A REMARK ON LOCALLY FLAT INFINITESIMAL CONNECTIONS.

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1. Let $P(B, G)$ be a differentiable principal fibre bundle over the base manifold B with the structural Lie group G. Suppose a connection Γ is defined by the differentiable distribution $x \rightarrow Q_x$, where $x \in P$ and Q_x is the subspace of the tangent space P_x of P at x [1] [2].*) Let G_x be the subspace of P_x tangent to the fibre through x. Denote the canonical projection of P onto B as π.

Now we prove the following probably known theorem for later use:

THEOREM 1. *Connection Γ in $P(B, G)$ is locally flat if and only if the differentiable distribution $x \rightarrow Q_x$ is involutive.*

PROOF. For any neighborhood U in B such that $\pi^{-1}(U)$ is isomorphic with $U \times G$, we take n vector fields $X_1, X_2, \ldots\ldots, X_n$ in B which span the tangent space T_u of B at each point $u \in U$. Then the lifts $X_1^*, X_2^*, \ldots\ldots, X_n^*$ of them span the horizontal subspace Q_x at each point $x \in \pi^{-1}(U)$. Let $v[X_i^*, X_j^*]$ be the vertical component of $[X_i^*, X_j^*]$. From the structure equation, and the relation $2d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ for each pair of vector fields X and Y it follows that $\omega_x([X_i^*, X_j^*]) = \omega_x(v[X_i^*, X_j^*]) = -2\Omega_x(X_i^*, X_j^*)$ hold at each point x, where ω is the connection form and Ω is the curvature form. Let Y_1, Y_2 be any two tangent vectors at $x \in P$ and hY_1, hY_2 be respectively their horizontal components. Then, as X_i^*'s span Q_x and hY_α $(\alpha = 1, 2)$ are linear combinations of them, $\Omega_x(Y_1, Y_2) = \Omega_x(hY_1, hY_2)$ vanishes provided all $\Omega_x(X_i^*, X_j^*)$'s vanish. Since the distribution $x \rightarrow Q_x$ is involutive if and only if all $[X_i^*, X_j^*]$'s are horizontal, from the above formula it follows that the considered distribution is involutive if and only if the curvature form is equal to zero, that is the connection is locally flat.

COROLLARY. *Let $P(B, G)$ be a principal fibre bundle over the simply connected base manifold B. If the distribution defining the connection is involutive, then $P(B, G)$ is the direct product $B \times G$.*

This corollary follows immediately from the above theorem and a corollary of [2] (p. 41).

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*) Numbers in brackets refer to the references at the end of the paper. Definitions and notations in the present paper are adopted from the book of K. Nomizu [2].

Before considering a generalization of the above theorem, we give a definition for preparation: Let B' be a submanifold of B, then the principal fibre bundle $P'(B', G)$ can be seen as a sub-bundle of $P(B, G)$. If $x' \in P'(B', G)$ and $\pi(x') = u' \in B'$, the inverse image of the tangent space $T'_{u'}$ of B' under the isomorphism π from $Q_{x'}$ onto the tangent space $T_{u'}$ of B is denoted as $Q'_{x'}$. Then the distribution $x' \to Q'_{x'}$ defines a connection Γ' in $P'(B', G)$ which is called the naturally induced connection of Γ. Then we have the following:

THEOREM 2. *For the base manifold B of $P(B, G)$ to have a system of locally flat (in the sense of naturally induced connection) submanifolds (one and only one of them through each point of B), it is necessary and sufficient that the distribution $x \to Q_x$ defining the connection admits an involutive differentiable subdistribution $x \to Q'_x$ ($Q'_x \subset Q_x$ at each x) satisfying $R_a Q'_x = Q'_{xa}$ for any $a \in G$ and $x \in P$, where R_a represents the mapping $x \to xa$ as well as its differential.*

PROOF. Suppose the distribution $x \to Q_x$ admits an involutive differentiable subdistribution $x \to Q'_x$ satisfying $R_a Q'_x = Q'_{xa}$. The canonical projection π from P onto B maps Q_x isomorphically onto T_u, the tangent space of B at $u = \pi(x)$. By this mapping the subdistribution $x \to Q'_x$ is mapped to a differentiable distribution $u \to T'_u \subset T_u$, which is independent of the choice of x covering u as $R_a Q'_x = Q'_{xa}$. Let U be a neighborhood in B such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. In U we can take s vector fields X_1, $X_2, \ldots\ldots, X_s$ so that these vector fields span T_u' at each $u \in B$. Let $X_1^*, X_2^*, \ldots\ldots, X_s^*$ be their lifts, then these vector fields span Q'_x at every $x \in \pi^{-1}(U)$. Since Q' is involutive, all $[X_i^*, X_j^*]$'s $(1 \leqq i, j \leqq s)$ are horizontal and by the structure equation we have $\omega_x([X_i^*, X_j^*]) = -2\Omega_x(X_i^*, X_j^*) = 0$ for every x and $1 \leqq i, j \leqq s$. Since $\pi[X_i^*, X_j^*] = [\pi X_i^*, \pi X_j^*] = [X_i, X_j]$ (because X_i^* and X_i $(i = 1, \ldots\ldots, s)$ are π-related) and that $[X_i^*, X_j^*]$'s are linear combinations of X_k^*'s it is clear that $[X_i, X_j]$'s are linear combinations of X_k's $(1 \leqq i, j, k \leqq s)$; that is T_u' is involutive. Hence there exists a family of submanifolds, one and only one of them through each point $u \in B$. Let $B'(u_0)$ be a maximal integral manifold of the considered distribution T_u' passing through $u_0 \in B$. Consider the principal fibre sub-bundle $P'(B', G)$. Then the distribution $x' \to Q'_{x'}$ for $x' \in P'$ defines a connection Γ' in P', and Γ' is locally flat. In fact, let ω' and Ω' be respectively the connection form and curvature form of Γ', then by structure equation, we have $\omega'_{x'}([X_i^*, X_j^*]) = \omega'_{x'}(v[X_i^*, X_j^*]) = -2\Omega'_{x'}(X_i^*, X_j^*)$. Since ω and ω' act on G_x by the same way, we have $0 = \omega_{x'}(v[X_i^*, X_j^*]) = \omega'_{x'}v([X_i^*, X_j^*]) = -2\Omega'_{x'}(X_i^*, X_j^*)$ at every $x' \in P'$ and $1 \leqq i, j \leqq s$. Thus the curvature form Ω' vanishes at every point $x' \in$

P'. Hence Γ' is locally flat. Similarly every integral manifold is locally flat.

Conversely, suppose B has a system of locally flat submanifolds (one and only one through each point), then the tangent spaces of the submanifolds at each point of B form a differentiable distribution T_u', which is involutive [3]. In a suitable neighborhood, let T_u' be spanned by s vector fields $X_1, X_2, \ldots\ldots, X_s$ at each point of the neighborhood. Then their lifts $X_1^*, X_2^*, \ldots\ldots, X_s^*$ give rise to a differentiable distribution Q_x' in P, and evidently $R_a Q_x' = Q_{xa}'$. As mentioned above, the distribution $x \to Q_x'$ defines a connection Γ' in $P(B', G)$, where B' is any submanifold and $x \in P'$. Then $\omega_x'(v[X_i^*, X_j^*]) = \omega_x'([X_i^*, X_j^*]) = -2\Omega_x'(X_i^*, X_j^*) = 0$, as B' is locally flat. Hence we have $\omega_x([X_i^*, X_j^*]) = \omega_x'([X_i^*, X_j^*]) = 0$ for every point $x \in P$, since x is contained in a certain $P'(B', G)$. Thus $[X_i^*, X_j^*]$'s are horizontal $(1 \leqq i, j \leqq s)$. Finally, from $\pi[X_i^*, X_j^*] = [X_i, X_j]$, we see that $[X_i^*, X_j^*]$ is a linear combination of X_k^*'s $(1 \leqq i, j, k \leqq s)$. Thus $x \to Q_x'$ is involutive, and Theorem 2 is proved.

2. Let $E(B, F, G, P)$ be an associated fibre bundle of the differentiable principal fibre bundle $P(B, G)$ having standard fibre F on which G acts effectively. It is known that there exists a natural one-to-one correspondence between the set of connections in P and the set of connections in E [2].

A connection in $E(B, F, G, P)$ is called locally flat if the corresponding connection in $P(B, G)$ is locally flat.

We like to prove analogous results of above theorems for the associated fibre bundle $E(B, F, G, P)$ of $P(B, G)$.

LEMMA. *Let the distributions $x \to Q_x$ and $e \to Q_e$ define the corresponding connections in $P(B, G)$ and $E(B, F, G, P)$ respectively. Then a differentiable subdistribution $x \to Q_x'$ of $x \to Q_x$ satisfying $R_a Q_x' = Q_{xa}'$ in P is involutive, if and only if the corresponding differentiable subdistribution $e \to Q_e'$ of $e \to Q_e$ satisfying $\pi Q_e' = \pi Q_f'$ for $\pi(e) = \pi(f)$ is involutive.*

PROOF. Given a distribution Q_x in P, the distribution defining the corresponding connection in E is obtained by the following way: For any point e_0 of E, take a point $x_0 \in P$ such that $\pi(e_0) = u_0 = \pi(x_0)$. As $x_0 \in P$ may be regarded as a mapping of F onto the fibre F_{u_0}, there is a $\xi_0 \in F$ such that $x_0 \cdot \xi_0 = e_0$. Consider a differentiable mapping ϕ: $x \in P \to x \cdot \xi_0 \in E$ for the fixed element ξ_0. Then the tangent subspace Q_{e_0} is defined to be the image of the horizontal subspace Q_{x_0} by the differential of ϕ. By this mapping ϕ, the subdistribution Q_{x_0}' $(\subset Q_{x_0})$ is mapped into a subdistribution $Q_{e_0}'(\subset Q_{e_0})$. From the conditions $R_a Q_x = Q_{xa}$ and $R_a Q_x' = Q_{xa}'$, it follows that Q_{e_0}' as well as Q_{e_0} do not depend on the choice of x_0 and ξ_0. Let

$\pi(e_0) = \pi(f_0)$, then $f_0 = x_0 \cdot (a\xi_0) = (x_0 a^{-1}) \cdot \xi_0$. So if $e_t = x_t \cdot \xi_0$ is the integral curve from e_0 of the distribution $e \to Q_e$ which covers u_t (of B through u_0), then $f_t = (x_t a^{-1}) \cdot \xi_0$ is the integral curve from f_0 covering u_t. If the tangent of e_t at e_0 is contained in Q'_{e_0}, then the tangent of f_t at f_0 is contained in Q'_{f_0}, and both these tangents cover the tangent of u_t at u_0. Therefore, we have $\pi Q'_{e_0} = \pi O'_{f_0}$. Finally, if the subdistribution $x \to Q'_x$ is involutive, the subdistribution $e \to Q'_e$ is also involutive. In fact, for a suitable coordinate neighborhood U of B, we can take s vector fields X_1^*, $X_2^*, \ldots\ldots, X_s^*$ which span Q'_x at each point x of $\pi^{-1}(U)$, then the vector fields $\phi X_1^*, \ldots\ldots, \phi X_s^*$ span Q'_e at each point e of $\pi^{-1}(U)$ in E. It is evident that if $X_1^*, X_2^*, \ldots\ldots, X_s^*$ are involutive, then $\phi X_1^*, \phi X_2^*, \ldots\ldots, \phi X_s^*$ are also involutive.

Conversely, given a connection in E, the distribution defining the corresponding connection in P is constructed by the following way: Let $\tau = \{u_t \mid 0 \leqq t \leqq 1\}$ be an arbitrary curve in B and x_0 be a point in P such that $\pi(x_0) = u_0$. Let C_t be a family of isomorphisms of the fibre F_{u_0} onto F_{u_t} then corresponding to u_t and x_0, there is a uniquely determined curve x_t in P such that $C_t(x_0 \cdot \xi) = x_t \cdot \xi$ for every $\xi \in F$. We define Q_{x_0} to be the set of tangent vectors to the curves x_t which correspond to all curves u_t starting at u_0 in B. Let Q'_{x_0} be the subset of Q_{x_0} which consists of the tangent vectors to the curves x'_t corresponding to all the curves u'_t whose tangent vectors at u_0 are contained in $\pi Q'_{e_0}$. It can be easily shown that $R_a Q_{x_0} = Q_{x_0 a}$ and $R_a Q'_{x_0} = Q'_{x_0 a}$ hold. It is also easy to show that $x \to Q_x$ is a connection in P from which the original connection $e \to Q_e$ is derived in the above manner, and $e \to Q'_e$ is derived from $x \to Q'_x$ at the same time. Hence $x \to Q'_x$ is involutive if $e \to Q'_e$ is involutive, as the vector fields spanning $x \to Q'_x$ and those spanning $e \to Q'_e$ are ϕ-related and ϕ is an isomorphism of Q_x onto Q_e.

From this lemma and the above Theorems 1 and 2, we have respectively the following:

THEOREM 3. *Connection* Γ *in* $E(B, F, G, P)$ *is locally flat if and only if the distribution* $e \to Q_e$ *is involutive* [4].

THEOREM 4. *For the base manifold* B *of* $E(B, F, G, P)$ *to have a system of locally flat (in the sense of naturally induced connection) submanifolds (one and only one of them through each point of* B*), it is necessary and sufficient that the distribution* $e \to Q_e$ *defining the connection in* E *admits an involutive differentiable subdistribution* $e \to Q'_e$ *satisfying* $\pi Q'_e = \pi Q'_f$ *for* $\pi(e) = \pi(f)$.

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