma\alpha}$$

$$= \sum_\alpha v_\sigma A_\alpha \varphi \otimes \varepsilon_{\sigma\alpha} = V_\sigma \left(\sum_\alpha A_\alpha \varphi \otimes \varepsilon_\alpha\right),$$

whence the proof is completed.

REMARK 3. In this theorem it may be noticed that the invariant of $\mathcal{B}$ equals to the cardinal of G. This fact is easily verified.

CORLOLARY. *Let $\mathcal{A}$ be a finite factor with the invariant $C = 1$ and let $\mathcal{B}$ a subfactor of $\mathcal{A}$ such that $\mathcal{B}' \cap \mathcal{A} =$ (scalar multiples of the identity), $\mathcal{G}$ a unitary group in $\mathcal{A}$ leaving $\mathcal{B}$ invariably. If there exists a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ whose elements are orthogonal to $\mathcal{B}$ except the unit, and $\mathcal{A}$ is generated by $\mathcal{B}$ and $\mathcal{G}_0$, then $\mathcal{A}$ is spatially isomorphic to the crossed product* $(\mathbf{B}, G)$ *where* $\mathbf{B}$ *is a factor with the invariant $C = 1$ and isomorphic to $\mathcal{B}$ and G is a group of outer automorphisms of* $\mathbf{B}$ *isomorphic to $\mathcal{G}_0$.*

In fact, it is easy to see that $\mathcal{G}_0$ conserves the subfactor $\mathcal{B}$, and if $U_\alpha \neq I$ in $\mathcal{G}_0$ defines an inner automorphisms of $\mathcal{B}$, there is a unitary operator $U \in \mathcal{B}$ such that $U_\alpha U \in \mathcal{B}'$, and hence $U_\alpha = \lambda U \in \mathcal{B}$ for a scalar λ. This contradicts to the fact that U_α is orthogonal to $\mathcal{B}$.

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