ON FREE MODULAR LATTICES II

KENSUKE TAKEUCHI

(Received Feb. 25, 1957)

The present paper is a continuation of the author's previous paper [8] under the same title, and it concerns some scattered topics related to the word problem for free modular lattices. § 1 is devoted to the decision of the free modular lattice generated by $2 + 1 + 1$. § 2 deals with the word problem restricted to the case that ranks $\leqq 2$, and § 3 with a solution of the word problem for free distributive lattices. In the course of these investigations it being desirable to clarify the situation where the word problem stands, the author wishes to present in the Appendix a formulation of the word problem for an arbitrary abstract algebra.

1. The free modular lattice generated by $2 + 1 + 1$. The free modular lattice generated by $2 + 1 + 1$ is of dimension 16 and has 138 elements. This is the answer to Problem 29 proposed in Birkhoff [1], which was obtained independently by Thrall and Duncan [9] and the author. (Schützenberger [6] also solved the finiteness problem for more general cases.) The present paper contains its Hasse diagram, No. 32 of its list being rewritten from the author's original list (written in his communication with Thrall), which was pointed out by Thrall and Duncan [9] as the discrepancies between their list and the author's. The Hasse diagram, together with the positions of the generators, completely solves the word problem for this case.

The list of elements in the free modular lattice generated by four elements $x < z$ and p, q.

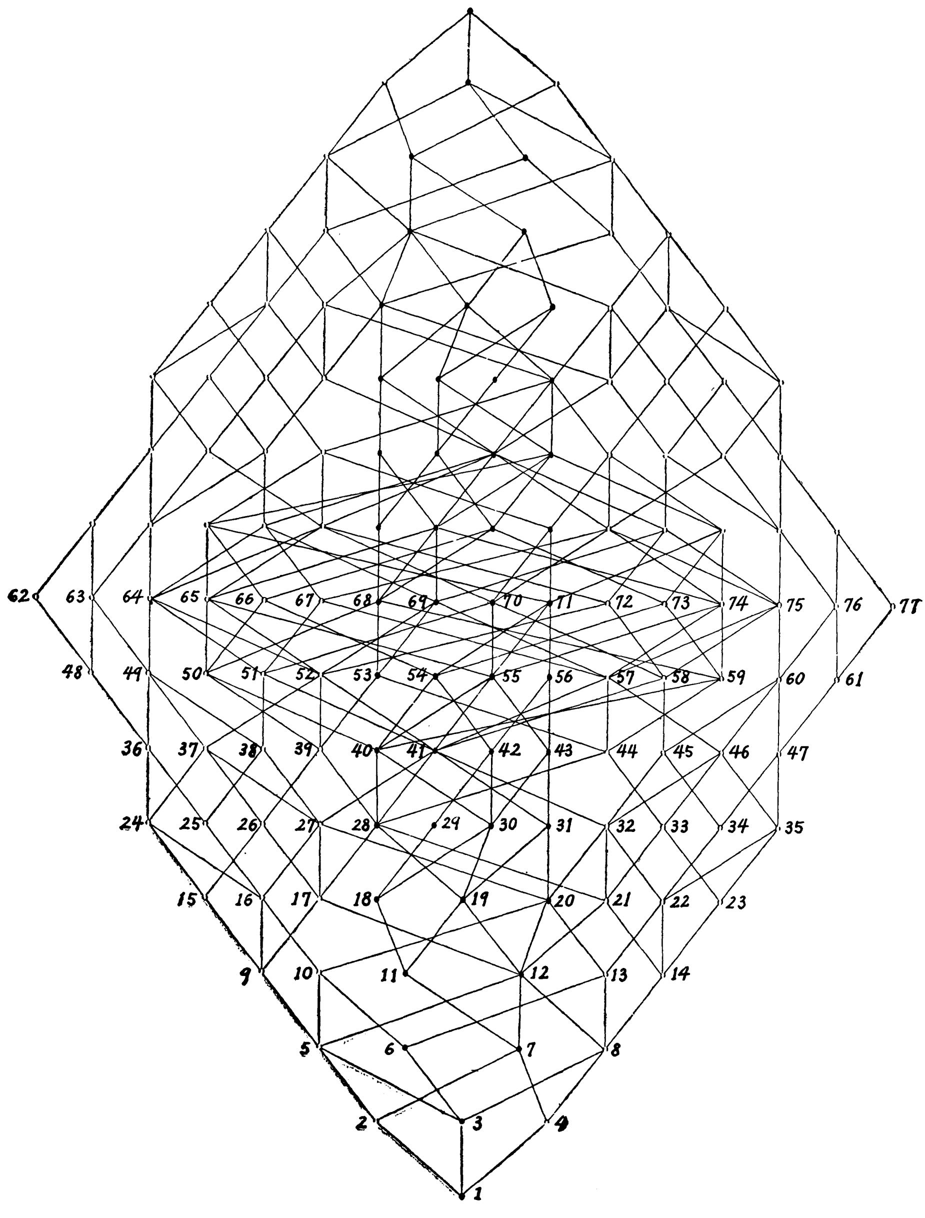
Notes: 1) For brevity, we shall write as $a \cup b \cap c$ in place of $(a \cup b) \cap c$ or $a \cup (b \cap c)$, when $a \leqq c$.

2) Dark spots in the diagram will represent the elements which are symmetric with respect to p and q.

Dimension	No.	
0	1	$x \cap p \cap q$
1	2	$x \cap p$
	3	$z \cap p \cap q$
	4	$x \cap q$
2	5	$(z \cap p \cap q) \cup (x \cap p)$
	6	$p \cap q$

Dimension	No.	
	7	$(x \cap p) \cup (x \cap q)$
	8	$(z \cap p \cap q) \cup (x \cap q)$
3	9	$p \cap (z \cap q \cup x)$
	10	$(x \cap p) \cup (p \cap q)$
	11	$x \cap (p \cup (z \cap q)$
	12	$(z \cap p \cap q) \cup (x \cap p) \cup (x \cap q)$
	13	$(x \cap q) \cup (p \cap q)$
	14	$q \cap (z \cap p \cup x)$
4	15	$z \cap p$
	16	$(x \cup q) \cap (z \cap p) \cup (p \cap q)$
	17	$(x \cup q) \cap (z \cap p) \cup (x \cap q)$
	18	$x \cap (p \cup q)$
	19	$(z \cap p \cap q) \cup (x \cap (p \cup (z \cap q)))$
	20	$(x \cap p) \cup (x \cap q) \cup (p \cap q)$
	21	$(x \cup p) \cap (z \cap q) \cup (x \cap p)$
	22	$(x \cup p) \cap (z \cap q) \cup (p \cap q)$
	23	$z \cap q$
5	24	$(z \cap p) \cup (p \cap q)$
	25	$p \cap (x \cup q)$
	26	$(z \cap p) \cup (x \cap q)$
	27	$(p \cap q) \cup z \cap ((x \cup q) \cap p \cup (x \cap q))$
	28	$(x \cup p) \cap (x \cup q) \cap ((z \cap p) \cup (z \cap q))$
	29	x
	30	$z \cap ((p \cup q) \cap x \cup (p \cap q))$
	31	$(p \cap q) \cup x \cap (p \cup (z \cap q))$
	32	$(p \cap q) \cup z \cap ((x \cup p) \cap q \cup (x \cap p))$
	33	$(z \cap q) \cup (x \cap p)$
	34	$q \cap (x \cup p)$
	35	$(z \cap q) \cup (p \cap q)$
6	36	$p \cap (x \cup q) \cup (z \cap p)$
	37	$(z \cap p) \cup (x \cap q) \cup (p \cap q)$
	38	$(x \cap q) \cup p \cap (x \cup q)$
	39	$(x \cup p) \cap ((z \cap p) \cup (z \cap q))$
	40	$z \cap (x \cup p) \cap (x \cup q) \cap (p \cup q)$
	41	$(x \cup p) \cap (x \cup q) \cap ((z \cap p) \cup (z \cap q) \cup (p \cap q))$
	42	$z \cap (p \cap q) \cup x$
	43	$(p \cap q) \cup x \cap (p \cup q)$
	44	$(x \cup q) \cap ((z \cap p) \cup (z \cap q))$
	45	$(x \cap p) \cup q \cap (x \cup p)$

Dimension	No.	
	46	$(z \cap q) \cup (x \cap p) \cup (p \cap q)$
	47	$q \cap (x \cup p) \cup (z \cap q)$
7	48	$p \cap (z \cup q)$
	49	$(z \cap p) \cup ((x \cap q) \cup p \cap (x \cup q))$
	50	$(z \cap p \cup x) \cap (p \cup q)$
	51	$(x \cup p) \cap (x \cup q) \cap (p \cup (z \cap q))$
	52	$(x \cup p) \cap ((z \cap p) \cup (z \cap q) \cup (p \cap q))$
	53	$(z \cap p) \cup (z \cap q)$
	54	$z \cap (x \cup p) \cap (x \cup q)$
	55	$(x \cup p) \cap (x \cup q) \cap ((p \cup q) \cap z \cup (p \cap q))$
	56	$x \cup (p \cap q)$
	57	$(x \cup q) \cap ((z \cap p) \cup (z \cap q) \cup (p \cap q))$
	58	$(x \cup p) \cap (x \cup q) \cap (q \cup (z \cap p))$
	59	$(z \cap q \cup x) \cap (p \cup q)$
	60	$(z \cap q) \cup ((x \cap p) \cup q \cap (x \cup p))$
	61	$q \cap (z \cup p)$
8	62	p
	63	$(z \cup q) \cap p \cup (x \cap q)$
	64	$((z \cap p \cup x) \cup q) \cap p \cup (q \cap (z \cap p \cup x))$
	65	$(z \cap p) \cup ((p \cap q) \cup x \cap (p \cup q))$ or $(x \cup p) \cap ((p \cup q) \cap z \cup (p \cap q))$
	66	$z \cap p \cup x$
	67	$(z \cap q) \cup p \cap (x \cup q)$
	68	$(z \cap p) \cup (z \cap q) \cup (x \cap (p \cup q))$
	69	$(z \cap p) \cup (z \cap q) \cup (p \cap q)$
	70	$(x \cup p) \cap (x \cup q) \cap (p \cup q)$
	71	$(x \cup p) \cap (x \cup q) \cap (z \cup (p \cap q))$
	72	$(z \cap p) \cup q \cap (x \cup p)$
	73	$z \cap q \cup x$
	74	$(z \cap q) \cup ((p \cap q) \cup x \cap (p \cup q))$ or $(x \cup q) \cap ((p \cup q) \cap z \cup (p \cap q))$
	75	$((z \cap q \cup x) \cup p) \cap q \cup (p \cap (z \cap q \cup x))$
	76	$(z \cup p) \cap q \cup (x \cap p)$
	77	q



2. The case that ranks $\leqq 2$. In the previous paper [8] the author prepared numerous devices attempting to solve the word problem and established a few facts on free modular lattices. We assume here a knowledge of the notations used in that paper. In the present section of the present paper the author gives the affirmative answer to one of his conjectures, i. e., any two distinct canonical words of ranks $\leqq 2$ cannot be identified as elements of the free modular lattice.

In what follows, generators are denoted by x, y,..... , words (identified by commutativity and associativity) by a, b,......, the set of words, the free lattice and the free modular lattice, with g generators (See Remark 1 on the Appendix), by $P(g)$, $L(g)$ and $M(g)$ respectively, their a-elements by a, $\bar{a}$ and $\bar{\bar{a}}$ respectively and equalities of their elements by $\equiv$, $=$ and $\sim$ respectively, the canonical words for a by κa, the set of distinct generators contained in a and its cardinal number by $G(a)$ and $g(a)$ respectively, the length, the rank and the type (Σ or Π) of a by $S(a)$, $R(a)$ and $T(a)$ respectively.

LEMMA A. *$R(\kappa a) = R(a)$ implies $T(\kappa a) = T(a)$.*

PROOE. If $T(\kappa a) \neq T(a)$, then, since, among three kinds of *replacements* (Whitman [11], p.329) getting to κa starting from a, the only one that is stated as

$$\text{`if } b^i \leqq \sum_{k \neq i} b^k, \text{ then replace } b \text{ with} \sum_{k \neq i} b^k\text{'}$$

may change the type, we have $R(\kappa a) < R(a)$.

LEMMA B. *If $\theta: (x_0/f(x_1, \ldots, x_{g-1}))$ is a substitution of x_0 with $f(x_1, \ldots x_{g-1})$, then $a \sim b$ implies $\theta a \sim \theta b$.*

PROOF. We can evidently construct a deduction (see Appendix (3)) of $\theta a \sim \theta b$ having the same *analysis* as that of $a \sim b$.

LEMMA C. $x_0 \cup (x_1 \cap x_2) \nsim (x_0 \cup x_1) \cap (x_0 \cup x_2)$,

$x_0 \cup (x_1 \cap \ldots \cap x_{g-1}) \nsim (x_0 \cup x_1) \cap \ldots \cap (x_0 \cup x_{g-1})$.

PROOF. See the Hasse diagram of $M(3)$, or, considering $\theta: (x_{g-1}/x_1)$, apply induction on g.

The proof of our theorem is mainly based upon the following three theorems.

THEOREM 0 (*Whitman* [11] [12]). *$a \leqq b$ if and only if*

$$\begin{array}{lll} (0) & R(a) = 0 = R(b) & \& \quad a \equiv b \\ \textit{or } (1) & T(a) = \Sigma & \& \quad \forall i \ a^i \leqq b \\ \textit{or } (1) & T(b) = \Pi & \& \quad \forall k \ a \leqq b_k \\ \textit{or } (2) & T(a) = \Pi & \& \quad T(b) = \Sigma \\ & \multicolumn{2}{l}{\& \{\exists i \ a_i \leqq b \textit{ or } \exists k \ a \leqq b^k\}} \end{array}$$

THEOREM 1 (*Takeuchi* [8][1]). *Let* $R(\kappa a) \leqq 1$. *Then, for any* b, $a \sim b$ *if and only if* $a = b$.

THEOREM 2 (*ibid.*). *Let* $R(\kappa a) = 2$. *If* $T(\kappa a) = \Sigma\,(\Pi)$, $\bar{a}$ *is the least* (*the greatest*) *element of* $\bar{\bar{a}}$.

Now we shall enter to the proof of our

THEOREM. *Suppose that* $R(\kappa a) \leqq 2 \geqq R(\kappa b)$ *and* $a \neq b$. *Then* $a \not\sim b$.

PROOF by *reductio ad absurdum*. Without loss of generality we can restrict ourselves to the case that $R(\kappa a) = 2 = R(\kappa b)$ by Theorem 1, and further to that

$$a \equiv \kappa a \equiv \sum_{i=0}^{s} a^i,\ \ b \equiv \kappa b \equiv \prod_{k=0}^{t} b_k (s,\ t \geqq 1)$$

and $a < b$ by Theorem 2, where our proof is devided into five steps as follows.

(I) Suppose that $R(a) = 2 = R(b)$, $T(a) = \Sigma$ and $T(b) = \Pi$. Then $a \leqq b$ if and only if $\forall i, k\ \exists j, l\ a^i{}_j \equiv b_k{}^l$.

PROOF. Using Theorem 0, we can resolve the relation $a \leqq b$ into those among generators as

$$\begin{aligned} a \leqq b &\rightleftarrows \forall i, k\ a^i \leqq b_k \\ &\leftrightarrows \forall i, k\ \exists j, l\ a^i{}_j \equiv b_k{}^l. \end{aligned}$$

Theree exists therefore at least one set of generators $\{x^i_k|$ for all pairs $(i, k)\}$ such that $\forall i, k\ \exists j, l\ a^i{}_j \equiv x^i_k \equiv b_k{}^l$. Now let us define as $\overset{*}{a} \equiv \sum_i \prod_k x^i_k$ and $\overset{*}{b} \equiv \prod_k \sum_i x^i_k$.

(II) Suppose further that $\kappa a \equiv a \sim b \equiv \kappa b$. Then $a = \overset{*}{a}$, $b = \overset{*}{b}$ and $G(a) = G(b)$.

PROOF. It is clcar that $a \leqq \overset{*}{a} \leqq \overset{*}{b} \leqq b$. Hence, by Theorem 2, we have $a = \overset{*}{a}$ and $b = \overset{*}{b}$. Thus, together with the canonicality of a and of b, we have

$$G(a) = G(\overset{*}{a}) = G(\overset{*}{b}) = G(b), \text{ putting, } = G = \{\underset{0}{x}, \ldots, \underset{g-1}{x}\}.$$

The remained steps of our proof will be accomplished by means of induction on $g = g(a) = g(b)$.

(III) $\forall i, k \{R(a^i) = 1 = R(b_k)\}$.

1) It must he noticed that the presencc of the following modular law

$$(Y \cap Z) \cup \{X \cap (Y \cup Z)\} \sim \{(Y \cap Z) \cup X\} \cap (Y \cup Z)$$

justifies to formulate the identification of words as stated in [8], although the formulation fairly differs from that appearing in the Appendix of the present paper.

PROOF. Suppose the contrary, i. e., that $\exists i\ R(a^i) = 0$, say $a^0 = \underset{0}{x}$, then, from the canonicality of $a, i \neq 0$ implies $G(a^i) \not\ni \underset{0}{x}$. If $s = 1$, we have

$$a \equiv \underset{0}{x} \cup (\underset{1}{x} \cap \cdots \cap \underset{g-1}{x})\ \&\ b \geqq (\underset{0}{x} \cup \underset{1}{x}) \cap \cdots \cap (\underset{0}{x} \cap \underset{g-1}{x}),$$

a contradiction to Lemma C. If $s > 1$, then considering θ: $(\underset{0}{x}/\underset{1}{x} \cap \cdots \cap \underset{g-1}{x})$ and using Lemma A, we have

$$R(\kappa\theta a) = 2\ \&\ T(\kappa\theta a) = \Sigma\ \&\ \{R(\kappa\theta b) \neq 2 \text{ or}$$
$$T(\kappa\theta b) \neq \Sigma\}\ \&\ g(\kappa\theta a) \leqq g - 1 \geqq g(\kappa\theta b),$$

a contradiction to our induction hypothesis.

(IV) $\forall x\{x \in G \rightarrow \exists i(a^i \not\leqq x)\ \&\ \exists k(x \not\geqq b_k)\}$.

PROOF. $\forall i\ a^i \leqq x \rightarrow a \geqq x \rightarrow b \cup x \sim a \cup x \sim x$, by Theorem 1, $b \cup x = x \rightarrow b \leqq x \rightarrow \exists k\ b_k \leqq x \rightarrow \exists k\ b_k \equiv x$, a contradiction to (III).

(V) The last absurdity.

PROOF. Applying θ: $(\underset{0}{x}/\underset{1}{x} \cap \cdots \cap \underset{g-1}{x})$ and using (III) and (IV), we have

$$R(\kappa\theta a) = 2\ \&\ T(\kappa\theta a) = \Sigma\ \&\ R(\kappa\theta b) = 2\ \&$$
$$T(\kappa\theta b) = \Pi\ \&\ g(\kappa\theta a) \leqq g - 1 \geqq g(\kappa\theta b),$$

a contradication to our induction hypotheeis, q. e. d.

REMARK. Restrict our attention to the case that $g = 4$. In $P(4)$ there exist 4 canonical words of ranks 0, 22 of ranks 1 and 232 of ranks 2. Our theorem states therefore that these 258 distinct elements in $L(4)$ are to fall into different classes under the modular law.

COROLLARY. *Any (finite or infinite) set of identities*

$$\sigma_\nu(X, Y, \ldots, Z) = \tau_\nu(X, Y, \ldots, Z)$$

with indeterminates $X, Y, \ldots, Z$, *satisfying*

$$\{R(\kappa\sigma_\nu) \text{ or } R(\kappa\tau_\nu) > 2\} \rightarrow \{R(\kappa\sigma_\nu) \text{ or } R(\kappa_\nu) \leqq 1\}$$

for every ν, *is not equivalent to the modular identity under the postulates of lattice. Or, more precisely, unless it forms a set of derived identities of the postulates of lattice, it is neither equivalent to, nor weaker than the modular identity.*

3. The word problem for free distributive lattices. The importance of the word pro lem for free modular lattices was emphasized in Birkhoff [2]. Recently, Schützenberger published his note [7], an error (concerning his Lemma II) being pointed out in Math. Reviews, vol. 15 (1954), p. 192.[2]

2) The author, too, has a counter example to his Lemma I which is stated, in our terminology, that if, for each lattice operation ($\cup$ or $\cap$) and for each generator, its number of repetition in u and in v is equal, and if u and v is transferable by his modular law

$$(A \cup B) \cap \{C \cup (A \cap D)\} \sim \{(A \cup B) \cap C\} \cup (A \cap D),$$

Whitman's investigations [11] [12] on free lattices suggest, the author believes, that it is worthwhile to consider whether we can get a solution of the word problem for free modular lattices by weakening the condition (2) of Theorem 0 (of the previous section of this paper) and by strengthening the corresponding one for free distributive lattices. (See Remark (3) on the Appendix.) It may be of interest, in these circumstances, to restate the known decision process for free distributive lattices in accordance with that of Whitman for free lattices, in the following way:

THEOREM D *Let* $\lhd$ *denote the inclusion relation of the free distributive lattice. Then* $a \lhd b$ *if and only if*

$$\begin{array}{lll} (0) & R(a)=1 \ \& \ R(b)=1 & \& \quad a \leqq b \\ or\ (1\,\Sigma) & T(a)=\Sigma & \& \ \forall i \ \ a^i \lhd b \\ or\ (1\,\Pi) & T(b)=\Pi & \& \ \forall k \ \ a \lhd b_k \\ or\ (2) & (R(a)\geqq 2 \text{ or } R(b)\geqq 2) & \& \ T(a)=\Pi \ \& \ (T(b)=\Sigma \\ & \& \ \forall a \ \ a \lhd b \ \& \ \forall b' \ \ a \lhd b, & \end{array}$$

where a *(or* b'*) denotes any word by removing from* a*(or* b*), with* $R(a)$ *(or* $R(b)) \geqq 2$, *one of its second component* a_i^j*(or* $b^k{}_l$*).*

PROOF. It is evident that our condition ((0) or (1 Σ) or (1 Π) or (2)) and the distributive law together imply $a \lhd b$, for

$$\begin{aligned} a &\equiv (a_i^j \cup \sum_{l \neq j} a_i^l) \cap \prod_{k \neq i} a_k \\ &\lhd (a_i^j \cap \prod_{k \neq i} a_k) \cup (\sum_{l \neq j} a_i^l \cap \prod_{k \neq i} a_k) \lhd b. \end{aligned}$$

Conversely, define an inclusion relation $\subset$, as in Whitman's paper [11], that $a \subset b$ if and only if one of the conditions (0), (1 Σ), (1 Π), (2) is true with $\lhd$ replaced by $\subset$. We now show, in the following three steps, that our system forms a distributive lattice (Sce Appendix (4)). (In every step, we omit the proof of the *basis* of the induction, which is almost evident.)

(I) $\subset$ is weaker than $\leqq$, i. e., $a \leqq b$ implies $a \subset b$.

PROOF by induction on $S(a)+S(b)$.
If $T(a)=\Sigma$ (or dually $T(b)=\Pi$), then we have

(where the indeterminate C never stands for the empty word,) then the canonicality of u implies that of v. The author's counter example is thus: Let x, y, z, c, d be generators and put

$$\begin{aligned} a &\equiv \{x \cup y) \cap c\} \cup (x \cap y), \quad b \equiv (x \cup y) \cap z, \\ u &\equiv (a \cup b) \cap \{c \cup (a \cap d)\}, \\ v &\equiv \{(a \cup b) \cap c\} \cup (a \cap d). \end{aligned}$$

Then, u and v clearly satisfies the premise of the Lemma and u is proved to be canonical, but v is not canonical. since $(a \cup b) \cap c$ is already not.

$$a \leqq b \to \forall i \ a^i \leqq b \to \forall i \ a^i \subset b \to a \subset b.$$

If $T(a) = \Pi$ and $T(b) = \Sigma$, then we have

$$\begin{aligned} a \leqq b &\to \forall a' \ a' \leqq b \ \& \ \forall b' \ a \leqq b' \\ &\to \forall a' \ a' \subset b \ \& \ \forall b' \ a \subset b' \to a \subset b', \end{aligned}$$

for it is evident that $a' \leqq a$ for all a' and $b \leqq b'$ for all b.

(II) Our system forms a lattice, i. e., for which we have only to prove that $a \subset b$ and $b \subset c$ together imply $a \subset c$.

PROOF by induction on $S(a) + S(b) + S(c)$:
If $T(a) = \Sigma$ (or dually $T(c) = \Pi$), then we have

$$\begin{aligned} &a \subset b \ \& \ b \subset c \to \forall i \ a^i \subset b \ \& \ b \subset c \\ &\to \forall i \ a^i \subset c \to a \subset c. \end{aligned}$$

If $T(a) = \Pi$ and $T(c) = \Sigma$, then, supposing for example that $T(b) = \Sigma$, we have

$$\begin{aligned} &a \subset b \ \& \ b \subset c \to \forall a' \ a' \subset b \ \& \ \forall i \ b^i \subset c \\ &\to (\forall a' \ a' \subset b \ \& \ b \subset c) \ \& \ (a \subset b \ \& \ \forall i \ \forall c' \ b^i \subset c) \\ &\to \forall a' \ a' \subset b \subset c \ \& \ \forall c' \ a \subset b \subset c' \to a \subset c. \end{aligned}$$

(III) Our system forms distributive lattice, i. e., Bowden's distributive law (Birkhoff [1], p.184, Ex. 3.)

$$\underset{*}{a} \subset \overset{*}{a} \ \& \ \underset{*}{b} \subset \overset{*}{b} \ \& \ \underset{*}{c} \subset \overset{*}{c} \to (\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c} \subset \overset{*}{a} \cup (\overset{*}{b} \cap \overset{*}{c})$$

holds in our system.

PROOF by induction on $S(\underset{*}{a}) + S(\underset{*}{b}) + S(\underset{*}{c}) + S(\overset{*}{a}) + S(\overset{*}{b}) + S(\overset{*}{c})$. Define $d^\dagger$ for $d \equiv \underset{*}{a}, \underset{*}{b}, \underset{*}{c}$ as

$$d^\dagger \equiv \begin{cases} d' \text{ or the empty word, for } T(d) = \Pi, \\ \sum\limits_{i \neq k} d^i \cup (d^k) \text{ or } \sum\limits_{i \neq k} d^i, \text{ for } T(d) = \sum, \end{cases}$$

then any $\{(\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c}\}'$ can be expressed in one of the following three forms:

$$\{(a \cup b) \cap c\} \equiv \begin{cases} (\underset{*}{a^\dagger} \cup \underset{*}{b}) \cap \underset{*}{c} \\ (\underset{*}{a} \cup \underset{*}{b^\dagger}) \cap \underset{*}{c} \\ (\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c^\dagger} \end{cases}$$

In any case, we have, by our induction-hypothesis,

$$\begin{aligned} &\{(\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c}\} \subset \overset{*}{a} \cup (\overset{*}{b} \cap \overset{*}{c}), \text{ and dually,} \\ &(\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c} \subset \{\overset{*}{a} \cup (\overset{*}{b} \cap \overset{*}{c})\}, \end{aligned}$$

therefore,

$$(\underset{*}{a} \cup \underset{*}{b}) \cap \underset{*}{c} \subset \overset{*}{a} \cup (\overset{*}{b} \cap \overset{*}{c}), \text{ q. e. d.}$$

APPENDIX: **The formulation of word problems for abstract algebras.**

By an $\mathfrak{A}$-algebra we here understand a set, having the set F of finitary operations (fixed for $\mathfrak{A}$, so that the concept of F may be interpreted being included in that of $\mathfrak{A}$), and satisfying the set $\mathfrak{A}$ of axioms, each of which is either an identity with a finite number of indeterminates, or an implication the premise of which being a conjunction of identities and the conclusion an identity.

Given a set G of generators and a set $\mathfrak{R}$ of relations, i. e., equalities between words, we can define, in the usual way, an $\mathfrak{A}_{\mathfrak{R}}$-algebra and the free $\mathfrak{A}_{\mathfrak{R}}$-algebra generated by G.

Next, let $G, F, \mathfrak{A}$ and $\mathfrak{R}$, each being finite, be given. We then formulate, along the following steps, the word problem for the free $\mathfrak{A}_{\mathfrak{R}}$-algebra generated by G, as one of the metamathematics for a formal system $\mathfrak{S}$ constructed from G, F, $\mathfrak{A}$ and $\mathfrak{R}$.

(1) Words of $\mathfrak{S}$ are defined inductively, the elements of G being regarded as individual symbols and those of F as operation symbols, where the empty word is sometimes included.

(2) Deduction rules of $\mathfrak{S}$ are defined, each having one of the following seven types, being regarded the equality symbol contained in $\mathfrak{A}$ and $\mathfrak{R}$ as a proposition symbol, and letters X, Y,......some of which are already contained in $\mathfrak{A}$ as indeterminates, as metamathematical letters standing for words.

i) $\dfrac{}{a = b}$ if $a = b$ is a relation in $\mathfrak{R}$,

ii) $\dfrac{}{X = X}$,

iii) $\dfrac{X = Y}{Y = X}$,

iv) $\dfrac{X = Y,\ Y = Z}{X = Z}$,

v) $\dfrac{X_1 = Y_1, \ldots\ldots, X_n = Y_n}{\alpha(X_1, \ldots, X_n) = \alpha(Y_1, \ldots, Y_n)}$ if α is an n-ary operation in F,

vi) $\dfrac{}{\sigma(X, \ldots, Z) = \tau(X, \ldots, Z)}$ if $\sigma = \tau$ is an identity in $\mathfrak{A}$,

vii) $\dfrac{\sigma_1(X, \ldots, Z) = \tau_1(X, \ldots, Z), \ldots, \sigma_m(X, \ldots, Z) = \tau_m(X, \ldots, Z)}{\sigma(X, \ldots, Z) = \tau(X, \ldots, Z)}$

if $\sigma_1 = \tau_1$ & ...... & $\sigma_m = \tau_m \rightarrow \sigma = \tau$ is an implication in $\mathfrak{A}$.

Deduction rules of types (i), (ii) and (vi) mean the immediate assertibility, and the others the immediate deduci ility. (A generalization of Turing's formulation [10])

(3) Finite sequence of equalities

$$a_1 = b_1, \dots, a_s = b_s$$

is said to be a deduction (of $a_s = b_s$), if each equality of the sequence is an *immediate consequence* of preceding ones. If there exists a deduction of $a = b$, $a = b$ is said to be deduci le and we write $\vdash a = b$.

(4) If there exists a general recursive predicate R (Kleene [5], the formulation being based on Church's Thesis), defined for all pairs (a, b) of words, satisfying

$$\forall a \ \forall b \{R(a, b) \leftrightarrows \ \vdash a = b\},$$

where quantifiers as well as other logical symbols being understood intuitively, then the word problem is said to be solvable and R a solution.

From the so-called *universality* of free algebras, a half

$$\forall a \ \forall b \{ \vdash a = b \rightarrow R(a, b)\}$$

of the previous expression may be restated as follows:

'R is not only an equivalence relation but also a congruence relation and the system with the identification by R forms an $\mathfrak{A}_{\mathfrak{R}}$-algebra.' (This is a generalization of Whitman's proof idea [11].)

The word problem is said to be unsolvable, if for any general recursive predicate R

$$\exists a \ \exists b \{R(a, b) \not\leftrightarrows \ \vdash a = b\}.$$

(5) If there exists a general recursive predicate R defined for all finite set $\mathfrak{R}$ of relations and for all pairs (a, b) of words, satisfying

$$\mathrm{A}\mathfrak{R} \ \forall a \ \forall b \{R(\mathfrak{R}, a, b) \leftrightarrows \ \vdash_{\mathfrak{R}} a = b\}$$

(where the meaning of $\vdash_{\mathfrak{R}}$ may be clear), the word problem is said to be solvable for the class $\mathfrak{A}$ of algebras. (Evans [3] [4])

A finite set $\mathfrak{R}$ of relations being found such that the word problem for that individual free $\mathfrak{A}_{\mathfrak{R}}$-algebra is unsolva le, the word problem is said to be unsolvable for the class $\mathfrak{A}$ of algebras. As was indicated in Turing [10], this unsolva ility is, as well as that for individual algebras, stronger than the negation of the corresponding solvability. For, in all cases with the established unsolvability, the intuitive existence of $\mathfrak{R}$ and that of (a, b) for every R which intuitively refutes the equivalence between R and $\vdash$, is assured by Kleene's Enumeration Theorem or its equivalent forms (Kleene [5]).

REMARK:

1) It is clear that the restriction for G to be finite is inessential for the identification of words. We may furthermore say:

Let A and A^* be the free $\mathfrak{A}_{\mathfrak{R}}$-algebras generated by G and by G^* respectively and let $G \subset G^*$. Then the words from G are identified in A^*, (if and) only if they are in A.

2) The restriction for $\mathfrak{R}$ to be finite is, in a sense, essential, i.e., this enables us to put borderlands, so to speak, among several $\mathfrak{A}$'s, so that the priority of $\mathfrak{A}$ over $\mathfrak{R}$ is indeed assured.

3) There exists a duality between the condition for words to be identified by $\mathfrak{A}_{\mathfrak{R}}$ and the condition $\mathfrak{A}_{\mathfrak{R}}$ that rules the system thus identified. It is the former that the word problem concerns, and there is no, at least visible, correlation between its recursiveness and its 'strength or weakness'.

REFERENCES

[1] G. BIRKHOFF, Lattice Theory, revised edition, New York, 1948.

[2] ———, Some problems in lattice theory, Proc. Internat. Congress Math., 1950, (1952) vol. 2, pp. 4-7.

[3] T. EVANS, The word problem for abstract algebras, J, London Math. Soc., vol. 26 (1951), pp. 64-71.

[4] ———, Embeddability and the word problem, ibid., vol. 28(1953), pp. 76-80.

[5] C. KLEENE, Introduction to Metamathematics, Van Nostrand, New York, 1952.

[6] M P. SCHÜTZENBERGER, Construction du treillis modulaire engendré par deux éléments et une chaîne finie discrète, C, R. Acad. Sci. Paris t. 235(1952), pp. 926-928.

[7] ———, Le problème des mots dans les treillis modulaires libres, C. R. Acad. Sci. Paris, t. 237(1953), pp. 507-508.

[8] K. TAKEUCHI, On free modular lattices, Jap. J. Math., vol 21(1951), pp 53-65.

[9] R. M. THRALL AND D. G. DUNCAN, Note on free modular lattices, Amer. J. Math., vol, 75(1953), pp. 627-632.

[10] A. M. TURING, The word problem in semi-groups with cancellation, Ann. Math., vol. 52 (1950), pp. 49-1505.

[11] P. M, WHITMAN, Free lattices, Ann. Math., vol. 42(1941), pp. 325-330.

[12] ———, Free lattices II, ibid., vol. 43(1942), pp. 104-115.

MATHEMATICAL INSTITUTE
FACULTY OF SCIENCE
TOKYO METROPOLITAN UNIVERSITY.