ON ALMOST-ANALYTIC VECTORS IN CERTAIN ALMOST-HERMITIAN MANIFOLDS[1)]

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(Received February 20 1959)

0. Introduction. On an n-dimensional differentiable manifold M with local coordinate systems $\{x^i\}$[2)], a tensor field $\varphi_j{}^i$ of type (1, 1) such that

$$\varphi_r{}^i\varphi_j{}^r = -\delta_j^i \tag{0. 1}$$

is called an almost-complex structure. It is a well known fact[3)] that a manifold M with an almost-complex structure $\varphi_j{}^i$ always admits a positive definite Riemannian metric tensor g_{ji} such that

$$g_{rs}\varphi_j{}^r\varphi_i{}^s = g_{ji}. \tag{0. 2}$$

The pair $(\varphi_j{}^i, g_{ji})$ satisfying (0. 1) and (0. 2) is called an almost-Hermitian structure and the manifold M with the structure $(\varphi_j{}^i, g_{ji})$ is called an almost-Hermitian manifold.

Let M be an almost-Hermitian manifold, then a differential form $\varphi = \varphi_{ji}dx^jdx^i$, where $\varphi_{ji} = \varphi_j{}^r g_{ri}$, is associated to the structure. If the form φ is closed, the structure is called an almost-Kählerian structure. In this case, the tensor φ_{ji} is harmonic of order two.

On the other hand, A. Frölicher[4)] proved that there exists an almost-complex structure on the six dimensional sphere S^6. And T. Fukami and S. Ishihara[5)] proved that the structure on S^6 is an almost-Hermitian one satisfying the following relation

$$\nabla_k\varphi_{ji} + \nabla_j\varphi_{ki} = 0, \tag{0. 3}$$

where and throughout this paper ∇_k denotes the operator of covariant derivative with respect to the Riemannian connection.

The last equation expresses the fact that the tensor φ_{ji} is a Killing tensor of order two.[6)]

In my previous paper,[7)] I treated almost-analytic vectors in almost-

1) This paper was prepared in a term when the present author was ordered to study at

Kählerian manifolds. By an analogous method we shall discuss about almost-analytic vectors in almost-Hermitian manifolds in which the equation (0. 3) is valid. After preliminaries in § 1, we shall introduce in § 2 almost-analytic vectors in our manifold. In § 3 it will be obtained a necessary condition in order that a vector v is a contravariant almost-analytic vector. Similarly § 4 is devoted to covariant almost-analytic vectors. In § 5 and § 6, integral formulas will be obtained in the case where our manifold is compact.

1. Preliminaries. In this paper, by M we shall always mean an n-dimensional differentiable manifold with a fixed almost-Hermitian structure $(\varphi_j^{\ i}, g_{ji})$ such that

(1. 1) $$\nabla_k \varphi_{ji} + \nabla_j \varphi_{ki} = 0,$$

where $\varphi_{ji} = \varphi_j^{\ r} g_{ri}$. We shall call such a manifold K-space, for convenience.

By (0. 1) and (0. 2), φ_{ji} is skew symmetric with respect to j and i. By (1. 1), $\nabla_k \varphi_{ji}$ is also skew symmetric with respect to all indices.

Transvecting (1. 1) with g^{ji}, it follows that

(1. 2) $$\nabla^r \varphi_{ri} = 0.$$

In this section we shall use (1. 2) but shall not use (1. 1), so the results which will be obtained in this section are true in almost-Hermitian manifolds with the relation (1. 2).

Let $R_{kji}^{\ \ \ h}$ be the Riemannian curvature tensor i. e.

$$R_{kji}^{\ \ \ h} = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kr} \right\} \left\{ {r \atop ji} \right\} - \left\{ {h \atop jr} \right\} \left\{ {r \atop ki} \right\},$$

where $\partial_i = \partial/\partial x^i$, and put

$$R_{ji} = R_{rji}^{\ \ \ r}, \qquad R_{kjih} = R_{kji}^{\ \ \ r} g_{rh}$$

and

(1. 3) $$R^*_{kj} = \frac{1}{2} \varphi^{pq} R_{pqij} \varphi_k^{\ i},$$

where $\varphi^{pq} = \varphi_r^{\ q} g^{rp}$.

Applying the Ricci's identity to $\varphi_i^{\ h}$, we obtain the identity

$$\nabla_k \nabla_j \varphi_i^{\ h} - \nabla_j \nabla_k \varphi_i^{\ h} = R_{kjr}^{\ \ \ h} \varphi_i^{\ r} - R_{kji}^{\ \ \ r} \varphi_r^{\ h}.$$

Transvecting the last equation with g^{ji} and using (1. 2), we find

$$\nabla^r \nabla_j \varphi_r^{\ h} = R_{kjr}^{\ \ \ h} \varphi^{kr} + R_j^{\ r} \varphi_r^{\ h}.$$

As φ^{kr} is skew symmetric with respect to k and r, we get

$$\nabla^r \nabla_j \varphi_r^{\ h} = \frac{1}{2} \varphi^{pq} R_{pqj}^{\ \ \ h} + R_j^{\ r} \varphi_r^{\ h},$$

from which we obtain

(1. 4) $$\nabla^r \nabla_j \varphi_{rh} = \frac{1}{2} \varphi^{pq} R_{pqjh} + R_j{}^r \varphi_{rh}.$$

A vector field v is called a *contravariant almost-analytic* vector or simply an *analytic* vector if its contravariant components satisfy the equations

(1. 5) $$\underset{v}{\pounds} \varphi_j{}^i \equiv v^r \nabla_r \varphi_j{}^i - \varphi_j{}^r \nabla_r v^i + \varphi_r{}^i \nabla_j v^r = 0,$$

where $\underset{v}{\pounds}$ is the operator of Lie derivative.

A vector field u is called a *covariant almost-analytic* vector or simply a *covariant analytic* vector if its covariant components satisfy the equations

(1. 6) $$\nabla_j(\varphi_i{}^r u_r) = u_r \nabla_i \varphi_j{}^r + \varphi_j{}^r \nabla_r u_i.$$

LEMMA 1. 1.[8] *In a compact almost-Hermitian manifold M in which the equation* (1. 2) *is valid, if scalar functions f and g satisfy the equation*

$$\nabla_i f = \varphi_i{}^r \nabla_r g,$$

then the functions are both constant over M.

Let v be an analytic vector, u a covariant analytic vector and put

$$g = u_l v^l \text{ and } f = \varphi_r{}^l u_l v^r,$$

then by virtue of Lemma 1. 1 and definitions, we get easily the following

THEOREM 1. 2. *In a compact almost-Hermitian manifold M in which the equation* (1. 2) *is valid, the inner product of an analytic vector and a covariant analytic vector is constant over the whole M.*

From (1. 6) we have

(1. 7) $$(\nabla_j \varphi_i{}^r - \nabla_i \varphi_j{}^r) u_r = \varphi_j{}^r \nabla_r u_i - \varphi_i{}^r \nabla_j u_r$$

for a covariant analytic vector u. And again from (1. 6) we have

(1. 8) $$\nabla_j(\varphi_i{}^r u_r) = \nabla_i(\varphi_j{}^r u_r) - \varphi_j{}^r \nabla_i u_r + \varphi_j{}^r \nabla_r u_i.$$

Now we shall define a vector field $\tilde{u}$ by the equation

$$\tilde{u}_i = \varphi_i{}^t u_t$$

for any vector field u, then it is equivalent to define

$$\tilde{u}^i = - \varphi_t{}^i u^t.$$

Thus (1. 8) becomes the following form :

(1. 9) $$\nabla_j \tilde{u}_i - \nabla_i \tilde{u}_j = \varphi_j{}^r (\nabla_r u_i - \nabla_i u_r)^{9)}.$$

The equations (1. 6), (1. 7) and (1. 9) are equivalent to each other.

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8) S. Tachibana [5].

9) In (1.9), ∇ may be replaced by ∂.

By transvection (1. 6) with g^{ji} we get easily

$$\nabla^r \tilde{u}_r = 0. \tag{1. 10}$$

By virtue of (1. 9) and (1. 10) we have

THEOREM 1. 3. *In an almost-Hermitian manifold M in which the equation* (1. 2) *is valid, if a covariant analytic vector u is closed i. e.* $\nabla_j u_i = \nabla_i u_j$, *then $\tilde{u}$ is harmonic.*

2. Identities. In the following we suppose that the manifold M is always a K-space, that is, (1. 1) holds good.

From (1. 1) and (1. 4), we get directly

$$\nabla^r \nabla_r \varphi_{ji} = -\frac{1}{2} \varphi^{pq} R_{pqji} - R_j{}^r \varphi_{ri}. \tag{2. 1}$$

If we notice the skew symmetry with respect to j and i in (2. 1), we see that

$$R_j{}^r \varphi_{ri} + R_i{}^r \varphi_{rj} = 0,$$

from which we get

$$R_{rs} \varphi_j{}^r \varphi_i{}^s = R_{ji}. \tag{2. 2}$$

In the next place, from (1. 1) we have

$$\nabla_k \varphi_{ji} = \nabla_i \varphi_{kj}.$$

Transvecting the last equation with φ^{kj} and taking account of (0. 1) and (0. 2), we find

$$(\nabla_k \varphi_{ji}) \varphi^{kj} = 0.$$

If we operate ∇_r to the last equation, then we have easily

$$(\nabla_i \varphi_{kj}) \nabla_r \varphi^{kj} = -(\nabla_r \nabla_k \varphi_{ji}) \varphi^{kj}.$$

By the Ricci's identity and skew symmetry of φ^{kj}, after some calculation, we get

$$(\nabla_i \varphi_{kj}) \nabla_r \varphi^{kj} = R^*_{ri} - 2 R^*_{ir} + R_{ri}, \tag{2. 3}$$

where R^*_{kj} is defined by (1. 3). As the left hand side is symmetric with respect to i and r, we see that

$$R^*_{ir} = R^*_{ri} \tag{2. 4}$$

holds good. Consequently (2. 3) becomes

$$(\nabla_i \varphi_{kj}) \nabla_r \varphi^{kj} = R_{ir} - R^*_{ir}. \tag{2. 5}$$

Hence we have

THEOREM 2. 1. *In a K-space M, the inequality*

$$R_{ji} v^j v^i \geqq R^*_{ji} v^j v^i$$

is valid for any vector field v.

As an application of the last theorem, we shall give the following

THEOREM 2. 2.[10] *If $n \geqq 4$ and a K-space M is conformally flat, then the Ricci's quadratic form $R_{ji}v^j v^i$ can not be negative definite.*

PROOF. From the assumption, the curvature tensor of M has the following form[11]

$$R_{kjih} = \frac{1}{n-2}(g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} - R_{jh}g_{ki}) - \frac{R}{(n-1)(n-2)}(g_{kh}g_{ji} - g_{jh}g_{ki}).$$

Hence we have

$$R^*_{ji} = \frac{1}{n-2}\left(2R_{ji} - \frac{R}{n-1}g_{ji}\right),$$

from which it follows that

$$R_{ji} - R^*_{ji} = \frac{1}{n-2}\left\{(n-4)R_{ji} + \frac{R}{n-1}g_{ji}\right\}. \tag{2. 6}$$

If the Ricci's quadratic form $R_{ji}v^j v^i$ is negative definite, then $R < 0$. Hence (2. 6) contradicts to the Theorem 2. 1.

COROLLARY. *If $n \geqq 4$, there does not exist a K-space of constant curvature with $R < 0$.*

The Nijenhuis' tensor of an almost-complex structure is defined by

$$N_{ji}{}^h = \varphi_j{}^l(\nabla_l\varphi_i{}^h - \nabla_i\varphi_l{}^h) - \varphi_i{}^l(\nabla_l\varphi_j{}^h - \nabla_j\varphi_l{}^h).$$ [11]

By virtue of (1. 1), in K-space M, the last equation turns to

$$N_{ji}{}^h = -4\varphi_r{}^h\nabla_j\varphi_i{}^r.$$

If we put

$$N_{jih} = N_{ji}{}^r g_{rh}, \tag{2. 7}$$

then it follows that

$$N_{jih} = -4(\nabla_j\varphi_i{}^r)\varphi_{rh}.$$

It is easily seen that N_{jih} is skew symmetric with respect to all indices.

Let v be a vector field and define $N(v)_h$ by the equation

$$N(v)_h = \frac{1}{4}(\Delta^j v^i)N_{jih}, \tag{2. 8}$$

10) If the manifold M is compact, the theorem is a direct consequence of Theorem 4. 2 in p. 80 of K. Yano, and S. Bochner [6].

11) K. Yano [7].

where $\nabla^j v^t = g^{jr}\nabla_r v^t$. Then, by virtue of (0. 1), (0. 2), (1. 1) and (2. 7), we have

$$N(v)_h = (\nabla^j v^t)\,(\nabla_r \varphi_{jt})\varphi_h^{\ r}. \tag{2. 9}$$

In the next place we shall prepare a lemma which is useful in § 6.

From $\tilde{v}_i = \varphi_i^{\ t} v_t$, we have

$$\nabla^r\nabla_r\tilde{v}_i = (\nabla^r\nabla_r\varphi_{it})v^t + 2(\nabla^r\varphi_i^{\ t})\nabla_r v_t + \varphi_i^{\ t}\nabla^r\nabla_r v_t.$$

Transvecting $\tilde{v}^i = -\varphi_t^{\ i} v^t$ with the last equation and taking account of (1. 1), (2. 1), (2. 2) and (2. 9), we find

$$\tilde{v}^i\nabla^r\nabla_r\tilde{v}_i = v^i\{\nabla^r\nabla_r v_i + R^*_{ri}v^r - R_{ri}v^r + 2\,N(v)_i\}.$$

Hence we get the following equation

$$\begin{aligned}(\nabla^r\nabla_r\tilde{v}_i - R_{ri}\tilde{v}^r)\tilde{v}^i = (\nabla^r\nabla_r v_i - R_{ri}v^r)v^i \\ + \{2\,N(v)_i - (R_{ri} - R^*_{ri})v^r\}v^i.\end{aligned} \tag{2. 10}$$

On the other hand, the following theorem is well known.[12]

In a compact orientable Riemannian manifold V_n, the integral formula

$$\int_{V_n}[(\nabla^r\nabla_r u_i - R_{ri}u^r)u^i + S(u)]d\sigma = 0$$

is valid for any vector field u, where $S(u)$ is given by

$$S(u) = \frac{1}{2}(\nabla^r u^s - \nabla^s u^r)\,(\nabla_r u_s - \nabla_s u_r) + (\nabla^r u_r)^2.$$

As an almost-Hermitian manifold is an orientable Riemannian one, the theorem is applicable to our K-space M. If we put $\tilde{v}_i = u_i$, then, by virtue of (2. 10), we get the following

LEMMA 2. 3. *In a compact K-space M, the integral formula*

$$\int_M[\{\nabla^r\nabla_r v_i - R_{ri}v^r + 2\,N(v)_i - (R_{ri} - R^*_{ri})v^r\}v^i + S(\tilde{v})]d\sigma = 0 \tag{2. 11}$$

is valid for any vector field v.

3. Contravariant almost-analytic vectors. Consider an analytic vector v, then it holds that

$$\underset{v}{\pounds}\varphi_j^{\ i} \equiv v^r\nabla_r\varphi_j^{\ i} - \varphi_j^{\ r}\nabla_r v^i + \varphi_r^{\ i}\nabla_j v^r = 0, \tag{3. 1}$$

which is equivalent to the following equation

$$v^r\nabla_r\varphi_{ji} - \varphi_j^{\ r}\nabla_r v_i - \varphi_i^{\ r}\nabla_j v_r = 0. \tag{3. 2}$$

Operating $\nabla^j = g^{jp}\nabla_p$ to (3. 2), we have

12) For example, K. Yano [7].

$$(\nabla^j v^r)\nabla_r\varphi_{ji} + v^r\nabla^j\nabla_r\varphi_{ji} - \varphi_j{}^r\nabla^j\nabla_r v_i - (\nabla^j\varphi_i{}^r)\nabla_j v_r - \varphi_i{}^r\nabla^j\nabla_j v_r = 0.$$

On taking account of (1. 1), (1. 4) and

$$\varphi^{jr}\nabla_j\nabla_r v_i = -\frac{1}{2}\varphi^{pq}R_{pqi}{}^r v_r,$$

we find that the equation

$$\nabla^r\nabla_r v_i + R_{ri}v^r = 0 \tag{3. 3}$$

holds good for an analytic vector v. Hence we have

THEOREM 3. 1. *In a compact K-space, if an analytic vector v satisfies $\nabla^r v_r = 0$, then it is a Killing vector.*

In an n-dimensional Riemannian manfold V_n, if a vector field v satisfies

$$\underset{v}{\pounds} g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 2\phi g_{ji}, \tag{3. 4}$$

where ϕ is a scalar function, then it is called a conformal Killing vector. A conformal Killing vector v satisfies

$$\nabla^r\nabla_r v_i + R_{ri}v^r + \frac{n-2}{n}\nabla_i\nabla_r v^r = 0,\text{[13)]} \tag{3. 5}$$

by virtue of the Ricci's identity and (3. 4).

Now we suppose that $n > 2$ and M is compact. If a conformal Killing vector v is at the same time analytic, then we have from (3. 3) and (3. 5)

$$\nabla_i\nabla_r v^r = 0,$$

from which it follows that $\nabla_r v^r = \text{const.}$. As M is compact, $\nabla_r v^r = 0$ by virtue of the Green's theorem. Hence, we have the following

THEOREM 3. 2. *If an n-dimensional K-space $(n > 2)$ is compact, a conformal Killing vector which is at the same time analytic is a Killing vector.*

In V_n, a vector field v which satisfies the equation

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} \equiv \nabla_j\nabla_i v^h + R_{rji}{}^h v^r = \delta_j^h\psi_i + \delta_i^h\psi_j, \tag{3. 6}$$

where ψ_i is a certain vector, is called a projective Killing vector. For a projective Killing vector v, we have

$$\nabla^r\nabla_r v_i + R_{ri}v^r = \frac{2}{n+1}\nabla_i\nabla_r v^r,$$

from which we can obtain the following

THEOREM 3. 3. *In a compact K-space, if a projective Killing vector*

13) Cf. K. Yano [7].

is at the same time analytic, then it is a Killing vector.

By an analogous method[14] as in almost-Kählerian manifold, we have easily the following

THEOREM. 3. 4. *In a compact K-space, the integral*

$$\int_M (R_{ri} v^r v^i) d\sigma$$

is positive or zero for any analytic vector v.

COROLLARY. *If a compact K-space is an Einstein space with $R < 0$, then there does not exist a non-trivial analytic vector.*

In a compact almost-Kählerian manifold, the equation (3. 3) is a sufficient condition in order that v is an analytic vector. But in a K-space, the equation (3. 3) is not sufficient. In the next place we shall obtain another equation which must be satisfied by an analytic vector.

If we operate $\varphi^{kj}\nabla_k$ to (3. 2) then we get

$$\varphi^{kj}\nabla_k(v^r\nabla_r\varphi_{ji} - \varphi_j{}^r\nabla_r v_i - \varphi_i{}^r\nabla_j v_r) = 0.$$

The left hand side is the sum of the following six terms $a_1, \dots\dots, a_6$.

$$a_1 = \varphi^{kj}(\nabla_k v^r)\nabla_r\varphi_{ji} = -(\nabla^k v^r)(\nabla_j\varphi_{kr})\varphi_i{}^j = -N(v)_i,$$

$$a_2 = \varphi^{kj}v^r\nabla_k\nabla_r\varphi_{ji} = 0.$$

The validity of the last equality owes to (1. 1), the Ricci's identity and (2. 4).

$$a_3 = -\varphi^{kj}(\nabla_k\varphi_j{}^r)\nabla_r v_i = 0,$$

$$a_4 = -\varphi^{kj}\varphi_j{}^r\nabla_k\nabla_r v_i = \nabla^r\nabla_r v_i,$$

$$a_5 = -\varphi^{kj}(\nabla_k\varphi_i{}^r)\nabla_j v_r = -N(v)_i,$$

$$a_6 = -\varphi^{kj}\varphi_i{}^r\nabla_k\nabla_j v_r = R^*_{ri}v^r.$$

Hence we have

$$\nabla^r\nabla_r v_i + R^*_{ri}v^r - 2N(v)_i = 0 \tag{3. 7}$$

for an analytic vector v. From (3. 3) and (3. 7), we get the following equation

$$(R_{ri} - R^*_{ri})v^r = -2N(v)_i. \tag{3. 8}$$

We shall see in §5 that, in a compact K-space, (3. 3) and (3. 8) constitute a set of sufficient conditions in order that v is an analytic vector.

4. Covariant almost-analytic vectors. In § 1, a covariant analytic vector field was defined. In the present section we shall obtain equations which must be satisfied by such vectors.

14) S. Tachibana [5].

For covariant analytic vector v, we have

$$(\nabla_j\varphi_i^{\ r} - \nabla_i\varphi_j^{\ r})v_r = \varphi_j^{\ r}\nabla_r v_i - \varphi_i^{\ r}\nabla_j v_r.$$

On account of (1. 1), the last equation is equivalent to

$$2\,v^r\nabla_r\varphi_{ji} - \varphi_j^{\ r}\nabla_r v_i + \varphi_i^{\ r}\nabla_j v_r = 0. \tag{4. 1}$$

As an analytic vector v is defined by (3. 2) i. e.

$$v^r\nabla_r\varphi_{ji} - \varphi_j^{\ r}\nabla_r v_i - \varphi_i^{\ r}\nabla_j v_r = 0,$$

if we notice the similarity of (4. 1) and the last equation, then we shall be able to avoid some complication in the following calculation.

If we operate ∇^j to (4. 1), it follows that

$$2(\nabla^j v^r)\nabla_r\varphi_{ji} + 2\,v^r\nabla^j\nabla_r\varphi_{ji} - \varphi_j^{\ r}\nabla^j\nabla_r v_i + (\nabla^j\varphi_i^{\ r})\nabla_j v_r + \varphi_i^{\ r}\nabla^j\nabla_j v_r = 0.$$

Hence, by virtue of (1. 1) and (1. 4), we obtain

$$\nabla^r\nabla_r v_i + R^*_{ri}v^r - 2\,R_{ri}v^r + 3\,N(v)_i = 0. \tag{4. 2}$$

Next we operate $\varphi^{kj}\nabla_k$ to (4. 1) and obtain the equation

$$\varphi^{kj}\nabla_k(2\,v^r\nabla_r\varphi_{ji} - \varphi_j^{\ r}\nabla_r v_i + \varphi_i^{\ r}\nabla_j v_r) = 0.$$

The left hand member is the sum of the following six terms $a_1', \ldots\ldots, a_6'$. Making use of the notation in § 3, we have

$$a_1' = 2\,a_1, \qquad a_2' = 2\,a_2, \qquad a_3' = a_3,$$

$$a_4' = a_4, \qquad a_5' = -\,a_5, \qquad a_6' = -\,a_6.$$

Therefore we get

$$\nabla^r\nabla_r v_i - R^*_{ri}v^r - N(v)_i = 0. \tag{4. 3}$$

From (4. 2) and (4. 3), it follows that the equation

$$(R_{ri} - R^*_{ri})v^r = 2\,N(v)_i \tag{4. 4}$$

holds good. Substituting (4. 4) in (4. 2), we obtain

$$\nabla^r\nabla_r v_i - R_{ri}v^r + N(v)_i = 0 \tag{4. 2}$$

for a covariant analytic vector v.

5. The integral formula. In this section we shall obtain a integral formula concerning a vector field in a compact K-space and prove a theorem which gives a necessary and sufficient condition for an analytic vector.

Let v be a vector field and introduce a tensor $a(v)_{jk}$ by

$$\begin{aligned} a(v)_{jk} &= (\underset{v}{\pounds}\varphi_j^{\ l})\varphi_{lk} \\ &= v^r(\nabla_r\varphi_j^{\ l})\varphi_{lk} - \varphi_j^{\ r}(\nabla_r v^l)\varphi_{lk} - \nabla_j v_k. \end{aligned} \tag{5. 1}$$

For simplicity, we shall denote a_{jk} instead of $a(v)_{jk}$ in the following. $a_{jk} = 0$ is equivalent to the fact that the vector v is an analytic vector.

In the first place we shall compute $\nabla^j a_{jk}$, which is the sum of the following six terms $A_1, \ldots\ldots, A_6$.

$$A_1 = (\nabla^j v^r)(\nabla_r \varphi_j{}^l)\varphi_{lk} = N(v)_k,$$
$$A_2 = v^r(\nabla^j \nabla_r \varphi_j{}^l)\varphi_{lk} = (R^*_{rk} - R_{rk})v^r,$$
$$A_3 = v^r(\nabla_r \varphi_j{}^l)\nabla^j \varphi_{lk} = (R_{rk} - R^*_{rk})v^r,$$
$$A_4 = -\varphi_j{}^r(\nabla^j \nabla_r v^l)\varphi_{lk} = -R^*_{rk}v^r,$$
$$A_5 = -\varphi_j{}^r(\nabla_r v^l)\nabla^j \varphi_{lk} = N(v)_k,$$
$$A_6 = -\nabla^j \nabla_j v_k.$$

Then we get

$$\nabla^j a_{jk} = -(\nabla^j \nabla_j v_k + R^*_{rk}v^r) + 2\,N(v)_k. \tag{5. 2}$$

In the next place we shall compute

$$\nabla^j(a_{jk}v^k) = v^k \nabla^j a_{jk} + a_{jk}\nabla^j v^k.$$

If we substitute (5. 1) and (5. 2) in the last equation, we have after some calculation,

$$\nabla^j(a_{jk}v^k) = -(\nabla^j \nabla_j v_k + R^*_{rk}v^r) + N(v)_k v^k + \varphi^{jr}\varphi^{kl}(\nabla_j v_k)\nabla_r v_l - (\nabla_j v_k)\nabla^j v^k. \tag{5. 3}$$

Now if we put $a^2(v) = a_{jk}a^{jk}$, then

$$a^2(v) = [v^r(\nabla_r \varphi_j{}^l)\varphi_{lk} - \varphi_j{}^r(\nabla_r v^l)\varphi_{lk} - \nabla_j v_k] \times [v^p(\nabla_p \varphi^{js})\varphi_s{}^k - \varphi^{jp}(\nabla_p v^s)\varphi_s{}^k - \nabla^j v^k]$$

is the sum of the following nine terms $B_1, \ldots, B_9$.

$$B_1 = v^r(\nabla_r \varphi_j{}^l)\varphi_{lk}v^p(\nabla_p \varphi^{js})\varphi_s{}^k = (R_{rp} - R^*_{rp})v^r v^p,$$
$$B_2 = -v^r(\nabla_r \varphi_j{}^l)\varphi_{lk}\varphi^{jp}(\nabla_p v^s)\varphi_s{}^k = N(v)_r v^r,$$
$$B_3 = -v^r(\nabla_r \varphi_j{}^l)\varphi_{lk}\nabla^j v^k = B_2,$$
$$B_4 = -\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}v^p(\nabla_p \varphi^{js})\varphi_s{}^k = B_2,$$
$$B_5 = \varphi_j{}^r(\nabla_r v^l)\varphi_{lk}\varphi^{jp}(\nabla_p v^s)\varphi_s{}^k = (\nabla_p v_l)\nabla^p v^l,$$
$$B_6 = \varphi_j{}^r(\nabla_r v^l)\varphi_{lk}\nabla^j v^k = -\varphi^{jr}\varphi^{kl}(\nabla_j v_k)\nabla_r v_l,$$
$$B_7 = -(\nabla_j v_k)v^p(\nabla_p \varphi^{js})\varphi_s{}^k = B_2,$$
$$B_8 = \nabla_j v_k \varphi^{jp}(\nabla_p v^s)\varphi_s{}^k = B_6,$$
$$B_9 = (\nabla^j v^k)\nabla_j v_k = B_5.$$

Therefore we get

$$a^2(v) = (R_{rp} - R^*_{rp})v^r v^p + 4\,N(v)_r v^r - 2\,\varphi^{jr}\varphi^{kl}(\nabla_j v_k)\nabla_r v_l + 2\,(\nabla_j v_k)\nabla^j v^k. \tag{5. 4}$$

Consequently, from (5. 3) and (5. 4), we have

$$\nabla^j(a_{jk}v^k) + \frac{1}{2}a^2(v) = -(\nabla^j \nabla_j v_k + R_{rk}v^r)v^k$$

$$+\frac{3}{2}\{2\,N(v)_k + (R_{rk} - R^*_{rk})v^r\}v^k.$$

If M is compact, by integration of the last equation, we have the following

THEOREM 5. 1. *In a compact K-space M, the integral formula*

$$\int_M\left[(\nabla^j\nabla_j v_k + R_{rk}v^r)v^k + \frac{1}{2}a^2(v)\right]d\sigma = \frac{3}{2}\int_M[2\,N(v)_k + (R_{rk} - R^*_{rk})v^r]\,v^k d\sigma \tag{5. 5}$$

is valid for any vector field v, where

$$a^2(v) = a_{jk}a^{jk},\ a_{jk} = (\underset{v}{\pounds}\varphi_j{}^l)\varphi_{lk},$$

$$N(v)_i = \frac{1}{4}(\nabla^p v^q)N_{pqi}$$

and N_{pqi}, R^*_{ri} *are given by* (2. 7) *and* (1. 3) *respectively.*

In § 3 we have seen that an analytic vector v satisfies the following equations

$$\nabla^r\nabla_r v_i + R_{ri}v^r = 0, \tag{5. 6}$$

$$2\,N(v)_i + (R_{ri} - R^*_{ri})v^r = 0. \tag{5. 7}$$

Now consider a vector field v satisfying (5. 6) and (5. 7). Then if M is compact, we have $a^2(v) = 0$ i. e. $a_{jk} = 0$ by virtue of (5. 5), so v is analytic. Thus we have the

THEOREM 5. 2. *In a compact K-space M, a necessary and sufficient condition in order that a vector v be analytic is that equations* (5. 6) *and* (5. 7) *are both satisfied.*

6. Another integral formula. Consider a vector field v and put

$$b(v)_{jk} = (2\,v^r\nabla_r\varphi_j{}^l - \varphi_j{}^r\nabla_r v^l - \varphi_r{}^l\nabla_j v^r)\varphi_{lk}$$
$$= 2\,v^r(\nabla_r\varphi_j{}^l)\varphi_{lk} - \varphi_j{}^r(\nabla_r v^l)\varphi_{lk} + \nabla_j v_k.$$

$b(v)_{jk} = 0$ is equivalent to the fact that the vector v is covariant analytic. For simplicity we write b_{jk} instead of $b(v)_{jk}$. Using the notation in § 5, $\nabla^j b_{jk}$ is the sum of the following six terms A_1', ..., A_6'.

$$A_1' = 2\,A_1, \qquad A_2' = 2\,A_2, \qquad A_3' = 2\,A_3,$$
$$A_4' = A_4, \qquad A_5' = A_5, \qquad A_6' = -\,A_6.$$

Hence we have

$$\nabla^j b_{jk} = \nabla^j\nabla_j v_k - R^*_{rk}v^r + 3\,N(v)_k.$$

After some calculation we get

(6. 1) $$\nabla^j(b_{jk}v^k) = (\nabla^j\nabla_j v_k - R^*_{rk}v^r)v^k + N(v)_k v^k + \varphi^{jr}\varphi^{kl}(\nabla_j v_k)\nabla_r v_l + (\nabla_j v_k)\nabla^j v^k.$$

If we put $b^2(v) = b_{jk}b^{jk}$, then it is the sum of the following nine terms $B_1, \dots, B_9'$.

$$B_1' = 4B_1, \quad B_2' = 2B_2, \quad B_3' = -2B_3,$$
$$B_4' = 2B_4, \quad B_5' = B_5, \quad B_6' = -B_6,$$
$$B_7' = -2B_7, \quad B_8' = -B_8, \quad B_9' = B_9.$$

Hence we have

(6. 2) $$b^2(v) = 4(R_{rk} - R^*_{rk})v^r v^k + 2(\nabla_j v_k)\nabla^j v^k + 2\varphi^{jr}\varphi^{kl}(\nabla_j v_k)\nabla_r v_l.$$

Therefore, from (6. 1) and (6. 2), we get

$$\nabla^j(b_{jk}v^k) - \frac{1}{2}b^2(v) = (\nabla^j\nabla_j v_k - R_{rk}v^r)v^k + \{N(v)_k - (R_{rk} - R^*_{rk})v^r\}v^k.$$

Thus we have the following

LEMMA 6. 1. *In a compact K-space M, the integral formula*

(6. 3) $$\int_M\left[\{\nabla^j\nabla_j v_i - R_{ri}v^r + N(v)_i - (R_{ri} - R^*_{ri})v^r\}v^i + \frac{1}{2}b^2(v)\right]d\sigma = 0$$

is valid for any vector field v.

On the other hand, in compact M, we have Lemma 2. 3. If we subtract (6. 3) from the twice of (2. 11), it follows the following

THEOREM 6. 2. *In a compact K-space M, the integral formula*

$$\int_M[(\nabla^r\nabla_r v_i - R_{ri}v^r + N(v)_i) + \{2N(v)_i - (R_{ri} - R^*_{ri})v^r\}]\,v^i d\sigma = \int_M\left[\frac{1}{2}b^2(v) - 2S(\tilde{v})\right]d\sigma$$

is valid for any vector field v.

In § 4 we have seen that a covariant analytic vector v satisfies the following equations

(6. 4) $$\nabla^r\nabla_r v_i - R_{ri}v^r + N(v)_i = 0,$$

(6. 5) $$2N(v)_i - (R_{ri} - R^*_{ri})v^r = 0.$$

Now, let v be a covariant analytic vector, then (6. 4), (6. 5) and $b^2(v) = 0$ hold good. Hence, in compact M, $S(\tilde{v}) = 0$ by virtue of Theorem 6. 2. Thus, from the definition of $S(\tilde{v})$, the vector $\tilde{v}$ is harmonic. As $\tilde{v}$ is also a covariant analytic vector, v is also harmonic by the same argument. Conversely, let v and $\tilde{v}$ be both harmonic, then their components satisfy (1. 9) trivially, so v is a covariant analytic vector. Thus we have

THEOREM 6. 3. *In a compact K-space M, a necessary and sufficient condition in order that a vector v be covariant analytic is that v and $\tilde{v}$ are both harmonic.*

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