ON REPLICAS

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0. In the paper [5] the author defined the replicas in the case of algebraic group defined over a field of characteristic 0 and characterized algebraic subalgebras of Lie algebras of algebraic groups. Now we shall define replicas in general case and show that these two definitions are same if the field of definition is of characteristic 0 and that in the case of algebraic groups of matrices the replicas defined here are same with those which were defined in [2] by means of tensor invariants.

We shall use the terminology in [5].

1. Let G be a connected algebraic group with the Lie algebra $\mathfrak{g}$; let $\Omega(G)$ be the field of rational functions of G ; for a field k of definition for G let $k(G)$ be the subfield of $\Omega(G)$ consisting of all rational functions defined over k. For $D \in \mathfrak{g}$ the subset $\Omega^D(G)$ of those $f \in \Omega(G)$ such that $Df = 0$ is a subfield of $\Omega(G)$. Let $\mathfrak{g}(D)$ be the subalgebra of $\mathfrak{g}$ consisting of those $D' \in \mathfrak{g}$ such that $D'f = 0$ for any $f \in \Omega^D(G)$.

DEFINITION *Any element of* $\mathfrak{g}(D)$ *is called a replica of* D *in* $\mathfrak{g}$.

In the section 2 we shall show that if the characteristic of the ground field is 0, the concept of replicas is independent of the ambient algebraic Lie algebra $\mathfrak{g}$. For simplicity we shall take one fixed algebraic group G and consider replicas in the Lie algebra $\mathfrak{g}$ of G without reference to the ambient algebraic Lie algebra $\mathfrak{g}$.

For a field k of definition for G, put $k^D(G) = k(G) \cap \Omega^D(G)$. Suppose that D is defined over k. Clearly the compositum of $k^D(G)$ and Ω is contained in $\Omega^D(G)$. Conversely any $\bar{f} \in \Omega^D(G)$ is expressed as a rational function of elements of $k^D(G)$ with coefficients in K, where K is any extension field of k such that $\bar{f}$ is defined over K.

In fact we may suppose that K is finitely generated over k. It is sufficient to show the fact in the next two cases ; (i) K is finite algebraic over k, (ii) K is simply transcendental over k. For the case (i) let $a_1, \ldots\ldots a_r$ be a k-base for K, then we may express $\bar{f} = a_1 f_1 + \ldots\ldots + a_r f_r$ for some $f_i \in k(G)$. So we have $D\bar{f} = a_1 Df_1 + \ldots\ldots + a_r Df_r = 0$, and $Df_i = 0$ since Df_i is in $k(G)$ and K and $k(G)$ are linearly disjoint over k. For the case (ii) let t be a transcendental element over k such that $K = k(t)$. Since K and $k(G)$ are linearly disjoint over k, t is transcendental over $k(G)$. We may express $\bar{f} = F(t)/H(t)$ for some $F(t) = \Sigma_i f_i t^i$, $H(t) = \Sigma_j h_j t^j \in k(G)\,[t]$ such that $F(t)$ and $H(t)$ are relatively prime in $k(G)\,[t]$. As $D\bar{f} = 0$, we have

$$(1) \qquad DF(t)\cdot H(t) = F(t)\cdot DH(t).$$

If one of $DF(t)$ and $DH(t)$ is 0, (1) implies that the other is 0. So $DF(t) = \Sigma_i Df_i \cdot t^i = 0$ and $DH(t) = \Sigma_j Dh_j \cdot t^j = 0$, i. e. $Df_i = 0$ and $Dh_j = 0$. Thus we may suppose that $DF(t) \neq 0$ and $DH(t) \neq 0$. Since $F(t)$ and $H(t)$ are relatively prime, (1) implies that $DF(t)$ is divided by $F(t)$. But degree of $DF(t) \leqq$ degree of $F(t)$. So there exists c in $k(G)$ such that $DF(t) = cF(t)$, and therefore $DH(t) = cH(t)$. We have $Df_i = cf_i$ and $Dh_j = ch_j$. So $Df_i \cdot h_j - f_i \cdot Dh_j = 0$. Since we may suppose $f_i \neq 0$, we have $\bar{f} = \Sigma_i \{t^i/\Sigma_j(h_j/f_i)t^j\}$, where $D(h_j/f_i) = 0$.

Thus we have shown

LEMMA 1. *Let k be a field of definition for G. If $D \in \mathfrak{g}$ is defined over k, $\Omega^D(G)$ is the compositum of $k^D(G)$ and Ω.*

And therefore

PROPOSITION 1. *Let k be a field of definition for G. Then for D and $D' \in \mathfrak{g}$ defined over k, D' is a replica of D if and only if $k^D(G)$ is contained in $k^{D'}(G)$.*

Let H be an algebraic subgroup of G; let k be a field of defintion for G and H; let $k_H(G)$ be the subfield of $k(G)$ consisting of f such that $L_h^* f = f$ for any point h on H, then we have

LEMMA 2. *Let H_1 and H_2 be algebraic subgroups of G; let k be a field of definition for G, H_1 and H_2. Then H_1 contains H_2 if and only if $k_{H_2}(G)$ contains $k_{H_1}(G)$.*

PROOF. Suppose that $k_{H_1}(G)$ is a subfield of $k_{H_2}(G)$. Let φ_i be the natural mapping of G into the homogeneous space G/H_i which is generically surjective rational mapping defined over k such that for two generic points y_1 and y_2 over k on G, $\varphi_i(y_1) = \varphi_i(y_2)$ if and only if $y_1 \in H_i y_2$ (cf. [6] theorem 2). Since $k_{H_1}(G)$ is a subfield of $k_{H_2}(G)$, there exists a rational mapping ρ of G/H_2 into G/H_1 such that $\varphi_1 = \rho\circ\varphi_2$. Let H_{2j} be any irreducible component of H_2; let $h \times x$ be a generic point over k on $H_{2j} \times G$, then $\varphi_1(hx) = \rho(\varphi_2(hx)) = \rho(\varphi_2(x)) = \varphi_1(x)$; hence $hx \in H_1 x$ and $h \in H_1$; therefore H_{2j} is contained in H_1. The converse is trivial.

q. e. d.

2. In this section we assume that the characteristic of the universal domain is 0. For $D \in \mathfrak{g}$, let $\bar{G}(D)$ be the algebraic subgroup of G consisting of all y such that $L_y^* f = f$ for any $f \in \Omega^D(G)$. If k is a field of definition for G and D, by the Lemma 1, $\bar{G}(D)$ consists of all $y \in G$ such that $L_y^* f = f$ for any $f \in k^D(G)$ and therefore $\bar{G}(D)$ is k-closed. Hence the connected component of $\bar{G}(D)$ containing the unit element is defined over k. On the other hand $k^D(G)$ is right invariant i. e. for any rational point p over k on G, R_p^* maps $k^D(G)$ into itself and $k^D(G)$ is algebraically closed in $k(G)$; in fact, let $\bar{f} \in k(G)$ be algebraic over $k^D(G)$; let $P(X)$ be the irreducible polynomial in $k^D(G)[X]$ of $\bar{f}$, then, the characteristic of k being 0, $DP(\bar{f}) = P'(\bar{f}) \cdot D\bar{f} = 0$ implies $D\bar{f} = 0$. Since by the Lemma 1 $\bar{G}(D)$ consists of all y such that $L_y^* f = f$ for any $f \in \bar{k}^D(G)$, the theorem of [1] shows that $\bar{G}(D)$ is connected. Thus we have

LEMMA 3. *For any $D \in \mathfrak{g}$, $\bar{G}(D)$ is connected, and if k is a field of definition for G and D, $\bar{G}(D)$ is defined over k.*

Let $\bar{\mathfrak{g}}(D)$ be the Lie algebra of $\bar{G}(D)$, then we obtain

LEMMA 4. *$\bar{\mathfrak{g}}(D)$ is contained in $\mathfrak{g}(D)$*

PROOF. Let k be a field of definition for G and D; let $h \times x$ be a generic point over k on $\bar{G}(D) \times G$; for $f \in k^D(G)$, from the definition of $\bar{G}(D)$, $f(hx) = f(x)$. Hence $R_x^* f - f(x)$ is in $k(x)(G) \cap \mathfrak{m}_h$, where $\mathfrak{m}_h$ is the maximal ideal of the local ring $\mathfrak{o}_h$ of h in $\Omega(G)$. For any $D' \in \bar{\mathfrak{g}}(D)$ defined over k, $D'(R_x^* f - f(x)) \in k(x)(G) \cap \mathfrak{m}_h$ and $D'(R_x^* f - f(x))(h) = 0$. But $D'(R_x^* f - f(x))(h) = (D' R_x^* f)(h) = (R_x^* D' f)(h) = (D' f)(hx)$. Since hx is generic over k on G and $D'f$ is in $k(G)$, we have $D'f = 0$. The Lie algebra $\bar{\mathfrak{g}}(D)$ over Ω having a base consisting of invariant derivations defined over k, we have the lemma.

q. e. d.

LEMMA 5. *If an algebraic subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contains D, $\mathfrak{h}$ contains $\bar{\mathfrak{g}}(D)$.*

PROOF. Let H be the connected algebraic subgroup of G whose Lie algebra is $\mathfrak{h}$; let k be a field of definition for G, H and D; let $k_H(G)$ and $k_{\bar{G}(D)}(G)$ be the subfields of $k(G)$ consisting of all f such that $L_y^* f = f$ for any $y \in H$ and $\bar{G}(D)$, respectively. By the definition of $\bar{G}(D)$, $k^D(G)$ is a subfield of $k_{\bar{G}(D)}(G)$. Let $h \times x$ be a generic point over k on $H \times G$. If $f \in k_H(G)$, the notations being as in the proof of the lemma 4, $R_x^* f - f(x) \in k(x) \cap \mathfrak{m}_h$. D being defined over k and in $\mathfrak{h}$, $D(R_x^* f - f(x)) \in k(x)(G) \cap$

$\mathfrak{m}_h$. As in the proof of the lemma 4 we have $Df=0$. Thus we have proved that $k_H(G)$ is contained in $k^D(G)$ and a fortiori in $k_{\bar{G}(D)}(G)$. The Lemma 2 shows that H contains $\bar{G}(D)$ and $\mathfrak{h}$ contains $\bar{\mathfrak{g}}(D)$.

q. e. d.

LEMMA 6. *$\mathfrak{g}(D)$ is contained in $\bar{\mathfrak{g}}(D)$.*

PROOF. Let k be a field of definition for G and D; let $x \times y$ be a generic point over k on $G \times G$; let (ξ) be a coordinate functions of G relative to an affine variety V in which the unit element e has a representative. For any $f(\xi) \in k^D(G)$ there exists $L(X, Y) \in k[X, Y]$ such that $L(x,y) \neq 0$ and $L(x, y)\{f(xy)-f(y)\} \in k[x, y]$. Then we have an expression

$$(2) \qquad L(x, y)\{f(xy)-f(y)\} = \Sigma_i P_i(x) \cdot F_i(y),$$

where the summation runs over some $P_i(x) \in k[x]$ and $F_i(y) \in k[y]$ such that these finite quantities $F_i(y)$ are linearly independent over k. From the definition we have

$$(3) \qquad L(x, \xi)\{L_x^* f(\xi) - f(\xi)\} = \Sigma_i P_i(x) \cdot F_i(\xi).$$

Since Ω and $k(G)$ are linearly disjoint over k, for $z \in G$ which has a representative in V, $L(z, \xi)\{L_z^* f(\xi) - f(\xi)\} = 0$ if and only if all $P_i(z)=0$.

Let $\mathfrak{E}$ be the set of all $P(X) \in k[X]$ such that $P(x)$ appears as one of $P_i(x)$ in some expression (3) for some $f \in k^D(G)$; put $\mathfrak{A} = \mathfrak{P} + \mathfrak{E} \cdot k[X]$ where $\mathfrak{P}$ is the ideal in $k[X]$ determined by V. Then for a point z of G which has a representative in V, z is in $\bar{G}(D)$ if and only if z is a zero of $\mathfrak{A}$; in fact, if z is in $\bar{G}(D)$, $P(z)=0$ for $P(X) \in \mathfrak{P}$; if $P(X)$ is in $\mathfrak{E}$, $P(X)$ is one of $P_i(X)$ in some expression (3) for some $f(\xi) \in k^D(G)$; since $z \in \bar{G}(D)$, $L_z^* f(\xi) = f(\xi)$ and $P_i(z)=0$, hence $P(z)=0$; conversely suppose that z is a zero of $\mathfrak{A}$; let $x \times y$ be a generic point over $k(z)$ on $G \times G$; for $f(\xi) \in k^D(G)$, $f(xy)-f(y)$ is in the specialization ring of $z \times y$ in $k(x, y)$, hence there exists $L(X, Y) \in k[X, Y]$ such that $L(z, y) \neq 0$ and $L(x, y)\{f(xy)-f(y)\} \in k[x, y]$; let $L(x, y)\{f(xy)-f(y)\} = \Sigma_i P_i(x)\, F_i(y)$ be an expression of the type (2), then $L(x, \xi)\{L^* f(\xi) - f(\xi)\} = \Sigma_i P_i(x)\, F_i(\xi)$ is of the type (3); thus $L(z, \xi)\{L_z^* f(\xi) - f(\xi)\} = 0$ and $L_z^* f(\xi) = f(\xi)$ since $L(z, \xi) \neq 0$.

Let D' be in $\mathfrak{g}(D)$; let k be a field of definition for G, D and D', then by the Lemma 3, $\bar{G}(D)$ is defined over k; let $h \times x \times y$ be a generic point over k on $\bar{G}(D) \times G \times G$. We shall denote by the same letter D' the $k(y)$-derivation of $k(y)(x)$ induced naturally by the Ω-derivation D' of $\Omega(G)$. Let $P(X) \in \mathfrak{E}$, then there exists an expression of the type (2)

$$L(x, y)\{f(xy)-f(y)\} = \Sigma_i P_i(x) \cdot F_i(y)$$

for some $f(\xi) \in k^D(G)$ such that $P(x)$ is one of $P_i(x)$. Applying D' on this equation we have

$$D'L(x, y)\cdot\{f(xy)-f(y)\} + L(x, y)\cdot D'f(xy) = \Sigma_i D'P_i(x)\cdot F_i(y).$$

But $D'f(xy) = (D'R_y^* f(\xi))(x) = (R_y^* D'f(\xi))(x) = (D'f(\xi))(xy)$ and since D' is in $\mathfrak{g}(D)$ and xy is generic over k on G, we have $(D'f(\xi))(xy) = 0$, and we have

$$D'L(x, y)\cdot\{f(xy)-f(y)\} = \Sigma_i D'P_i(x)\cdot F_i(y).$$

On the other hand, $L(x, y)$ being in $k[x, y]$, we have

$$L(x, y) = \Sigma_j S_j(x)\cdot T_j(y), \qquad \text{for some } S_j(X),\ T_j(X) \in k[X],$$

and $D'L(x, y) = \Sigma_j D'S_j(x)\cdot T_j(y)$. Since D' is everywhere finite, there exists $Q(X) \in k[X]$ such that $Q(h) \neq 0$ and $Q(x)D'S_j(x)$, $Q(x)D'P_i(x) \in k[x]$. Thus the expression

$$Q(x)D'L(x, y)\cdot\{f(xy)-f(y)\} = \Sigma_i Q(x)D'P_i(x)\cdot F_i(y)$$

is of the type (2). Hence, let $\bar{P}_i(X) \in k[X]$ such that $\bar{P}_i(x) = Q(x)D'P_i(x)$, then $\bar{P}_i(X) \in \mathfrak{E}$ and $D'P_i(x) = \bar{P}_i(x)/Q(x)$ where $Q(h) \neq 0$. Clearly if $P(X) \in \mathfrak{P}$, then $D'P(x) = 0$. Thus, since D' is everywhere finite, we have shown that for any $P(X) \in \mathfrak{A}$ there exist $A(X) \in \mathfrak{A}$ and $B(X) \in k[X]$ such that $B(h) \neq 0$ and $D'P(x) = A(x)/B(x)$.

Let $\bar{\mathfrak{Q}}$ be the set of those $F(X) \in k[X]$ for which there exists $L(X) \in k[X]$ snch that $L(h) \neq 0$ and $L(X)F(X) \in \mathfrak{A}$. Then by the lemma 5 of [8] III_3 $\bar{\mathfrak{Q}}$ is $\bar{\mathfrak{P}}$-primary, where $\bar{\mathfrak{P}}$ is the prime ideal in $k[X]$ determined by $\bar{G}(D)$. So the argument which has run in the proof of the Proposition 2 of [5] shows that D' is in the Lie algebra $\bar{\mathfrak{g}}(D)$ of $\bar{G}(D)$.

q. e. d.

From the Lemmas 4 and 6 it follows that $\mathfrak{g}(D) = \bar{\mathfrak{g}}(D)$ and $\mathfrak{g}(D)$ is algebraic. Let G_D be the minimal connected algebraic subgroup of G whose Lie algebra contains D (cf. the Corollary 1 of the Proposition 2 of [5]) ; let $\mathfrak{g}_D$ be the Lie algebra of G_D. Then from the definition of $\mathfrak{g}_D$ and the Lemma 5 it follows that $\mathfrak{g}_D = \mathfrak{g}(D)$. And the Corollary 2 of the Proposition 2 of [5] shows that $G_D = \bar{G}(D)$. So by the Lemma 3 we have that G_D is defined over k if k is a field of definition for G and D. Thus we have the main theorem ;

THEOREM 1. *Let G be a connected algebraic group with the Lie algebra $\mathfrak{g}$. If the characteristic of the universal domain is 0, for any $D \in \mathfrak{g}$ there exists the minimal connected algebraic subgroup G_D of G whose Lie algebra contains D. If k is a field of definition for G and D, G_D is defined over k. G_D is the algebraic subgroup of G consisting of all p for which $L_p^* f = f$ for any $f \in k(G)$ such that $Df = 0$. The Lie algebra $\mathfrak{g}_D$ of G_D is the subalgebra* *of* $\mathfrak{g}$ *consisting of all* D' *such that if* $Df = 0$ *for* $f \in k(G)$ *then* $D'f = 0$, *i. e. all replicas of D.*

By this theorem, in the case of characteristic 0 replicas defined in this paper are same with those in [5] and the concept of replicas is independent of the ambient algebra $\mathfrak{g}$.

3. Any algebraic group of matrices is an algebraic subgroup of $GL(n, \Omega)$ for some positive integer n. We shall take the Lie algebra of $GL(n, \Omega)$ as the ambient algebraic Lie algebra as far as algebraic groups of matrices are concerned. Let (u_{ij}) be the coordinate functions of $GL(n, \Omega)$. For any matrix $A = (\alpha_{ij}) \in \mathfrak{gl}(n, \Omega)$, which is the Lie algebra of $GL(n, \Omega)$ defined by Chevalley [3], we denote by $d(A)$ an Ω-derivation of $\Omega(u)$ such that $d(A)u_{ij} = \Sigma_{q=1}^{n}\alpha_{iq}u_{qj}$. Then $A \to -d(A)$ is an Ω-isomorphism of $\mathfrak{gl}(n, \Omega)$ onto the Lie algebra of $GL(n, \Omega)$ (cf. [5]). For $A \in \mathfrak{gl}(n, k)$ Chevalley [2] defined the replicas of A as follows ; let $\mathfrak{M}$ be a vector space over k on which A operates such as $Au_i = \Sigma_{j=1}^{n}\alpha_{ji}u_j$ where $u_1, \ldots\ldots, u_n$ is a base of $\mathfrak{M}$ over k ; let $\mathfrak{M}_{r,s} = \underbrace{\mathfrak{M}^* \otimes \mathfrak{M}^* \otimes \ldots\ldots \otimes \mathfrak{M}^*}_{r\text{-times}} \otimes \underbrace{\mathfrak{M} \otimes \mathfrak{M} \otimes \ldots\ldots \otimes \mathfrak{M}}_{s\text{-times}}$, where $\mathfrak{M}^*$ is the dual space of $\mathfrak{M}$ and $\otimes$ means the tensor product over k ; let $A_{r,s} = \underbrace{(-{}^tA) \dotplus (-{}^tA) \dotplus \ldots\ldots \dotplus (-{}^tA)}_{r\text{-times}} \dotplus \underbrace{A \dotplus A \dotplus \ldots\ldots \dotplus A}_{s\text{-times}}$, where $\dotplus$ means the Kronecker sum, then $A_{r,s}$ operate naturally on $\mathfrak{M}_{r,s}$; $A' \in \mathfrak{gl}(n, k)$ is called to be a (r, s)-replica of A if $A'_{r,s}\bar{u} = 0$ for $\bar{u} \in \mathfrak{M}_{r,s}$ such that $A_{r,s}\bar{u} = 0$, and A' is called to be a replica of A if A' is a (r, s)-replica for any non-zero pair r and s. We call a matrix A' to be a c-replica of A if A' is a replica of A in the sense of Chevalley. If the characteristic of the ground field is 0, the theorem 1 shows that the two definitions of replica are same if we identify a matrix $A \in \mathfrak{gl}(n, \Omega)$ and the element $-d(A)$ of the Lie algebra of $GL(n, \Omega)$. In the following we shall show directly this fact for the non-zero characteristic case at the same time with the characteristic zero case.

In the following sections we assume the algebraic closedness of k without loss of generality by the Proposition 1.

For a set of non-negative integers $e_{ij}(1 \leqq i, j \leqq n)$ let $u^e = u_{11}{}^{e_{11}}u_{12}{}^{e_{12}}\ldots\ldots u_{nn}{}^{e_{nn}}$; for a non-negative integer q let $\mathfrak{M}_q = \Sigma_{e_{11}+e_{12}+\ldots+e_{nn}=q}k\cdot u^e$ and $\bar{\mathfrak{M}}_q = \Sigma_{j\leqq q}\mathfrak{M}_j$, then $\mathfrak{M}_q$ and $\bar{\mathfrak{M}}_q$ are $d(A)$-invariant for any $A \in \mathfrak{gl}(n, k)$; let A_q and $\bar{A}_q$ be the matrix representations of the restrictions of $d(A)$ to $\mathfrak{M}_q$ and $\bar{\mathfrak{M}}_q$, respectively. Then we obtain

LEMMA 7. *If* A *is semisimple or nilpotent, then* A_q *is semisimple or nilpotent, respectively.*

PROOF. If A is semisimple, we may suppose that A is a diagonal matrix $(s_1, \ldots\ldots, s_n)$. Then $d(A)u_{ij} = s_i u_{ij}$, hence $d(A)u^e = s(e)u^e$, where $s(e) = \Sigma_{i,j=1}^n s_i e_{ij}$. Therefore A_q is diagonal, hence A is semisimple. Now in the set of elements of the base (u^e) for $\mathfrak{M}_q$ we introduce an order as follows; for $\mathfrak{M}_1$, $u_{ij} < u_{st}$ if and only if $i < s$ or $i = s$ and $j < t$. For $q > 1$, $u^e < u^{e'}$ if and only if there exist integers s and t such that $1 \leqq s$, $t \leqq n$ and $e_{ij} = e'_{ij}$ for $u_{ij} < u_{st}$ and $e_{st} > e'_{st}$. If A is nilpotent, we may suppose that $\alpha_{ij} = 0$ if $j \neq i - 1$. Then $d(A)u^e = \Sigma\, \alpha_{ii-1} e_{ij} u_{11}{}^{e_{11}} \ldots\ldots u_{i-1j}{}^{e_{i-1j}+1} \ldots\ldots u_{ij}{}^{e_{ij}-1} \ldots\ldots u_{nn}{}^{e_{nn}} = \Sigma\, \alpha_{ii-1} e_{ij} u^{\tau_{ij}(e)}$. Since $u^{\tau_{ij}(e)} < u^e$, A_q is nilpotent.

q. e. d.

Clearly $\overline{A}_q = A_1 \oplus \ldots\ldots \oplus A_q$, where $\oplus$ means the direct sum. Hence we have

LEMMA 8. *If A is semisimple or nilpotent, $\overline{A}_q$ is semisimple or nilpotent, respectively.*

LEMMA 9. *Let A and $B \in \mathfrak{gl}(n, k)$. If $[A, B] = 0$, then $[A_q, B_q] = 0$ and $[\overline{A}_q, \overline{B}_q] = 0$.*

PROOF. Let $f(u) \in k(u)$. Then $d(A)f(u) = \Sigma_{i,j=1}^n \partial f/\partial u_{ij} A u_{ij}$ and $d(B)d(A) f(u) = \Sigma_{i,j,s,t=1}^n \partial^2 f/\partial u_{st}\partial u_{ij} B u_{st} A u_{ij} + \Sigma_{i,j=1}^n \partial f/\partial u_{ij} BA u_{ij} = d(A)d(B)f(u)$. Hence $[d(A), d(B)] = 0$ and restricting $d(A)$ and $d(B)$ to $\mathfrak{M}_q$ and $\overline{\mathfrak{M}}_q$ we have the lemma. q. e. d.

For any matrix $A \in \mathfrak{gl}(n, k)$ there exist uniquely the semisimple matrix S and the nilpotent matrix N such that $A = S + N$ and $[S, N] = 0$. We shall call this decomposition of A the canonical decomposition of A. Since for any A and $B \in \mathfrak{gl}(n, k)$, $d(A + B) = d(A) + d(B)$, from the Lemmas 7, 8 and 9 follows the next lemma;

LEMMA 10. *If $A = S + N$ is the canonical decomposition of A, then $A_q = S_q + N_q$ and $\overline{A}_q = \overline{S}_q + \overline{N}_q$ are the canonical decompositions of A_q and $\overline{A}_q$, respectively.*

4. Let $S = \mathrm{diag.}(s_1, \ldots\ldots, s_n)$ in $\mathfrak{gl}(n, k)$; put $s(e) = \Sigma_{i,j=1}^n s_i e_{ij}$ for (e_{ij}). Another diagonal matrix $S' = (s'_1, \ldots\ldots, s'_n)$ is said to be a linear specialization of S if $\lambda_1 s'_1 + \ldots\ldots + \lambda_n s'_n = 0$ for any set of integers λ_i such that $\lambda_1 s_1 + \ldots \ldots + \lambda_n s_n = 0$.

Let $f = \Sigma_e \alpha_e u^e$ and $g = \Sigma_e \beta_e u^e$ be elements of $k[u]$ such that $d(S) f \neq 0$ and $d(S) g \neq 0$. Then we have

$$d(S)f \cdot g - f \cdot d(S)g = \Sigma_\rho\, \Sigma_{0 \leqq e \leqq \rho} (\alpha_e \beta_{\rho-e} - \alpha_{\rho-e}\beta_e)\, s(e) u^\rho,$$

where for (e_{ij}), (e'_{ij}), $e \leqq e'$ means $e_{ij} \leqq e'_{ij}$ for any pair i and j. Now if there exists non-zero $s(e)$ in the above equation, we express

$$\Sigma_{0\leqq e\leqq \rho}(\alpha_e\beta_{\rho-e} - \alpha_{\rho-e}\beta_e)\, s(e) = \Sigma_{i=1}^{r}\gamma_i x_i,$$

where x_i is one of non-zero $s(e)$ and $x_i \neq x_j$ if $i \neq j$.

Suppose that $d(S)(f/g) = 0$, then a simple calculation shows that

$$d(S)^q f\cdot g - f\cdot d(S)^q g = 0 \qquad \text{for } q = 1, 2, \ldots\ldots$$

and

$$\Sigma_{0\leqq e\leqq \rho}(\alpha_e\beta_{\rho-e} - \alpha_{\rho-e}\beta_e)\, s(e)^q = \Sigma_{i=1}^{r}\gamma_i x_i^{\,q} = 0$$

for $q = 1, 2, \ldots\ldots$ But $\det(x_i^{\,j})_{1\leqq i,\, j\leqq r} = x_1\ldots\ldots x_r\ \Pi_{i>j}(x_i - x_j) \neq 0$. So we have $\gamma_i = 0$.

Now suppose that S' is a linear specialization of S. Then if $s(e) = 0$, $s'(e) = 0$ and if $s(e) = s(e')$, then $s'(e) = s'(e')$. And it is easily seen that

$$\Sigma_{0\leqq e\leqq \rho}(\alpha_e\beta_{\rho-e} - \alpha_{\rho-e}\beta_e)\, s'(e) = 0.$$

Thus we have $d(S')(f/g) = 0$, i. e.

LEMMA 11. *Let S and S' be diagonal matrices in $\mathfrak{gl}(n, k)$; let f and g be in $k[u]$ such that $d(S)f \neq 0$ and $d(S)g \neq 0$. If S' is a linear specialization of S and $d(S)(f/g) = 0$, then $d(S')(f/g) = 0$.*

Now we have

PROPOSITION 2. *Let S be a semisimple matrix in $\mathfrak{gl}(n, k)$. Then for any matrix S' in $\mathfrak{gl}(n, k)$, S' is a c-replica of S if and only if $d(S')$ is a replica of $d(S)$.*

PROOF. We may suppose that S is diag. $(s_1, \ldots\ldots, s_n)$. Suppose that S' is a c-replica of S, then from the Theorems 1 and 3 of [2] it follows that S' is diag. $(s'_1, \ldots\ldots, s'_n)$ and it is a linear specialization of S. If $f = \Sigma_e \alpha_e u^e$ is in $k[u]$ such that $d(S)f = 0$, then $s(e) = 0$ for $\alpha_e \neq 0$, so $s'(e) = 0$ for $\alpha_e \neq 0$ and therefore $d(S')f = 0$. If $d(S)(1/f) = 0$, it is easily seen that $d(S')(1/f) = 0$. Thus from the Lemma11 it follows that $d(S')$ is a replica of $d(S)$.

Conversely suppose that $d(S')$ is a replica of $d(S)$. If $n = 1$, $S' = (s'_{ij})$ is diagonal. If $n > 1$, $d(S)u_{i1}/u_{i2} = 0$ for $i = 1, 2, \ldots\ldots, n$. So we have $d(S')(u_{i1}/u_{i2}) = 0$ and $d(S')u_{i1}\cdot u_{i2} - u_{i1}\cdot d(S')u_{i2} = 0$. Thus we have $s'_{ij} = 0$ for $i \neq j$, i. e. S' is diag. $(s'_1, \ldots\ldots, s'_n)$. If $\lambda_1, \ldots\ldots, \lambda_n$ are integers such that $\lambda_1 s_1 + \ldots\ldots + \lambda_n s_n = 0$, then $d(S)\,\Pi_{i=1}^{n} u_{ii}^{\lambda_i} = (\Sigma_{i=1}^{n}\lambda_i s_i)\,\Pi_{i=1}^{n} u_{ii}^{\lambda_i} = 0$ and therefore $d(S')\,\Pi_{i=1}^{n} u_{ii}^{\lambda_i} = 0$, and $\lambda_1 s'_1 + \ldots\ldots + \lambda_n s'_n = 0$. From the Theorem 3 of [2] it follows that S' is a c-replica of S.

q. e. d.

Thus, identifying a matrix $A \in \mathfrak{gl}(n, \Omega)$ with the element $-d(A)$ of the Lie algebra of $GL(n, \Omega)$, we have shown that the two definitions of replicas of semisimple matrices are same.

5. For a positive integer $s \leqq n$ let $\mathfrak{M}^{(s)}$ be the vector subspace of $k[u]$ spanned by $u_{i_1 1} u_{i_2 2} \cdots\cdots u_{i_s s}$ for $1 \leqq i_j \leqq n$, then $\mathfrak{M}^{(s)}$ is $d(A)$-invariant for any $A \in \mathfrak{gl}(n, k)$. Moreover there exists a k-isomorphism Ψ_s of $\mathfrak{M}_{0,s}$ onto $\mathfrak{M}^{(s)}$ such that $\Psi_s(u_{i_1} \otimes u_{i_2} \otimes \cdots\cdots \otimes u_{i_s}) = u_{i_1 1} u_{i_2 2} \cdots\cdots u_{i_s s}$. From the definition it follows that $\Psi_s A_{0,s} = d(A) \Psi_s$ for any $A \in \mathfrak{gl}(n, k)$. Thus we have

LEMMA 12. *Let A and $A' \in \mathfrak{gl}(n, k)$. For A' to be a $(0, s)$-replica of A it is necessary and sufficient that if $d(A)f = 0$ for $f \in \mathfrak{M}^{(s)}$ then $d(A')f = 0$, where s is a positive integer $\leqq n$.*

Let N be a nilpotent matrix in $\mathfrak{gl}(n, k)$, then by the theorem of Chevalley and Tuan (cf. [7]) $N' \in \mathfrak{gl}(n, k)$ is a c-replica of N if and only if $N' = \alpha N$ for some $\alpha \in k$ for the characteristic 0 case and $N' = \alpha_0 N + \alpha_1 N^p + \cdots\cdots + \alpha_r N^{p^r}$ for some $\alpha_i \in k$ for the modular case. In the later case $d(N^{p^i}) = d(N)^{p^i}$, hence if N' is a c-replica of N, $d(N')$ is a replica of $d(N)$. Conversely if $d(N')$ is a replica of $d(N)$, by the Lemma 12 N' is a $(0, s)$-replica of N for positive integer $s \leqq n$. So if $n \geqq 4$, by the lemma 1' of [4] N' is nilpotent and by the Theorem 1' of [4] N' is a c-replica. In the case of $n \leqq 3$, N' is a c-replica of N; in fact, if $n = 1$, the assertion is trivial; if $n = 2$, then $N^2 = 0$ and by the Lemma 12 and the Lemma 1' of [4] $N' = \alpha N$ for some $\alpha \in k$; in the case of $n = 3$, if $N^2 = 0$, the argument in the case of $n = 2$ gives the assertion; otherwise, we may suppose that $N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; since $N^3 = 0$, by the Lemma 12 and by the Lemma 1' of [4], $N' = \alpha N + \beta N^2$ for some $\alpha, \beta \in k$; if the characteristic of k is 2, the theorem of Chevalley and Tuan gives the assertion; otherwise, applying $d(N)$ and $d(N')$ on $2 u_{11} u_{31} - u_{21}^2$, we have $\beta = 0$. Thus we have

PROPOSITION 3. *Let N be a nilpotent matrix in $\mathfrak{gl}(n, k)$. Then for any matrix $N' \in \mathfrak{gl}(n, k)$, N' is a c-replica of N if and only if $d(N')$ is a replica of $d(N)$.*

Thus, identifying a matrix $A \in \mathfrak{gl}(n, \Omega)$ with the element $-d(A)$ of the Lie algebra of $GL(n, \Omega)$, we have shown that the two definitions of replicas of nilpotent matrices are same.

6. Let $N_0 = (n_{ij}) \in \mathfrak{gl}(n, k)$ such that $n_{i i-1} = 1$ and other $n_{ij} = 0$; let $A = \lambda E + N_0$ for some $\lambda \in k$, where E is the unit matrix of $\mathfrak{gl}(n, k)$. Then $d(A) u_{1j} = \lambda u_{1j}$ and $d(A) u_{ij} = \lambda u_{ij} + u_{i-1 j}$ for $i > 1$. Let $U = \det(u_{ij}) \in k[u]$, then we have

$$(4) \qquad d(A) U = n\lambda \cdot U.$$

In fact, we shall show (4) by the induction on the degree n. If $n = 1$, it is trivial ; let U_{ij} be the cofactor of the (i, j)-component u_{ij} of (u_{ij}). Then $d(A)U = d(A) \Sigma_{j=1}^n u_{nj}U_{nj} = \Sigma_{j=1}^n \lambda u_{nj}U_{nj} + \Sigma_{j=1}^n u_{n-1\,j}U_{nj} + \Sigma_{j=1}^n u_{nj}d(A)U_{nj}$. By the induction assumption $d(A)U_{nj} = (n-1)\lambda U_{nj}$ and we have (4).

In particular we have

$$d(N_0)U = 0. \tag{5}$$

As $d(A)\Pi_{j=1}^n u_{1j} = n\lambda\cdot\Pi_{j=1}^n u_{1j}$, we have

$$d(A)(U/\Pi_{j=1}^n u_{1j}) = 0 \qquad \text{for } A = \lambda E + N_0. \tag{6}$$

LEMMA 13. *Let A and $A' \in \mathfrak{gl}(n, k)$. If $A = \lambda E + N_0$ and $d(A')$ is a replica of $d(A)$, then $A' = \mu E + N'$, where $\mu \in k$ and N' is nilpotent and triangular.*[1]

PROOF. If $n = 1$, this lemma is trivial. Suppose that $n > 1$. From $d(A)(u_{11}/u_{12}) = 0$, it follows that $d(A')(u_{11}/u_{12}) = 0$ and $\alpha'_{1j} = 0$ for $j > 1$, where $A' = (\alpha'_{ij})$. Put $\alpha'_{11} = \mu$. For a positive integer $r \leqq n$ let $U_r = \det(u_{ij})_{1\leqq i,j\leqq r}$ and let U_{rj} be the cofactor of u_{rj} of U_r. From (6) $d(A')(U_r/\Pi_{j=1}^r u_{1j}) = 0$. So

$$\begin{aligned} &d(A')U_r\cdot\Pi_{j=1}^r u_{1j} - U_r\cdot d(A')\Pi_{j=1}^r u_{1j} \\ &= (d(A')U_r - r\mu U_r)\,\Pi_{j=1}^r u_{1j} = 0, \end{aligned}$$

hence

$$d(A')U_r = r\mu U_r \qquad 1 \leqq r \leqq n. \tag{7}$$

But

$$d(A')U_r = d(A')\Sigma_{j=1}^r u_{rj}U_{rj} = \Sigma_{j=1}^r d(A')u_{rj}U_{rj} + \Sigma_{j=1}^r u_{rj}d(A')U_{rj},$$

where the first term $= \alpha'_{rr}U_r + \Sigma_{t=r+1}^n \alpha'_{rt}\Sigma_{j=1}^n u_{tj}U_{rj}$ and the second term $= (r-1)\mu U_r$. Since $U_{rj} \in k[u_{11}, u_{12}, \ldots\ldots, u_{rr}]$, U_r and $u_{tj}U_{rj}$ $(t = r+1, r+2, \ldots\ldots n)$ are linearly independent over k. So by (7) $\alpha'_{rr} = \mu$ and $\alpha'_{rt} = 0$ if $t > r$.

q. e. d.

LEMMA 14. *Let $A_i = \lambda_i E_i + N_i$, where $\lambda_i \in k$, E_i is unit matrix of some degree and N_i is nilpotent and triangular matrix of the same degree ; let $A = A_1 \oplus A_2 \oplus \cdots\cdots \oplus A_r$ (direct sum). If $d(A')$ is a replica of $d(A)$, then $A' = A'_1 \oplus A'_2 \oplus \cdots\cdots \oplus A'_r$, where A'_i is of the same type with A_i.*

PROOF. We may suppose that $d(A)u_{ij} = \lambda_1 u_{ij}$ for $1 \leqq i \leqq s_1$, $1 \leqq j \leqq n$ and $d(A)u_{ij} = \lambda_1 u_{ij} + u_{i-1\,j}$ for $s_1 < i < t_2$, $1 \leqq j \leqq n$. Then the argument in the proof of the Lemma 13 gives the Lemma 14 for A_1 and A'_1. Similarly we obtain the Lemma 14.

q. e. d.

1) A matrix (a_{ij}) is called to be triangular if $a_{ij} = 0$ for $i < j$.

LEMMA 15. *Let $A = S + N$ be the canonical decomposition of A. Suppose that $d(A')$ is a replica of $d(A)$. Then if A' is semisimple or nilpotent, $d(A')$ is a replica of $d(S)$ or $d(N)$, respectively.*

PROOF. We may suppose that A and A' are of the type in the Lemma 14 and that if A' is semisimple $A'_q = \mu_q E_q$ for $q = 1, 2, \ldots\ldots, r$ and therefore A' is diagonal. For any q there exists $u_{i(q)1}$ such that $d(A)u_{i(q)1} = \lambda_q u_{i(q)1} = d(S)u_{i(q)1}$. Let $\gamma_1, \ldots\ldots, \gamma_r$ be a family of integers such that $\Sigma_{q=1}^r \gamma_q \lambda_q = 0$, then $d(S)\Pi_{q=1}^r u_{i(q)1}^{\gamma_q} = d(A)\Pi_{q=1}^r u_{i(q)1}^{\gamma_q} = 0$. Since $d(A')$ is a replica of $d(A)$, $d(A')\ \Pi_{q=1}^r u_{i(q)1}^{\gamma_q} = 0$ and therefore $\Sigma_{q=1}^r \gamma_q \mu_q = 0$. It is easily seen that A' is a linear specialization of S. By the Theorem 3 of [2] and the Proposition 2 $d(A')$ is a replica of $d(S)$.

If A' is nilpotent, then $A'_q = N'_q$. We may suppose that $\lambda_i \neq \lambda_j$ for $i \neq j$. Let $\mathfrak{E}_q$ be the set of all u_{ij} which belongs to the eigenspace of λ_q ; let $\mathfrak{U}_q = \Sigma_{u_{ij} \in \mathfrak{E}_q} k \cdot u_{ij}$. For a set $e = (e_1, \ldots\ldots, e_r)$ of non-negative integers, put $\mathfrak{M}_e = \mathfrak{U}_1^{e_1} \ldots\ldots \mathfrak{U}_r^{e_r}$. Then the vector space $\mathfrak{M}_e$ is invariant under $d(A)$, $d(A')$, $d(S)$ and $d(N)$. For $1 \leqq s \leqq n$, $\mathfrak{M}^{(s)} = \Sigma_{e_1 + \ldots + e_r = s} \mathfrak{M}_e \cap \mathfrak{M}^{(s)}$ (direct sum). For $f = \Sigma f_e \in \mathfrak{M}^{(s)}$ where $f_e \in \mathfrak{M}_e$ we have that $d(N)f = 0$ if and only if $d(N)f_e = 0$ for all e. But $d(A)f_e = d(S)f_e + d(N)f_e = \lambda(e)f_e + d(N)f_e$ where $\lambda(e) = \Sigma_{q=1}^r \lambda_q e_q$. For any e put $u_e = \Pi_{q=1}^r u_{i(q)1}^{e_q}$, then we have $d(S)u_e = \lambda(e)u_e$, $d(N)u_e = 0$ and $d(A')u_e = 0$. Thus $d(N)f_e = 0$ if and only if $d(A)(f_e/u_e) = 0$. and $d(A')(f_e/u_e) = 0$ if and only if $d(A')f_e = 0$. Since $d(A')$ is a replica of $d(A)$, it is easily seen that if $d(N)f = 0$ for $f \in \mathfrak{M}^{(s)}$ then $d(A')\, f = 0$. By the Lemma 12 A' is a $(0, s)$-replica of N for $s \leqq n$, and therefore if $n \neq 3$ or if $n = 3$ and the characteristic of k is 2, the argument in the proof of the Proposition 3 gives the lemma. If $n = 3$ and the characteristic of k is not 2, we may suppose that $N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $A = \lambda E + N$, where E is the unit matrix in $\mathfrak{gl}(3, k)$. As in the proof of the Proposition 3 we have that $A' = \alpha N + \beta N^2$ for some $\alpha, \beta \in k$. Applying $d(A)$ and $d(A')$ on $(2u_{11}u_{31} - u_{21}^2)/u_{11}^2$ we have the lemma.

q. e. d.

LEMMA 16. *For a canonical decomposition $A = S + N$,*

$$k^{d(A)}(G) = k^{d(S)}(G) \cap k^{d(N)}(G),$$

where $G = GL(n, \Omega)$.

PROOF. For any $f/g \in k^{d(A)}(G)$ we have $d(A)f \cdot g - f \cdot d(A)g = 0$ and further $d(A)^r f \cdot g - f \cdot d(A)^r g = 0$ for any positive integer r. We may suppose that f and g are in $\overline{\mathfrak{M}}_i$ for some integer i. So $\overline{A}_i^r f \cdot g - f \cdot \overline{A}_i^r g = 0$. By the

Lemma 10, $\bar{A}_i = \bar{S}_i + \bar{N}_i$ is the canonical decomposition of $\bar{A}_i$ and there exists a polynomial $F(X) \in k[X]$ such that $S_i = F(\bar{A}_i)\bar{A}_i$. Thus we have $F(\bar{A}_i)\bar{A}_i f \cdot g - f \cdot F(\bar{A}_i)\bar{A}_i g = 0$ and therefore $d(S)(f/g) = 0$. Similarly $d(N)(f/g) = 0$.

Since $d(A) = d(S) + d(N)$, the converse is trivial.

q. e. d.

Let $A = S + N$ and $A' = S' + N'$ be canonical decompositions, then from the Lemmas 15 and 16 it follows that $d(A')$ is a replica of $d(A)$ if and only if $d(S')$ is a replica of $d(S)$ and $d(N')$ is a replica of $d(N)$. By the Theorem 5 of [2] and the Propositions 2 and 3 we obtain

THEOREM 2. *For A and $A' \in \mathfrak{gl}(n, k)$, A' is a c-replica of A if and only if $d(A')$ is a replica of $d(A)$.*

Thus identifying a matrix $A \in \mathfrak{gl}(n, \Omega)$ with the element $-d(A)$ of the Lie algebra of $GL(n, \Omega)$, we have shown that the two definitions of replicas are same.

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