ON THE COMPLETE CONVERGENCE OF THE RIEMANN SUM

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1. In the present note $f(t)$, $-\infty < t < +\infty$, will denote a function with period 1 and $(L)\int_0^1 f(t)\,dt = 0$. Then we put, for $k = 1, 2, \ldots\ldots$,

$$F_k(t, f) = k^{-1}\sum_{\nu=1}^{k} f(t + \nu k^{-1})$$

which is known as the Riemann sum of the function $f(t)$.

B. Jessen has proved that if $n_k \mid n_{k+1}$, then for almost all t,

$$\lim_{k\to\infty} F_{n_k}(t, f) = 0. \tag{1.1}$$

Since then the problem whether (1.1) holds or not for a given increasing sequence $\{n_k\}$ has been discussed. But it is not yet answered for the question "Does there exist a function $f(t)$ such that $f(t) \in L_2(0, 1)$ and $F_k(t,f)$ does not converge to 0 almost everywhere in t ?"

On the other hand various sufficient conditions of (1.1) without assuming any arithmetical property of $\{n_k\}$ have been given ([3], [4], [5] and [6]) and from them it follows that, for any $\varepsilon > 0$,

$$\sum \mid (0 \leqq t \leqq 1,\ |F_{n_k}(t,f)| > \varepsilon) \mid < +\infty. \tag{1.2}$$

Following P.L. Hsu and H. Robbins [1] if (1.2) holds, then we say that $F_{n_k}(t,f)$ converges to 0 "*completely.*" However, $F_{n_k}(t,f)$ converges to 0 completely if and only if $F_{n_k}(t + h_k, f)$ converges to 0 almost everywhere in t for any sequence of real numbers $\{h_k\}$. The necessity is obvious. To prove the sufficiency consider $\{F_{n_k}(t + t_k(w), f)\}$ where $\{t_k(w)\}$ is a sequence of independent random variables such that Prob. $\{t_k(w) \quad x\} = x$ for $0 \leqq x \quad 1$. If $F_{n_k}(t + t_k(w), f)$ converges to 0 almost everywhere in t for any fixed w, then $F_{n_k}(t + t_k(w), f)$ converges to 0 with probability one for suitably fixed t. From the lemma of Borel-Cantelli, (1.2) follows.

Hence if $f(t) \in R(0, 1)$, then $F_k(t,f)$ converges to 0 completely. Further by the above mentioned remark, there exist functions $f(t)$ such that $f(t) \overline{\in} R(0, 1)$ and $F_k(t,f)$ converges to 0 completely. Therefore it seems to be natural that we consider the complete convergence of the Riemann sum.

2. The following theorem shows that an analogous theorem to that of

B. Jessen does not hold with respect to the complete convergence.

THEOREM. *Let $\{n_k\}$ be any increasing sequence of positive integers. Then there exists a function $f(t)$ such that $f(t) \in L_p(0,1)$, $1 \leqq p < +\infty$, and $F_{n_k}(t,f)$ does not converge to 0 completely.*

PROOF. Let us put, for $m = 1, 2, \ldots\ldots$,

$$a_m = \{2m^{-1} \log^{-2}(m+1)\}^{1/2} \tag{2.1}$$

and

$$N_m = \prod_{k=1}^{m} n_k. \tag{2.2}$$

Then the series $\sum_{m=1}^{\infty} a_m \cos 2\pi N_m t$ converges to a function $f(t)$ of the class $L_p(0,1)$, $1 \leqq p < +\infty$ (c. f. [7]) and we have

$$F_{n_k}(t,f) = \sum_{m=k'}^{\infty} a_m \cos 2\pi N_m t \qquad (k' \leqq k).$$

Hence for the proof of the theorem it is sufficient to prove that, for $k > k_0$,

$$|(0 \leqq t \leqq 1,\ R_k(t) > 2^{-1})| \geqq k^{-1}, \tag{2.3}$$

where

$$R_k(t) = \sum_{m=k}^{\infty} a_m \cos 2\pi N_m t. \tag{2,3'}$$

Next let us expand all real numbers t, $0 \leqq t \leqq 1$, as follows:

$$t = \sum_{k=1}^{\infty} \varphi_k(t) N_k^{-1} \qquad (\varphi_k(t) = 0, 1, \ldots\ldots, n_k - 1), \tag{2.4}$$

and put

$$\theta_m(t) = \varphi_{m+1}(t) n_{m+1}^{-1} \tag{2.4'}$$

and

$$S_k(t) = \sum_{m=k}^{\infty} a_m \cos 2\pi\theta_m(t). \tag{2.4''}$$

Then we have

$$\begin{aligned} |R_k(t) - S_k(t)| &\leqq \sum_{m=k}^{\infty} a_m\, |\cos 2\pi N_m t - \cos 2\pi\theta_m(t)| \\ &\leqq 2\pi \sum_{m=k}^{\infty} a_m N_m \sum_{l=m+2}^{\infty} n_l . N_l^{-1} \leqq 2\pi \sum_{m=k}^{\infty} a_m (m+1)^{-1} \leqq A a_k, \end{aligned} \tag{2.5}$$

where A is a constant independent of k. Hence we have, by (2. 5),

$$\left|\int_0^1 S_k(t)\,dt\right| \leqq Aa_k \tag{2.6}$$

and

$$\left|\int_0^1 S_k^2(t)\,dt - \left(\int_0^1 S_k(t)dt\right)^2 - \int_0^1 R_k^2(t)dt\right| \tag{2.6'}$$
$$\leqq Aa_k\int_0^1 |S_k(t) + R_k(t)|\,dt + A^2a_k^2$$
$$\leqq Aa_k\left(2\int_0^1 |R_k(t)|\,dt + Aa_k\right) + A^2a_k^2$$
$$\leqq 2Aa_k\left(\int_0^1 R_k^2(t)\,dt\right)^{1/2} + 2A^2a_k^2 \leqq Ba_k$$

where B is a constant independent of k.

On the other hand $S_k(t) - \int_0^1 S_k(t)\,dt$ is the sum of independent functions $a_m\{\cos 2\pi\theta_m(t) - \int_0^1 \cos 2\pi\theta_m(t)\,dt\}$ and by (2.1) (2.3′) and (2.6′), it is seen that

$$\left|\int_0^1 S_k^2(t)\,dt - \left(\int_0^1 S_k(t)\,dt\right)^2 - \log^{-1}(k+1)\right| \leqq Ba_k. \tag{2.7}$$

Hence we apply the lemma of A. N. Kolmogorov [2] to $S_k(t) - \int_0^1 S_k(t)\,dt$ and obtain, for $k > k_0$,

$$\left|\left(0 \leqq t \leqq 1,\ S_k(t) - \int_0^1 S_k(t)\,dt > 1\right)\right| \geqq k^{-1}. \tag{2.8}$$

By (2.5), (2.6) and (2.8) we can obtain (2.3).

We can not see whether there exists a bounded function or not whose Riemann sum does not converge to 0 completely.

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