ON THE JACOBSON RADICAL OF A SEMIRING

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The concept of the Jacobson radical of a semiring has been introduced internally by S. Bourne ([1]). Recently, by associating a suitable ring with the semiring, S. Bourne and H. Zassenhaus have defined the semiradical of the semiring ([2]). In [3] it has been proved that the concepts of the Jacobson radical and the semiradical coincide. Consequently some properties of the Jacobson radical of the semiring are reduced to those in the ring theory. For example, the fact "If R is the Jacobson radical of a semiring S with a unit element, then R_n is the Jacobson radical of the matrix semiring S_n ([1])" is deduced immediately from the corresponding result in the ring theory.

The purpose of this paper is to consider the Jacobson radical of a semiring from the point of view of the representation theory[1] without reducing it to the ring theory. In § 1 we shall describe some preliminary definitions and propositions. In § 2, we shall define the irreducible representations and the radical of a semiring and prove some fundamental properties of the radical which correspond to those in the ring theory. In § 3, the external notion of the radical will be related to internal one, at the same time, we shall see that the radical defined in this paper coincides with the Jacobson radical and with the semiradical of the semiring. In the last section we shall consider some of the results obtained in the preceding sections from the point of view of the ring theory and give some examples.

1. Preliminaries. In this paper we shall assume that a *semiring* $\mathfrak{A}$ is commutative relative to addition and has a zero element. A commutative additive semigroup $\mathfrak{M}$ with a zero element is called a *right* $\mathfrak{A}$ *semimodule* if and only if a law of composition on $\mathfrak{M} \times \mathfrak{A}$ into $\mathfrak{M}$ is defined which, for $x, y \in \mathfrak{M}$ and $a, b \in \mathfrak{A}$, satisfies (a) $(x+y)a = xa + ya$, (b) $x(a+b) = xa + xb$ and (c) $x(ab) = (xa)b$. Henceforth the term "$\mathfrak{A}$-semimodule" without modifier will always mean right $\mathfrak{A}$-semimodule. The semiring $\mathfrak{A}$ itself is an $\mathfrak{A}$-semimodule relative to right multiplication as semimodule composition. A subset $\mathfrak{N}$ of $\mathfrak{M}$ is called an $\mathfrak{A}$-*subsemimodule* of $\mathfrak{M}$ if and only if

1) This notion has been used by N. Jacobson ([5]).

(i) if $y, z \in \mathfrak{N}$, $y + z \in \mathfrak{N}$,

(ii) for all $y \in \mathfrak{N}$ and for all $a \in \mathfrak{A}$, $ya \in \mathfrak{N}$,

(iii) $\mathfrak{N}$ contains the zero of $\mathfrak{M}$.

$\mathfrak{J}$ is called a right ideal in $\mathfrak{A}$ if and only if it is an $\mathfrak{A}$-subsemimodule of $\mathfrak{A}$ as an $\mathfrak{A}$-semimodule.

DEFINITION 1. An equivalence relation ρ defined in an $\mathfrak{A}$-semimodule $\mathfrak{M}$ is called *linear* if and only if

(i) $x\rho x'$ and $y\rho y'$ imply $(x + y)\rho(x' + y')$.

(ii) $x\rho y$ implies $(xa)\rho(ya)$ for all a $\in \mathfrak{A}$.

We say that a linear equivalence relation ρ *admits the cancellation law* (*of addition*) if and only if

(iii) $(x + u)\rho(y + v)$ and $u\rho v$ imply $x\rho y$.

DEFINITION 2. Let $\mathfrak{N}$ be an $\mathfrak{A}$-subsemimodule of an $\mathfrak{A}$-semimodule $\mathfrak{M}$. x, $y \in \mathfrak{M}$ are called (*strongly*) *congruent modulo* $\mathfrak{N}$ and denoted by $x \equiv y$ $(\mathfrak{N})$ if and only if there exist $n_1, n_2 \in \mathfrak{N}$ such that $x + n_1 = y + n_2$. $x, y \in \mathfrak{M}$ are called *weakly congruent* modulo $\mathfrak{N}$ and denoted by $x[\equiv]y$ $(\mathfrak{N})$ if and only if there exist $n_1, n_2 \in \mathfrak{N}$ and $z \in \mathfrak{M}$ such that $x + n_1 + z = y + n_2 + z$.

Evidently both kinds of relations "congruent modulo $\mathfrak{N}$" are linear equivalence relations and "weak" one admits the cancellation law. In each case, defining the compositions in the obvious way, the equivalence classes modulo $\mathfrak{N}$ form an $\mathfrak{A}$-semimodule. The $\mathfrak{A}$-semimodule thus obtained is denoted by $\mathfrak{M}$-$\mathfrak{N}$ in the "strong" case and $\mathfrak{M}[-]\mathfrak{N}$ in the "weak" case. In $\mathfrak{M}[-]\mathfrak{N}$ the cancellation law of addition holds. If we put $\overline{\mathfrak{N}} = \{y \in \mathfrak{M}; y \equiv 0\,(\mathfrak{N})\}$ and $\widehat{\mathfrak{N}} = \{y \in \mathfrak{M};\ y[\equiv]0\ (\mathfrak{N})\}$, then $\overline{\mathfrak{N}}$ and $\widehat{\mathfrak{N}}$ are $\mathfrak{A}$-subsemimodules of $\mathfrak{M}$. $\mathfrak{N}$ is called (*weakly*) *closed* in $\mathfrak{M}$ if and only if $\overline{\mathfrak{N}} = \mathfrak{N}$ and *strongly closed* in $\mathfrak{M}$ if and only if $\widehat{\mathfrak{N}} = \mathfrak{N}$.

The following results are easily seen:

a) If $x \equiv y\,(\mathfrak{N})$, $x[\equiv]y\,(\mathfrak{N})$.

b) $x \equiv y\,(\mathfrak{N})$ if and only if and only if $x \equiv y\,(\overline{\mathfrak{N}})$. $x[\equiv]y\,(\mathfrak{N})$ if and only if $x[\widehat{\equiv}]y\,(\mathfrak{N})$.

c) $\widehat{\mathfrak{N}} \supseteqq \overline{\mathfrak{N}} \supseteqq \mathfrak{N}$. $\overline{\overline{\mathfrak{N}}} = \overline{\mathfrak{N}}$, $\widehat{\widehat{\mathfrak{N}}} = \widehat{\mathfrak{N}}$.

d) If $\mathfrak{N}_1 \supseteqq \mathfrak{N}_2$, then $\overline{\mathfrak{N}}_1 \supseteqq \overline{\mathfrak{N}}_2$ and $\widehat{\mathfrak{N}}_1 \supseteqq \widehat{\mathfrak{N}}_2$.

e) The zeroid $\mathfrak{Z}(\mathfrak{M}) = \widehat{\{0\}}$ of $\mathfrak{M}$ is the minimum strongly closed $\mathfrak{A}$-subsemimodule of $\mathfrak{M}$.

Let $\mathfrak{M}'$ be another $\mathfrak{A}$-semimodule which is homomorphic to $\mathfrak{M}$ via a mapping φ and $\mathfrak{N}'$ be an $\mathfrak{A}$-subsemimodule of $\mathfrak{M}'$.

f) $\varphi^{-1}(\mathfrak{N}')$ is closed if $\mathfrak{N}'$ is closed and it is strongly closed if $\mathfrak{N}'$ is strongly closed.

g) $\varphi^{-1}(\mathfrak{Z}(\mathfrak{M}')) \supseteqq \mathfrak{Z}(\mathfrak{M})$.

h) The relation "$\varphi(x) = \varphi(y)$" is a linear equivalence relation. If $x \equiv y$ $(\varphi^{-1}(0))$ then $\varphi(x) = \varphi(y)$, but the converse does not always hold.

Let $\nu(x) \in \mathfrak{M} - \mathfrak{N}$ and $\hat{\nu}(x) \in \mathfrak{M}[-]\mathfrak{N}$ be the equivalence classes represented by $x \in \mathfrak{M}$.

i) $\mathfrak{M} - \mathfrak{N}$ is homomorphic to $\mathfrak{M}$ via the mapping $x \to \nu(x)$ and $\mathfrak{M}[-]\mathfrak{N}$ is homomorphic to $\mathfrak{M} - \mathfrak{N}$ via the mapping $\nu(x) \to \hat{\nu}(x)$.

j) $\widehat{\mathfrak{N}} = \{y \in \mathfrak{M};\ \nu(y) \in \mathfrak{Z}(\mathfrak{M} - \mathfrak{N})\}$.

Put $\mathfrak{M}^* = \mathfrak{M}[-]\{0\}$ and denote by x^* the equivalence class represented by $x \in \mathfrak{M}$.

k) $\mathfrak{N}^* \equiv \{y^* \in \mathfrak{M}^*;\ y \in \mathfrak{N}\}$ is an $\mathfrak{A}$-subsemimodule of $\mathfrak{M}^*$ and $\mathfrak{M}^* - \mathfrak{N}^*$ is isomorphic to $(\mathfrak{M} - \mathfrak{N})[-]\{0\}$.

Let denote by $E(\mathfrak{M})$ the set of all linear equivalence relations defined in the $\mathfrak{A}$-semimodule $\mathfrak{M}$ which admit the cancellation law. For ρ_α, $\rho_\beta \in E(\mathfrak{M})$ we write $\rho_\beta \geqq \rho_\alpha$ if and only if, for $y, z \in \mathfrak{M}$, $y\rho_\alpha z$ implies $y\rho_\beta z$.

l) $(E(\mathfrak{M}), \geqq)$ is a lattice which has the maximum element ρ_1 and the minimum element ρ_0: $y\rho_1 z$ for every pair of $y, z \in \mathfrak{M}$, and $y\rho_0 z$ if and only if $y^* = z^*$.

DEFINITION 3. Let $\mathfrak{B}$ be an ideal in a semiring $\mathfrak{A}$. The semirings $\mathfrak{A}/\mathfrak{B}$ and $\mathfrak{A}[/]\mathfrak{B}$ are defined in the same fashion as $\mathfrak{M} - \mathfrak{N}$ and $\mathfrak{M}[-]\mathfrak{N}$ are defined, respectively.

Almost all the facts considered on $\mathfrak{A}$-semimodules are established for semirings with some modifications. For example, the zeroid of $\mathfrak{A}$ is the minimum strongly closed ideal in $\mathfrak{A}$.

DEFINITION 4. $\mathfrak{M}$ is called a *representation semimodule* of a semiring $\mathfrak{A}$ if and only if $\mathfrak{M}$ is an $\mathfrak{A}$-semimodule in which the cancellation law of addition holds. A homomorphism of a semiring $\mathfrak{A}$ into the endomorphism semiring of a semimodule, which has a zero element and in which the commutative law and the cancellation law of addition hold, is called a *representation* of $\mathfrak{A}$.

If $\mathfrak{M}$ is a representation semimodule of $\mathfrak{A}$ then, for an arbitrarily fixed $a \in \mathfrak{A}$, the mapping $x \to xa$ of $\mathfrak{M}$ into $\mathfrak{M}$ itself is an endomorphism a_R of the semimodule $\mathfrak{M}$, and the mapping $a \to a_R$ is a representation of $\mathfrak{A}$, which is called *the representation of* $\mathfrak{A}$ *associated with* $\mathfrak{M}$ and denoted by $(\mathfrak{A}, \mathfrak{M})$. Conversely, a representation of $\mathfrak{A}$ defines a representation semimodule in the obvious way.

m) If φ is a representation of $\mathfrak{A}$, then $\varphi(\mathfrak{A})$ is a semiring in which the cancellation law of addition holds.

n) The zeroid $\mathfrak{Z}(\mathfrak{A})$ of $\mathfrak{A}$ is represented by zero in every representation of $\mathfrak{A}$.

A representation semimodule $\mathfrak{M}$ of a semiring $\mathfrak{A}$ is called *faithful* and the associated representation $(\mathfrak{A}, \mathfrak{M})$ is called *faithful* if and only if $\mathfrak{Z}(\mathfrak{A}) = (0 : \mathfrak{M}) \equiv \{b \in \mathfrak{A}; \mathfrak{M}\, b = \{0\}\}$.

2. Irreducible representations and Radicals.

DEFINITION 5. A representation semimodule $\mathfrak{M}$ of a semiring $\mathfrak{A}$ with $\mathfrak{M} \neq \{0\}$ is called *irreducible* and the associated representation $(\mathfrak{A}, \mathfrak{M})$ is called *irreducible* if and only if, for an arbitrarily fxed pair of $u_1, u_2 \in \mathfrak{M}$ with $u_1 \neq u_2$ and any $x \in \mathfrak{M}$, there exist $a_1, a_2 \in \mathfrak{A}$ such that

$$x + u_1 a_1 + u_2 a_2 = u_1 a_2 + u_2 a_1. \tag{1}$$

A representation semimodule $\mathfrak{M}$ of a semiring $\mathfrak{A}$ is called *semi-irreducible* if and only if

(i) $\mathfrak{M}\mathfrak{A} \neq \{0\}$.

(ii) $\mathfrak{M}$ does not have any closed $\mathfrak{A}$-subsemimodule except $\{0\}$ and $\mathfrak{M}$ itself.

LEMMA 1. *Let $\mathfrak{B}$ be an ideal in $\mathfrak{A}$ and assume that $\mathfrak{M}$ is a representation semimodule of $\mathfrak{A}$ with $\mathfrak{M}\mathfrak{B} \neq \{0\}$.*

1) *If $\mathfrak{M}$ is semi-irreducible and u is an element of $\mathfrak{M}$, then $u = 0$ is equivalent to $ub = 0$ for all $b \in \mathfrak{B}$.*

2) *If $\mathfrak{M}$ is irreducible and u, v are elements of $\mathfrak{M}$, then $u = v$ is equivalent to $ub = vb$ for all $b \in \mathfrak{B}$.*

PROOF. 1) Assume $\mathfrak{M}$ is semi-irreducible. As is easily seen, $\mathfrak{M}_0 \equiv \{y \in \mathfrak{M} ; y\mathfrak{B} = \{0\}\}$ is a closed $\mathfrak{A}$-subsemimodule of $\mathfrak{M}$. Since $\mathfrak{M}\mathfrak{B} \neq \{0\}$, we have $\mathfrak{M}_0 \neq \mathfrak{M}$ and hence $\mathfrak{M}_0 = \{0\}$.

2) Assume that $\mathfrak{M}$ is irreducible and $u \neq v$. Since $\mathfrak{M}\mathfrak{B} \neq \{0\}$, we can find $y \in \mathfrak{M}$ and $b \in \mathfrak{B}$ with $yb \neq 0$. For this y, there exist $a_1, a_2 \in \mathfrak{A}$ such that

$$y + ua_1 + va_2 = ua_2 + va_1.$$

Hence

$$yb + ua_1b + va_2b = ua_2b + va_1b,\ a_ib \in \mathfrak{B}.$$

Since $yb \neq 0$ and since the cancellation law of addition holds in $\mathfrak{M}$, for at least one of a_ib, say b_0, we must have $ub_0 \neq vb_0$.

LEMMA 2. *A representation semimodule $\mathfrak{M} \neq \{0\}$ of $\mathfrak{A}$ is semi-irreducible if and only if $\overline{u\mathfrak{A}} = \mathfrak{M}$ for every non-zero $u \in \mathfrak{M}$, i.e., for an arbitrarily fixed non-zero $u \in \mathfrak{M}$ and any $x \in \mathfrak{M}$, there exist $a_1, a_2 \in \mathfrak{A}$ such that*

$$(1') \qquad x + ua_1 = ua_2.$$

PROOF. Assume $\mathfrak{M}$ is semi-irreducible. If $u \neq 0$ then, by Lemma 1, $u\mathfrak{A} \neq \{0\}$ and hence $\overline{u\mathfrak{A}} = \mathfrak{M}$. The converse is evident.

COROLLARY. *If $\mathfrak{M}$ is irreducible, then it is semi-irreducible.*

LEMMA 3. *If $\mathfrak{M}$ is an (semi-) irreducible representation semimodule of $\mathfrak{A}$ and $\mathfrak{N} \neq \{0\}$ is an arbitrary $\mathfrak{A}$-subsemimodule of $\mathfrak{M}$, then $\mathfrak{N}$ is (semi-) irreducible and $\varphi(\mathfrak{A})$ and $\psi(\mathfrak{A})$ are isomorphic via the correspondence $\varphi(a) \leftrightarrow \psi(a)$, where $\varphi = (\mathfrak{A}, \mathfrak{M})$ and $\psi = (\mathfrak{A}, \mathfrak{N})$.*

PROOF. Assume $\mathfrak{M}$ is (semi-) irreducible. Then, from Definition 5 (Lemma 2), it is easy to see that $\mathfrak{N}$ is (semi-) irreducible, where $\mathfrak{N}$ is an arbitrary non-zero $\mathfrak{A}$-subsemimodule of $\mathfrak{M}$. Assume further $\psi(a) = \psi(b)$. If v is a non-zero element of $\mathfrak{N}$ then, for an arbitrary $u \in \mathfrak{M}$, there exist $a_1, a_2 \in \mathfrak{A}$ such that $u + va_1 = va_2$.
Hence

$$ua + (va_1)a + (va_2)b = ub + (va_1)b + (va_2)a.$$

Since $(va_i)a = (va_i)b$, $i = 1, 2$, and the cancellation law of addition holds in $\mathfrak{M}$, we get $ua = ub$ for every $u \in \mathfrak{M}$ and hence $\varphi(a) = \varphi(b)$.

LEMMA 4. *Let $\mathfrak{B}$ be an ideal in $\mathfrak{A}$.*

1) *If $\mathfrak{M}$ is an (semi-) irreducible representation semimodule of $\mathfrak{A}$, then either $\mathfrak{M}\mathfrak{B} = \{0\}$ or $\mathfrak{M}$ is an (semi-) irreducible representation semimodule of $\mathfrak{B}$.*

2) *If $\mathfrak{M}$ is an irreducible representation semimodule of $\mathfrak{B}$, then there exists an irreducible representation semimodule $\mathfrak{M}'$ of $\mathfrak{A}$ such that $\varphi(\mathfrak{B})$ and $\varphi'(\mathfrak{B})$ are isomorphic via the correspondence $\varphi(b) \leftrightarrow \varphi'(b)$, where $\varphi = (\mathfrak{B}, \mathfrak{M})$ and $\varphi' = (\mathfrak{A}, \mathfrak{M}')$.*

PROOF. 1) Let $\mathfrak{M}$ be an irreducible representation semimodule of $\mathfrak{A}$ and u_1, u_2 be an arbitrarily fixed pair of elements in $\mathfrak{M}$ with $u_1 \neq u_2$. Suppose $\mathfrak{M}\mathfrak{B} \neq \{0\}$. Then $u_1 b \neq u_2 b$ for some $b \in \mathfrak{B}$ and hence, for any $x \in \mathfrak{M}$, there exist $a_1, a_2 \in \mathfrak{A}$ such that

$$x + u_1(ba_1) + u_2(ba_2) = u_1(ba_2) + u_2(ba_1)$$

and $ba_i \in \mathfrak{B}$, $i = 1, 2$. Therefore $\mathfrak{M}$ is irreducible as a representation semimodule of $\mathfrak{B}$.

If $\mathfrak{M}$ is semi-irreducible and $\mathfrak{M}\mathfrak{B} \neq \{0\}$, then we can see similarly as above that $\mathfrak{M}$ is semi-irreducible as a representation semimodule of $\mathfrak{B}$.

2) Let $\mathfrak{M}$ be an irreducible representation semimodule of $\mathfrak{B}$.
By Lemma 3, $\mathfrak{M}\mathfrak{B}$, say $\mathfrak{N}$, is an irreducible representation semimodule of $\mathfrak{B}$

and $\varphi(\mathfrak{B})$ and $\psi(\mathfrak{B})$ are isomorphic via the correspondence $\varphi(b) \leftrightarrow \psi(b)$, where $\psi = (\mathfrak{B}, \mathfrak{N})$. If $\Sigma u_i b_i = \Sigma u'_j b'_j$ for b_i, $b'_j \in \mathfrak{B}$ and $u_i, u'_j \in \mathfrak{M}$ then, for an arbitrary $a \in \mathfrak{A}$ and every $b \in \mathfrak{B}$,

$$\begin{aligned}(\Sigma u_i(b_i a))b &= \Sigma u_i(b_i ab) = (\Sigma u_i b_i)(ab) \\ &= (\Sigma u'_j b'_j)(ab) = \Sigma u'_j(b'_j ab) = (\Sigma u'_j(b'_j a))b\end{aligned}$$

and hence, by Lemma 1, $\Sigma u_i(b_i a) = \Sigma u'_j(b'_j a)$. Therefore, we can define a composition on $\mathfrak{N} \times \mathfrak{A}$ into $\mathfrak{N}$ by setting $(\Sigma u_i b_i)a \equiv \Sigma u_i(b_i a)$, where $u_i \in \mathfrak{M}$, $b_i \in \mathfrak{B}$ and $a \in \mathfrak{A}$. Thus, as is easily seen, the semimodule $\mathfrak{N}$ with the composition forms an $\mathfrak{A}$-semimodule $\mathfrak{M}'$ which, considering it as a $\mathfrak{B}$-semimodule, is isomorphic to the $\mathfrak{B}$-semimodule $\mathfrak{N}$. It is evident that $\mathfrak{M}'$ is irreducible as a representation semimodule of $\mathfrak{A}$.

DEFINITION 6. Let I be the set of all irreducible representation semimodules of a semiring $\mathfrak{A}$. $\mathfrak{R}(\mathfrak{A}) \equiv \bigcap_{\mathfrak{M} \in I} (0 : \mathfrak{M})$ is called the *radical* of $\mathfrak{A}$. It is understood that if I is vacuous, then $\mathfrak{A}$ is its own radical, in which case, we say that $\mathfrak{A}$ is a *radical semiring*.

A semiring $\mathfrak{A}$ is called *semisimple* if and only if $\mathfrak{R}(\mathfrak{A}) = \{0\}$.

The zeroid $\mathfrak{Z}(\mathfrak{A})$ of $\mathfrak{A}$ is contained in the radical $\mathfrak{R}(\mathfrak{A})$.

THEOREM 1. *The radical $\mathfrak{R}$ of a semiring $\mathfrak{A}$ is a strongly closed ideal in $\mathfrak{A}$.*

PROOF. It is evident that $\mathfrak{R}$ is an ideal in $\mathfrak{A}$. Let $\mathfrak{M}$ be an arbitrary irreducible representation semimodule of $\mathfrak{A}$. For an arbitrary element r of $\widehat{\mathfrak{R}}$, there exist $r_1, r_2 \in \mathfrak{R}$ and $s \in \mathfrak{A}$ such that $r + r_1 + s = r_2 + s$. Then for every $u \in \mathfrak{M}$

$$ur + ur_1 + us = ur_2 + us,$$

hence we get $ur = 0$, because we have $ur_1 = ur_2 = 0$ and the cancellation law of addition holds in $\mathfrak{M}$. Therefore $\widehat{\mathfrak{R}} = \mathfrak{R}$, i. e., $\mathfrak{R}$ is strongly closed in $\mathfrak{A}$

THEOREM 2.[2] *If $\mathfrak{B}$ is an ideal in a semiring $\mathfrak{A}$, then $\mathfrak{R}(\mathfrak{B}) = \mathfrak{B} \cap \mathfrak{R}(\mathfrak{A})$.*

The proof of this theorem follows immediately from Lemma 4.

COROLLARY.[3] *The radical of a semiring is a radical semiring, considering it as a semiring.*

THEOREM 3. *If $\mathfrak{R}$ is the radical of a semiring $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{R}$ and $\mathfrak{A}[/]\mathfrak{R}$ are both semisimple.*

2) This theorem is a refinement of Lemma 4 in [4].
3) This corollary corresponds to the theorem in [3].

This theorem is proved by the same method as in the ring theory,[4] because of Theorem 1.

THEOREM 4. *If* $\mathfrak{J}$ *is any semi-nilpotent right ideal in a semiring* $\mathfrak{A}$, *i.e., there exists a positive rational integer* n *such that* $\mathfrak{J}^n \subseteqq \mathfrak{Z}(\mathfrak{A})$, *then* $\mathfrak{J}$ *is contained in the radical of* $\mathfrak{A}$.

PROOF. Assume that a right ideal $\mathfrak{J}$ is semi-nilpotent and $\mathfrak{R} \not\supseteqq \mathfrak{J}$. Let $\mathfrak{M}$ be an irreducible representbtion semimodule of $\mathfrak{A}$ with $\mathfrak{M}\mathfrak{J} \neq \{0\}$. There exist $u \in \mathfrak{M}$ and a positive rational integer l such that $u\mathfrak{J}^l \neq \{0\}$ and $u\mathfrak{J}^{l+1} = \{0\}$. For a non-zero $v \in u\mathfrak{J}^l$, there exist $a_1, a_2 \in \mathfrak{A}$ such that $u + va_1 = va_2$. Since $va_1 i = va_2 i = 0$ for every $i \in \mathfrak{J}$, we have $ui = 0$ and hence $u\mathfrak{J} = \{0\}$, which is a contradiction.

COROLLARY. *If a semiring is semisimple, then it does not have a nilpotent right ideal.*

THEOREM 5. *If* $\mathfrak{R}$ *is the radical of a semiring* $\mathfrak{A}$ *and if* $\mathfrak{A}r\mathfrak{A} \subseteqq \mathfrak{R}$ *then* $r \in \mathfrak{R}$.

PROOF. Suppose $\mathfrak{A}r\mathfrak{A} \subseteqq \mathfrak{R}$ and $r \bar{\in} \mathfrak{R}$. Let $\mathfrak{M}$ be an irreducible representation semimodule of $\mathfrak{A}$ with $\mathfrak{M}r \neq \{0\}$. By Lemma 3, we have $\mathfrak{M}\mathfrak{A}r \neq \{0\}$ and hence, by Lemma 1, we get $\mathfrak{M}\mathfrak{A}r\mathfrak{A} \neq \{0\}$, which is a contradiction.

DEFINITION 7. A semiring $\mathfrak{A}$ is called *primitive* if and only if it has a faithful irreducible representation semimodule. An ideal $\mathfrak{P}$ in a semiring $\mathfrak{A}$ is called *primitive* if and only if $\mathfrak{A}/\mathfrak{P}$ is primitive.

$\mathfrak{P}$ is a primitive ideal in $\mathfrak{A}$ if and only if so is $\widehat{\mathfrak{P}}$.

LEMMA 5. $\mathfrak{P}$ *is a strongly closed primitive ideal in* $\mathfrak{A}$ *if and only if* $\mathfrak{P} = (0 : \mathfrak{M})$, *where* $\mathfrak{M}$ *is an irreducible representation semimodule of* $\mathfrak{A}$.

If an ideal $\mathfrak{P}$ is strongly closed then $\mathfrak{Z}(\mathfrak{A}/\mathfrak{P}) = \{0\}$, and if $\mathfrak{M}$ is a representation semimodule of $\mathfrak{A}$ then $(0 : \mathfrak{M})$ is a strongly closed ideal in $\mathfrak{A}$. Thus, Lemma 5 is proved in the same way as in the ring theory.[4]

From Lemma 5, we obtain the following theorem.

THEOREM 6. *The radical of a semiring* $\mathfrak{A}$ *is the intersection of all strongly closed primitive ideals in* $\mathfrak{A}$.

REMARK. We note in the following that the irreducibility of representation semimodules can be described in terms of equivalence relations. Let $\mathfrak{M} \neq \{0\}$ be a representation semimodule of $\mathfrak{A}$. The maximum element ρ_1 and the minimum element ρ_0 of $E(\mathfrak{M})$ are different and, for $y, z \in \mathfrak{M}$, $y\rho_0 z$ is equivalemt to $y = z$. Assume $\mathfrak{M}$ is irreducible. By an arbitrarily given $\rho_\alpha \in$

4) Cf. [5].

$E(\mathfrak{M})$ with $\rho_\alpha \neq \rho_0$, some two different elements of $\mathfrak{M}$ are united, say u_1, u_2. For an arbitrary $x \in \mathfrak{M}$, there exist $a_1, a_2 \in \mathfrak{A}$ such that $x + u_1 a_1 + u_2 a_2 = u_1 a_2 + u_2 a_1$. Since ρ_α admits the cancellation law and we have $(u_1 a_i)\rho_\alpha(u_2 a_i)$, $i = 1, 2$, we get $x\rho_\alpha 0$ and hence $\rho_\alpha = \rho_1$. Therefore $E(\mathfrak{M})$ consists of ρ_0 and ρ_1 only. Assume, conversely, that $\mathfrak{M}\mathfrak{A} \neq \{0\}$ and that $E(\mathfrak{M})$ consists of ρ_0 and ρ_1 only. We show firstly that, for $y, z \in \mathfrak{M}$, $y = z$ if and only if $ya = za$ for all $a \in \mathfrak{A}$. To see this, we take a binary relation ρ_α which is defined in $\mathfrak{M}$ as follows: $y\rho_\alpha z$ if and only if $ya = za$ for all $a \in \mathfrak{A}$. As is easily seen, $\rho_\alpha \in E(\mathfrak{M})$. Since $\mathfrak{M}\mathfrak{A} \neq \{0\}$, $yb \neq 0b$ for some $y \in \mathfrak{M}$ and some $b \in \mathfrak{A}$. Therefore $\rho_\alpha \neq \rho_1$ and hence $\rho_\alpha = \rho_0$. Next, let u_1, u_2 be an arbitrary pair of different elements in $\mathfrak{M}$ and ρ_β the binary relation which is defined in $\mathfrak{M}$ as follows: $y\rho_\beta z$ if and only if there exist $a_1, a_2 \in \mathfrak{A}$ such that

$$y + u_1 a_1 + u_2 a_2 = z + u_1 a_2 + u_2 a_1.$$

It is easy to see $\rho_\beta \in E(\mathfrak{M})$. Hence, either $\rho_\beta = \rho_0$ or $\rho_\beta = \rho_1$. For every $a \in \mathfrak{A}$, since $u_1 a + u_1 0 + u_2 a = u_2 a + u_1 a + u_2 0$ holds, we have $(u_1 a)\rho_\beta(u_2 a)$. While $(u_1 a)\rho_0(u_2 a)$ does not hold for some $a \in \mathfrak{A}$. Hence we get $\rho_\beta = \rho_1$. Thus we obtain the following condition for irreducibility: $\mathfrak{M}$ is irreducible if and only if (i) $\mathfrak{M}\mathfrak{A} \neq \{0\}$ and (ii) $E(\mathfrak{M})$ consists of ρ_0 and ρ_1 only. We obtain analogously a condition for semi-irreducibility: $\mathfrak{M}$ is semi-irreducible if and only if (i) $\mathfrak{M}\mathfrak{A} \neq \{0\}$ and (ii) Every $\rho \in E(\mathfrak{M})$ with $\rho \neq \rho_1$ does not unite any non-zero element of $\mathfrak{M}$ to zero.

Suppose now $\mathfrak{M}$ is semi-irreducible. Then $E_1 \equiv E(\mathfrak{M}) - \rho_1$ is an inductively ordered set and hence there exists a maximal element in E_1, say ρ. Evidently, the equivalence classes by ρ form an irreducible representation semimodule $\mathfrak{M}'$ of $\mathfrak{A}$. $\mathfrak{M}'$ is homomorphic to $\mathfrak{M}$ via the mapping φ which maps each $x \in \mathfrak{M}$ onto the equivalence class represented by x. As is easily seen, $\varphi^{-1}(0) \equiv \{0\}$ and hence $(0 : \mathfrak{M}) = (0 : \mathfrak{M}')$. Thus, we can define the radical as follows:

DEFINITION 6'. Let I' be the set of all semi-irreducible representation semimodules of a semiring $\mathfrak{A}$. The ideal $\mathfrak{R}(\mathfrak{A}) \equiv \bigcap_{\mathfrak{M} \in I'} (0 : \mathfrak{M})$ in $\mathfrak{A}$ is called the radical of $\mathfrak{A}$.

3. Quasi-regularity. Let $\mathfrak{A}$ be a semiring and E the set of all linear equivalence relations admitting the cancellation law which are defined in $\mathfrak{A}$, considered as an $\mathfrak{A}$-semimodule.

For an arbitrarily fixed pair of $i_1, i_2 \in \mathfrak{A}$, we take a binary relation $\rho(i_1, i_2)$[5] which is defined in $\mathfrak{A}$ as follows: $s\rho(i_1, i_2)t$ if and only if there exist j_1,

5) The notion of $\rho(i_1, i_2)$ corresponds to that of modular right ideals in the ring theory. Using this notion, we can obtain the analogues of the results concerning modular right ideals, but we shall omit the details. Cf. [5].

$j_2 \in \mathfrak{A}$ such that

$$s + j_1 + i_1 j_1 + i_2 j_2 = t + j_2 + i_1 j_2 + i_2 j_1. \tag{2}$$

LEMMA 6. 1) *$\rho(i_1, i_2) \in E$ and, for any $a_1, a_2 \in \mathfrak{A}$,*

$$(a_1 + i_1 a_1 + i_2 a_2)\rho(i_1, i_2)(a_2 + i_1 a_2 + i_2 a_1) \tag{3}$$

holds.

2) *If $i_1 \rho(i_1, i_2) i_2$ then $\rho(i_1, i_2) = \rho_1$ (the maximum element of E).*

PROOF. 1) It is evident that $\rho(i_1, i_2)$ satisfies the reflexive law, the symmetric law and the conditions (i) and (ii) in Definition 1. We have to show that $\rho(i_1, i_2)$ satisfies the transitive law and the condition (iii) in Definition 1. Suppose $r\rho(i_1, i_2)s$ and $s\rho(i_1, i_2)t$ and let j_1, j_2, h_1, h_2 be elements of $\mathfrak{A}$ with

$$r + j_1 + i_1 j_1 + i_2 j_2 = s + j_2 + i_1 j_2 + i_2 j_1,$$
$$s + h_1 + i_1 h_1 + i_2 h_2 = t + h_2 + i_1 h_2 + i_2 h_1.$$

We then have

$$r + k_1 + i_1 k_1 + i_2 k_2 = t + k_2 + i_1 k_2 + i_2 k_1,$$

where $k_1 = s + j_1 + h_1$, $k_2 = s + j_2 + h_2$. Therefore we get $r\rho(i_1, i_2)t$. Suppose, next, $(p + s)\rho(i_1, i_2)(q + t)$ and $s\rho(i_1, i_2)t$. Let j_1, j_2 and h_1, h_2 be elements of $\mathfrak{A}$ with

$$p + s + j_1 + i_1 j_1 + i_2 j_2 = q + t + j_2 + i_1 j_2 + i_2 j_1,$$
$$t + h_1 + i_1 h_1 + i_2 h_2 = s + h_2 + i_1 h_2 + i_2 h_1.$$

Putting $k_1 = s + t + j_1 + h_1$ and $k_2 = s + t + j_2 + h_2$, we have

$$p + k_1 + i_1 k_1 + i_2 k_2 = q + k_2 + i_1 k_2 + i_2 k_1,$$

which shows $p\rho(i_1, i_2)q$.

The relation (3) follows from the equality

$$(a_1 + i_1 a_1 + i_2 a_2) + a_2 + i_1 a_2 + i_2 a_1$$
$$= (a_2 + i_1 a_2 + i_2 a_1) + a_1 + i_1 a_1 + i_2 a_2.$$

2) Suppose $i_1 \rho(i_1, i_2) i_2$. Then, for every pair of $a_1, a_2 \in \mathfrak{A}$, we have

$$(i_1 a_1 + i_2 a_2)\rho(i_1, i_2)(i_1 a_2 + i_2 a_1).$$

Since $\rho(i_1, i_2)$ admits the cancellation law, using (3), we get $a_1 \rho(i_1, i_2) a_2$. Therefore we see $\rho(i_1, i_2) = \rho_1$.

DEFINITION 8. A right ideal $\mathfrak{J}$ in a semiring $\mathfrak{A}$ is called *quasi-regular* if and only if, for every pair of $i_1, i_2 \in \mathfrak{J}$, $i_1 \rho(i_1, i_2) i_2$ holds, i. e., $\rho(i_1, i_2) = \rho_1$. A right ideal $\mathfrak{J}$ is called *semi-regular*[6] if and only if, for every pair of i_1, $i_2 \in \mathfrak{J}$, there exist $j_1, j_2 \in \mathfrak{J}$ such that

$$i_1 + j_1 + i_1 j_1 + i_2 j_2 = i_2 + j_2 + i_1 j_2 + i_2 j_1. \tag{4}$$

6) The definition of semi-regular right ideals has been given in [1].

Suppose now $\rho(i_1, i_2) \neq \rho_1$, i.e., $i_1\overline{\rho(i_1, i_2)}i_2$.[7] Let $E(i_1, i_2)$ be the set of all $\rho_\beta \in E$ which satisfy

(i) $i_1\bar{\rho}_\beta i_2$.

(ii) $\rho_\beta \geqq \rho(i_1, i_2)$.

As is easily seen, $E(i_1, i_2)$ is an inductively ordered set and hence, by Zorn's lemma, $E(i_1, i_2)$ has a maximal element, say ρ. We see from Lemma 6, 2) that ρ is maximal in $E - \rho_1$. From (3) and the above condition (i), we get

$$(i_1^2 + i_2^2)\bar{\rho}(i_1 i_2 + i_2 i_1). \tag{5}$$

Let $\mathfrak{M}$ be the $\mathfrak{A}$-semimodule which consists of all equivalence classes by ρ. Since ρ admits the cancellation law and it is maximal in $E - \rho_1$, $\mathfrak{M}$ is a representation semimodule of $\mathfrak{A}$ and $E(\mathfrak{M})$ consists of only two trivial equivalence relations. Moreover (5) implies $\mathfrak{M}\mathfrak{A} \neq \{0\}$. Therefore, according to Remark in § 2, we see that $\mathfrak{M}$ is irreducible.

Thus we obtain the following lemma.

LEMMA 7. *If $i_1\overline{\rho(i_1, i_2)}i_2$, then there exists an irreducible representation semimodule $\mathfrak{M}$ of $\mathfrak{A}$ such that at least one of i_1 and i_2 does not belong to $(0 : \mathfrak{M})$.*

THEOREM 7. 1) *The radical $\mathfrak{R}$ of a semiring $\mathfrak{A}$ is both a semi-regular right and a semi-regular left ideal in $\mathfrak{A}$.*

2) *The radical $\mathfrak{R}$ of a semiring $\mathfrak{A}$ contains every quasi-regular right ideal in $\mathfrak{A}$.*

PROOF 1) Since $\mathfrak{R}$ is a radical semiring, it has no irreducible representation semimodule. Hence it is a semi-regular right ideal in $\mathfrak{A}$, because of Lemma 7. The left semi-regularity of $\mathfrak{R}$ is proved in the same way as Lemma 3 of [1] is proved.[8]

2) Suppose next $\mathfrak{R} \not\supseteq \mathfrak{J}$ and $\mathfrak{M}\mathfrak{J} \neq \{0\}$ where $\mathfrak{J}$ is a quasi-regular right ideal in $\mathfrak{A}$ and $\mathfrak{M}$ is an irreducible representation semimodule of $\mathfrak{A}$. Then, there exist $i \in \mathfrak{J}$ and $u \in \mathfrak{M}$ with $ui \neq 0$. For these i and u, we can find $a_1, a_2 \in \mathfrak{A}$ such that

$$u + uia_1 = uia_2. \tag{6}$$

As ia_1 and ia_2 are in $\mathfrak{J}$, there exist $j_1, j_2 \in \mathfrak{A}$ such that

$$ia_1 + j_1 + ia_1j_1 + ia_2j_2 = ia_2 + j_2 + ia_1j_2 + ia_2j_1. \tag{7}$$

Multiplying the both sides of (6) by j_1 and j_2, we have

$$uj_1 + uia_1j_1 = uia_2j_1$$

7) If $\rho_\alpha \in E$. We use the notation $s\bar{\rho}_\alpha\, t$ to show the negation of $s\rho_\alpha t$.

8) The proof is analogous to that of the corresponding the orem in the ring theory. Cf. [1] and [4].

and

$$uia_2j_2 = uj_2 + uia_1j_2,$$

respectively. Adding (6) and the last two equations, we have

$$u + u(ia_1 + j_1 + ia_1j_1 + ia_2j_2) = u(ia_2 + j_2 + ia_1j_2 + ia_2j_1).$$

Since the cancellation law of addition holds in $\mathfrak{M}$, using (7), we get $u = 0$. This contradicts $ui \neq 0$.

THEOREM 8. *The radical and the left radical of a semiring coincide.*

THEOREM 9. *If* $\mathfrak{A}$ *is a radical semiring and* r *is any element of* $\mathfrak{A}$, *then for every positive rational integer* t *either* $r^{t-1}\mathfrak{A} \supset r^t\mathfrak{A}$ *or* $r^t \in \mathfrak{Z}(\mathfrak{A})$.

PROOF. Evidently $r^{t-1}\mathfrak{A} \supseteqq r^t\mathfrak{A}$. Suppose $r^{t-1}\mathfrak{A} = r^t\mathfrak{A}$. Then $r^t = r^ts$ for some $s \in \mathfrak{A}$. Since $\mathfrak{A}$ is a radical semiring, there exist $j_1, j_2 \in \mathfrak{A}$ such that $s + j_2 + sj_1 = j_1 + sj_2$, hence $r^ts + r^tj_2 + r^tsj_1 = r^tj_1 + r^tsj_2$. Using $r^t = r^ts$, we have $r^t + r^t(j_1 + j_2) = r^t(j_1 + j_2)$ which shows $r^t \in \mathfrak{Z}(\mathfrak{A})$.

REMARK. We can see immediately from Theorem 7 that both the Jacobson radical and the semiradical of a semiring coincide with our radical of the semiring.

4. Consideration from the ring theory. Examples. Let $\mathfrak{A}$ be a semiring and $\mathfrak{A}^*$ the semiring $\mathfrak{A}[/]\{0\}$. As the cancellation law of addition holds in $\mathfrak{A}^*$, $\mathfrak{A}^*$ is imbeded in a ring $\widetilde{\mathfrak{A}}$ generated by $\mathfrak{A}^*$.[9] Let $\mathfrak{M}$ be a representation semimodule of $\mathfrak{A}$ and $\widetilde{\mathfrak{M}}$ the module generated by $\mathfrak{M}$. $\widetilde{\mathfrak{M}}$ is considered as an $\mathfrak{A}$-module and moreover as an $\widetilde{\mathfrak{A}}$-module in the obvious way. Conversely any $\widetilde{\mathfrak{A}}$-module is considered as an $\mathfrak{A}$-module.

a) $\varphi(\mathfrak{A})$ and $\widetilde{\varphi}$ $(\mathfrak{A})$ are isomorphic via the correspondence $\varphi(a) \leftrightarrow \widetilde{\varphi}(a)$, where $\varphi = (\mathfrak{A}, \mathfrak{M})$ and $\widetilde{\varphi} = (\mathfrak{A}, \widetilde{\mathfrak{M}})$.

b) A representation of a semiring $\mathfrak{A}$ is a homomorphism of $\mathfrak{A}$ into the endomorphism ring of a module.

c) $\mathfrak{M}$ is irreducible, if and only if $\widetilde{\mathfrak{M}}$ is irreducible as an $\widetilde{\mathfrak{A}}$-module.

d) An $\mathfrak{A}$-subsemimodule $\mathfrak{N}$ of $\mathfrak{M}$ is closed if and only if $\mathfrak{N} = \mathfrak{M} \cap \mathfrak{L}$ where $\mathfrak{L}$ is an $\widetilde{\mathfrak{A}}$-submodule of $\widetilde{\mathfrak{M}}$.

The radical of $\mathfrak{A}$ can be defined as follows:

DEFINITION 6''. Let $\mathfrak{A}$ be a semiring and I_0 the set of all irreducible $\widetilde{\mathfrak{A}}$-modules. The ideal $\mathfrak{R}(\mathfrak{A}) \equiv \bigcap_{\mathfrak{M} \in I_0} (0 : \mathfrak{M})$ is called the *radical* of $\mathfrak{A}$.

e) $\mathfrak{R}(\widetilde{\mathfrak{A}})$ is the Jacobson radical of the ring $\widetilde{\mathfrak{A}}$ and we have

9) Cf. [2].

$$\mathfrak{R}(\mathfrak{A}^*) = \mathfrak{R}(\widetilde{\mathfrak{A}}) \cap \mathfrak{A}^*, \; \mathfrak{R}(\mathfrak{A}) = \{r \in \mathfrak{A};\, r^* \in \mathfrak{R}(\mathfrak{A}^*)\}.$$

f) Let φ be a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. φ can be extended to a homomorphism $\widetilde{\varphi}$ of $\widetilde{\mathfrak{M}}$ onto $\widetilde{\mathfrak{M}}'$. The equivalence relations "$\varphi(x) = \varphi(y)$" and "$x \equiv y\,(\varphi^{-1}(0))$" are equivalent if and only if $\widetilde{\varphi}^{-1}(0)$ coincides with the $\widetilde{\mathfrak{A}}$-submodule of $\widetilde{\mathfrak{M}}$ generated by $\varphi^{-1}(0)$.

EXAMPLE 1. Let $\mathfrak{A}$ be the semiring of non-negative rational integers and ρ the binary relation defined in $\mathfrak{A}$ by which every non-zero elements m, n of $\mathfrak{A}$ are united and the zero element is united to only zero itself. ρ is a (two-sided) linear equivalence relation which does not admit the cancellation law. The equivalence classes by ρ form a semiring, say $\mathfrak{A}_1$, of order 2 with elements c_0, c_1 whose composition tables are

$+$	c_0	c_1
c_0	c_0	c_1
c_1	c_1	c_1

$\cdot$	c_0	c_1
c_0	c_0	c_0
c_1	c_0	c_1

The mapping φ_1 of $\mathfrak{A}$ onto $\mathfrak{A}_1$, which maps every non-zero element of $\mathfrak{A}$ onto c_1 and zero of $\mathfrak{A}$ onto c_0, is a homomorphism and $\mathfrak{A}/\varphi_1^{-1}(c_0)$ is isomorphic to the semiring $\mathfrak{A}$ which is not isomorphic to $\mathfrak{A}_1$. Let $\mathfrak{A}_2$ be the semiring of order 4 with elements b_0, b_1, b_2, b_3 whose composition tables are

$+$	b_0	b_1	b_2	b_3
b_0	b_0	b_1	b_2	b_3
b_1	b_1	b_1	b_2	b_3
b_2	b_2	b_2	b_2	b_2
b_3	b_3	b_3	b_2	b_3

$\cdot$	b_0	b_1	b_2	b_3
b_0	b_0	b_0	b_0	b_0
b_1	b_0	b_0	b_1	b_0
b_2	b_0	b_1	b_2	b_3
b_3	b_0	b_0	b_3	b_3

The mapping φ_2 of $\mathfrak{A}_2$ onto $\mathfrak{A}_1$, which maps b_0, b_1 onto c_0 and b_2, b_3 onto c_1, is a homomorphism and $\mathfrak{A}_2/\varphi_2^{-1}(0)$ consists of the equivalence classes $\{b_0, b_1\}$, $\{b_2\}$, $\{b_3\}$, hence $\mathfrak{A}_2/\varphi_2^{-1}(0)$ is not isomorphic to $\mathfrak{A}_1$. In this case, we have

$$\mathfrak{A}^* = \mathfrak{A},\; \mathfrak{R}(\mathfrak{A}) = \mathfrak{Z}(\mathfrak{A}) = \{0\},$$
$$\mathfrak{A}_i^* = \{0\},\; \mathfrak{R}(\mathfrak{A}_i) = \mathfrak{Z}(\mathfrak{A}_i) = \mathfrak{A}_i,\; i = 1, 2.$$

EXAMPLE 2. Let $\mathbf{P}$ be the field of rational numbers and $\mathbf{P}[x]$ the polynomial ring over $\mathbf{P}$ in the indeterminate x. We consider the semiring $\mathfrak{A}$ which consists of all polynomials of $\mathbf{P}[x]$ with non-negative rational coefficients. We then have $\mathfrak{A} = \mathfrak{A}^*$, $\widetilde{\mathfrak{A}} = \mathbf{P}[x]$, $\mathfrak{R}(\mathfrak{A}) = \{0\}$. Let $\widetilde{\varphi}$ be the natural homomorphism of $\mathbf{P}[x]$ onto the residue class ring $\mathbf{P}[x]/(x-1)$ and φ be the homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}' \equiv \widetilde{\varphi}(\mathfrak{A})$ which is induced by $\widetilde{\varphi}$. As is easily seen, $\varphi^{-1}(0) = \{0\}$ and the semiring $\mathfrak{A}/\varphi^{-1}(0)$ is not isomorphic to $\mathfrak{A}'$. The ideal $X_0 \equiv (x-1)$ in $\mathbf{P}[x]$ is a modular maximal right ideal with a left identity $e \equiv x$ and $L_0 \equiv X_0 \cap \mathfrak{A} = \{0\}$. X_0 can not be generated by L_0.[10]

EXAMPLE 3. In Example 2, let $\mathfrak{M}$ be the $\mathfrak{A}$-semimodule consisting of all residue classes in the difference $\widetilde{\mathfrak{A}}$-module $\mathbf{P}[x] - (x-1)^2$ which are represented by the elements in $\mathfrak{A}$. The representation semimodule $\mathfrak{M}$ of $\mathfrak{A}$ is not irreducible while it is semi-irreducible.

REFERENCES

[1] S. BOURNE, The Jacobson radical of a semiring, Proc. Nat. Acad. Sci., 37 (1951), 163-170.

[2] S. BOURNE AND H. ZASSENHAUS, On the semiradical of a semiring, ibid,. 44 (1958), 907-914.

[3] K. IIZUKA AND I. NAKAHARA, A note on the semiradical of a semiring, Kumamoto Jour. of Sci., 4 (1959), 1-3.

[4] N. JACOBASON, The radical and semi-simplicity for arbitrary rings, Amer. Jour. Math., 67 (1945), 300-320.

[5] ————, Stucture of rings. Amer. Math. Soc., Col., 37 (1956); Providence, R. I. : Amer. Math. Soc.

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10) Since we have this example, I cannot follow the proof of Theorem 7 in [2].