SOME INTEGRABILITY THEOREMS OF TRIGONOMETRIC SERIES AND MONOTONE DECREASING FUNCTIONS

SATORU IGARI

(Received October 7, 1959)

1. Let $\{\lambda_n\}$ be a decreasing sequence tending to zero as $n \to \infty$, and put

$$g_1(x) = \sum_{n=1}^{\infty} \lambda_n \cos nx, \qquad h_1(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx.$$

Let $g_2(x)$ and $h_2(x)$ be both non-increasing functions bounded below in $(0, \pi)$ and such that

$$xh_2(x) \in L(0, \pi), \qquad g_2(x) \in L(0, \pi). \tag{A}$$

We put $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi g_2(x) \cos nx\, dx,\ b_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi h_2(x) \sin nx\, dx.$

Denote by $L(x)$ a slowly increasing function, that is, $L(x)$ is positive, continuous in $x \geqq 0$ and for any fixed $t > 0$,

$$\frac{L(tx)}{L(x)} \to 1 \text{ as } x \to \infty.$$

S. Aljančić, R. Bojanić and M. Tomić established in the paper [2] that $x^{-\gamma}L(1/x)g_1(x) \in L(0, \pi)$ for $0 < \gamma < 1$, if and only if $\sum n^{\gamma-1}L(n)\lambda_n$ converges, and that $x^{-\gamma}L(1/x)h_1(x) \in L(0, \pi)$ for $0 < \gamma < 2$, if and only if $\sum n^{\gamma-1}L(n)\lambda_n$ converges.

D. Adamović proved in the paper [1] that $x^{\gamma-1}L(1/x)h_2(x) \in L(0, \pi)$ for $0 < \gamma < 2$, if and only if $\sum n^{-\gamma}L(n)b_n$ converges absolutely, and that $x^{\gamma-1}L(1/x)$ $g_2(x) \in L(0, \pi)$ for $0 < \gamma < 1$, if and only if $\sum n^{-\gamma}L(n)a_n$ converges absolutely.

And recently Chen Yung-Ming showed the interesting theorems which are related to the above results [3].

In this note, we shall make some inprovement of the inequalities in T. M. Flett [4] and apply it to the generalization of those four theorems.

In this note, the condition (A) is not assumed preliminarily.

If $p = 1$, our theorems $2 - 5$ coincide with the just mentioned theorems.

The method of proof in Theorem 1 is due to T. M. Flett [4]. Theorems $2 - 5$ correspond to Theorems $2 - 5$ in G. Sunouchi [5] and our proofs will go along the line of [5] respectively.

2. The slowly increasing function $L(x)$ has the following properties (for (I), (II), (III) see [2]):

(I) $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$ uniformly for $0 < a \leqq t \leqq b < \infty$.

(II) $x^\alpha L(x) \to \infty$, $x^{-\alpha} L(x) \to 0$ as $x \to \infty$ for every $\alpha > 0$.

(III) If we set for $\alpha > 0$

$$\bar{L}_1(x) = x^{-\alpha} \max_{0 \leqq t \leqq x} \{t^\alpha L(t)\},\quad \underline{L}_1(x) = x^\alpha \min_{0 \leqq t \leqq x} \{t^{-\alpha} L(t)\},$$

$$\bar{L}_2(x) = x^\alpha \max_{x \leqq t < \infty} \{t^{-\alpha} L(t)\},\quad \underline{L}_2(x) = x^{-\alpha} \min_{x \leqq t < \infty} \{t^\alpha L(t)\},$$

then $\bar{\underline{L}}_k(x) \sim L(x)$ as $x \to \infty$ $k = 1, 2$.

(IV) For $\alpha > 0$, we have

$$L(tu) \leqq A_1 t^{-\alpha} L(u) \text{ for every } u \geqq 0,\ 1 \geqq t > 0,$$
$$L(u/t) \leqq A_2 t^{-\alpha} L(u) \text{ for every } u \geqq 0,\ 1 \geqq t > 0,$$

where A_1, A_2 are positive constants depending only on α and L. We give a proof of (IV).
By the first equation in (III), we obtain

$$u^{-\alpha} \max_{0 \leqq tu \leqq u} \{(tu)^\alpha L(tu)\} \leqq A_1 L(u) \text{ for every } u \geqq 0,$$

where A_1 is a constant independent of u, and we get the first inequality.
For the second, a proof is similar.

3. The inequalities in the following Theorem 1 are to be interpreted as meaning, "if the integral on the right is finite, then that on the left is finite and satisfies the inequality." The letters B, B_1, B_2,...... are positive constants which depend only on L and parameters concerned in the particular problem in which it appears.

THEOREM 1. *Let $f(x) \geqq 0$ in $x \geqq 0$, and let $F(x) = \int^x f(u)du$. If $q \geqq p \geqq 1$ and $\gamma > -1$, then*

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{L\left(\frac{1}{t}\right)F(t)}{t}\right)^q dt\right\}^{1/q} \leqq B\left\{\int_0^\infty t^{-1-p\gamma}\left(L\left(\frac{1}{t}\right)f(t)\right)^p dt\right\}^{1/p}, \quad (1)$$

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{L(t)F(t)}{t}\right)^q dt\right\}^{1/q} \leqq B\left\{\int_0^\infty t^{-1-p\gamma}(L(t)f(t))^p dt\right\}^{1/p}. \quad (2)$$

PROOF. First, we show (1) in the case $q \geqq p > 1$.

Put
$$J=\left\{\int_0^\infty t^{-1-p\gamma}L^p\left(\frac{1}{t}\right)f^p(t)dt\right\}^{1/p},$$

and let λ be a constant such that $\lambda<1/p'$ $(1/p+1/p'=1)$. Applying Hölder's inequality with indices q, p' and $pq/(q-p)$, we have

$$\begin{aligned}
F(t) &= \int_0^t f(u)du \\
&= \int_0^t \left\{u^{\lambda+\frac{(1+p\gamma)(q-p)}{pq}} L^{-\frac{q-p}{q}}\left(\frac{1}{u}\right)f^{\frac{p}{q}}(u)\right\}\{u^{-\lambda}\}\left\{u^{-p-p\gamma}L^p\left(\frac{1}{u}\right)f^p(u)\right\}^{\frac{q-p}{pq}}du \\
&\leqq \left\{\int_0^t u^{q\lambda+\frac{(1+p\gamma)(q-p)}{p}}L^{-(q-p)}\left(\frac{1}{u}\right)f^p(u)du\right\}^{1/q}\left\{\int_0^t u^{-p'\lambda}du\right\}^{1/p'} \\
&\qquad\cdot\left\{\int_0^t u^{-1-p\gamma}L^p\left(\frac{1}{u}\right)f^p(u)du\right\}^{\frac{q-p}{pq}} \\
&\leqq B_1^{1/p'}J^{1-\frac{p}{q}}t^{-\lambda+\frac{1}{p'}}\left\{\int_0^t u^{q\lambda+\frac{(1+p\gamma)(q-p)}{p}}L^{-(q-p)}\left(\frac{1}{u}\right)f^p(u)du\right\}^{1/q},
\end{aligned}$$

where
$$B_1=\frac{1}{1-p'\lambda}.$$

Write $w=q(\lambda+\gamma+1/p)>0$ (This is possible since $\lambda<1/p'$).

We have $t^{-1-q\lambda-q}F^q(t)\leqq B_1^{q/p'}J^{(q-p)}t^{-1-w}\int_0^t u^{w-1-p\gamma}L^{-(q-p)}\left(\frac{1}{u}\right)f^p(u)du,$

whence
$$\begin{aligned}
&\int_0^\infty t^{-1-q\gamma-q}L^q\left(\frac{1}{t}\right)F^q(t)dt \\
&\leqq B_1^{q/p'}J^{(q-p)}\int_0^\infty t^{-1-w}L^q\left(\frac{1}{t}\right)dt\int_0^t u^{w-1-p\gamma}L^{-(q-p)}\left(\frac{1}{u}\right)f^p(u)du \\
&= \int_0^\infty L^{-(q-p)}\left(\frac{1}{u}\right)f^p(u)u^{-1-p\gamma}K(u)du,
\end{aligned}$$

where
$$\begin{aligned}
K(u) &= u^w\int_u^\infty t^{-1-w}L^q\left(\frac{1}{t}\right)dt \\
&= \int_0^1 T^{-1+w}L^q\left(\frac{T}{u}\right)dT \qquad \left(\text{putting } t=\frac{u}{T}\right) \\
&\leqq A_1^qL^q\left(\frac{1}{u}\right)\int_0^1 T^{-1+w-\varepsilon}dT \\
&\leqq B_2L^q\left(\frac{1}{u}\right),
\end{aligned}$$

provided that we take $\varepsilon>0$ such that $w-\varepsilon>0$.

Thus we obtain

$$\int_0^\infty t^{-1-q\gamma-q}L^q\left(\frac{1}{t}\right)F^q(t)dt \leqq B_1^{q\,p'}B_2 J^{(q-p)}\int_0^\infty u^{-1-p\gamma}L^p\left(\frac{1}{u}\right)f^p(u)du,$$

which proves (1). In the case $q \geqq p = 1$, put $\lambda = 0$, and the inequality (1) may be obtained by the similar arguments.

We can prove the inequality (2) by the similar way writing

$$K(u) = \int^1 T^{-1+w}L^q\left(\frac{u}{T}\right)dT$$
$$\leqq A_2^q L^q(u)\int^1 T^{-1+w-\epsilon}dT,$$

where $\varepsilon > 0$ is sufficiently small so that $w - \varepsilon > 0$.

THEOREM 2. *If* $\lambda_n \downarrow 0$, $p \geqq 1$ *and* $0 > \gamma > -1$, *then a necessary and sufficient condition that* $\sum n^{-1+p\gamma+p}L(n)\lambda_n^p$ *should converge is that* $x^{-1-p\gamma}L(1/x)g_1^p(x) \in L(0, \pi)$.

PROOF. If $x^{-1-p\gamma}L(1/x)g_1^p(x) \in L(0, \pi)$, we put by Zygmund's method [6: p. 213],

$$G_1(x) = \int_0^x g_1(t)dt$$
$$= \sum_{n=1}^\infty \frac{\lambda_n}{n}\sin nx$$

and then

$$G_1\left(\frac{\pi}{n}\right) = \sum_{m=1}^{n-1}\left(\frac{\lambda_m}{m} - \frac{\lambda_{m+n}}{m+n} + \frac{\lambda_{m+2n}}{m+2n} - \cdots\cdots\right)\sin\frac{m}{n}\pi$$
$$\geqq \sum_{m=1}^{n-1}\left(\frac{\lambda_m}{m} - \frac{\lambda_{m+n}}{m+n}\right)\sin\frac{m}{n}\pi$$
$$= B_3 \sum_{[\frac{n}{3}]+1}^{[\frac{2n}{3}]}\left(\frac{\lambda_m}{m} - \frac{\lambda_{m+n}}{m+n}\right)$$
$$\geqq B_4 \sum_{[\frac{n}{3}]+1}^{[\frac{2n}{3}]}\frac{\lambda_m}{m}$$
$$\geqq B_5\lambda_n.$$

So, putting $\widetilde{G}_1(x) = \int_0^x |g_1(t)|dt$, we have by inequality (1)

$$\sum_{n=2}^\infty n^{-1+p\gamma+p}L(n)\lambda_n^p \leqq B_6\sum_{n=2}^\infty n^{-1+p\gamma}\left\{nL^{\frac{1}{p}}(n)G_1\left(\frac{\pi}{n}\right)\right\}^p$$

$$\leqq B_7 \sum_{n=2}^{\infty} n^{-1+p\gamma}\left\{nL^{\frac{1}{p}}(n)\widetilde{G}_1\left(\frac{\pi}{n}\right)\right\}^p$$

$$\leqq B_8 \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/(n-1)} x^{-1-p\gamma}\left\{\frac{L^{\frac{1}{p}}(1/x)\widetilde{G}_1(x)}{x}\right\}^p dx$$

$$= B_8 \int_0^{\pi} x^{-1-p\gamma}\left\{\frac{L^{\frac{1}{p}}(1/x)\widetilde{G}_1(x)}{x}\right\}^{1/p} dx$$

$$\leqq B_9 \int_0^{\pi} x^{-1-p\gamma} L\left(\frac{1}{x}\right)|g_1(x)|^p dx$$

$$< \infty \qquad (\gamma > -1).$$

Thus the sufficiency part of Theorem is proved.
To show the necessity, we observe that

$$|g_1(x)| \leqq \left|\sum_{\nu=1}^{n} \lambda_\nu\right| + \left|\sum_{\nu=n+1}^{\infty} \lambda_\nu \cos \nu x\right|$$

$$\leqq P_n + \frac{\pi}{x}\lambda_n \qquad \left(P_n = \sum_{\nu=1}^{n}\lambda_\nu\right),$$

and so $|g_1(x)| \leqq B_{10}P_n$ for $\pi/(n+1) \leqq x \leqq \pi/n$.

If we set $p(x) = \lambda_n$ for $n-1 \leqq x < n \,(n = 1, 2, \cdots\cdots)$

and
$$P(x) = \int^x p(t)dt,$$

then we have by (2)
$$\int_0^{\pi} x^{-1-p\gamma} L\left(\frac{1}{x}\right)|g_1(x)|^p dx$$

$$= \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} x^{-1-p\gamma} L\left(\frac{1}{x}\right)|g_1(x)|^p dx$$

$$\leqq B_{11} \sum_{n=1}^{\infty} n^{-1+p\gamma} L(n) P_n^p$$

$$\leqq B_{11} L(1)\lambda_1^p + B_{12}\int_1^{\infty} x^{-1+p\gamma+p}\left\{\frac{L^{\frac{1}{p}}(x)P(x)}{x}\right\}^p dx$$

$$\leqq B_{11} L(1)\lambda_1^p + B_{13}\int_1^{\infty} x^{-1+p\gamma+p} L(x) p^p(x) dx$$

$$\leqq B_{14} \sum_{n=1}^{\infty} n^{-1+p\gamma+p} L(n)\lambda_n^p$$

$$< \infty \qquad (\gamma < 0).$$

Thus we complete the proof.

THEOREM 3. *If* $\lambda_n \downarrow 0$, $p \geqq 1$ *and* $1 > \gamma > -1$, *then a necessary and sufficient condition that* $\sum n^{-1+p\gamma+p}L(n)\lambda^p{}_n$ *should converge is that* $x^{-1-p\gamma}L(1/x)h_1^p(x) \in L(0, \pi)$.

PROOF. The sufficiency of condition may be obtained by a similar argument as the Theorem 1. To prove that the condition is necessary, we observe that for $\pi/(n+1) \leqq x \leqq \pi/n$,

$$
\begin{aligned}
|h_1(x)| &\leqq \sum_{\nu=1}^{n} |\lambda_\nu \sin \nu x| + \left| \sum_{\nu=n+1}^{\infty} \lambda_\nu \sin \nu x \right| \\
&\leqq B_{16}\left(\frac{\lambda_1 + 2\lambda_2 + \cdots\cdots + n\lambda_n}{n} + (n+1)\lambda_{n+1}\right) \\
&\leqq B_{16}\left(\frac{(\lambda_1 + 2\lambda_{n+1}) + 2(\lambda_2 + 2\lambda_{n+1}) + \cdots\cdots + n(\lambda_n + 2\lambda_{n+1})}{n}\right) \\
&\leqq 3B_{16}\frac{\lambda_1 + 2\lambda_2 + \cdots\cdots + n\lambda_n}{n} \\
&= 3B_{16}\frac{Q_n}{n} \qquad \left(Q_n = \sum_{\nu=1}^{n} \nu\lambda_\nu\right).
\end{aligned}
$$

If we set $q(x) = n\lambda_n$ for $n - 1 \leqq x < n$ $(n = 1, 2, \ldots\ldots)$

and
$$Q(x) = \int_0^x q(t)dt,$$

then by (2) we have

$$
\begin{aligned}
\int_0^\pi x^{-1-p\gamma}L\left(\frac{1}{x}\right)|h_1(x)|^p dx &= \sum_{n=1}^{\infty}\int_{\pi/(n+1)}^{\pi/n} x^{-1-p\gamma}L\left(\frac{1}{x}\right)|h_1(x)|^p dx \\
&\leqq B_{17}\sum_{n=1}^{\infty} n^{-1+p\gamma}L(n)\left\{\frac{Q(n)}{n}\right\}^p \\
&\leqq B_{17}L(1)\lambda_1^p + B_{18}\int_1^\infty x^{-1+p\gamma}\left\{\frac{L^{\frac{1}{p}}(x)Q(x)}{x}\right\}^p dx \\
&\leqq B_{17}L(1)\lambda_1^p + B_{19}\int_1^\infty x^{-1+p\gamma}L(x)q^p(x)dx \\
&\leqq B_{20}\sum_{n=1}^{\infty} n^{-1+p\gamma+p}L(n)\lambda_n^p \\
&< \infty \qquad (\gamma < 1).
\end{aligned}
$$

THEOREM 4. *If* $g_2(x)$, $a_n (n = 0, 1, 2, \ldots\ldots)$ *are defined in* § 1 *with the exception of* (*A*) *and* $p \geqq 1$, $0 > \gamma > -1$, *then a necessary and sufficient* *condition that* $\sum n^{-1+p\gamma+p}L(n)|a_n|^p$ *should converge is that* $x^{-1-p\gamma}L(1/x)g_2^p(x) \in L(0, \pi)$.

PROOF. It is sufficient to consider the case $g_2(x) \geqq 0$ in $(0, \pi)$.

Employing Zygmund's argument [6: p. 215], we shall prove the two inequalities: $|a_n| \leqq 4\ G_2(\pi/n)$, $\tilde{P}_n \geqq B_{21}g(\pi/n)$,

where $$\tilde{P}_n = \sum_{\nu=1}^{n} |a_\nu|, \qquad G_2(x) = \int_0^x g_2(t)dt.$$

In fact, writing

$$\frac{\pi}{2} a_n = \int_0^{\pi/n} g_2(x)\cos nx\,dx + \int_{\pi/n}^{\pi} g_2(x)\cos nx\,dx,$$

the last term on the right is, by the second mean value theorem, less than $g_2(\pi/n)(2/n) \leqq G_2(\pi/n)$ in absolute value and the first inequality is immediate. To prove the second inequality it is enough to notice that:

$$\begin{aligned}\tilde{P}_n &\geqq \frac{2}{\pi}\int^{\pi} g_2(t)\frac{\sin nt}{2\tan\frac{t}{2}}dt \\ &\geqq \frac{2}{\pi}\int_0^{\pi/n}\left[\frac{g_2(t)}{2\tan\frac{t}{2}} - \frac{g_2\left(t+\frac{\pi}{n}\right)}{2\tan\left(t+\frac{\pi}{n}\right)/2}\right]\sin nt\,dt \\ &\geqq B_{22}\int_0^{\pi/2n}\frac{g_2(t)}{t}\sin nt\,dt \\ &\geqq B_{23}\ g_2\left(\frac{\pi}{2n}\right) \\ &\geqq B_{23}\ g_2\left(\frac{\pi}{n}\right).\end{aligned}$$

The proof of Theorem 4 is then quite analogous to that of Theorem 2.

THEOREM 5. *If* $h_2(x)$ *and* $b_n(n = 1, 2, \ldots\ldots)$ *are defined in* § 1 *with the exception of* (A) *and* $p \geqq 1$, $0 > \gamma > -2$, *then a necessary and sufficient condition that* $\sum n^{-1+p\gamma+p}L(n)|b_n|^p$ *should converge is that* $x^{-1-p\gamma}L(1/x)h_2^p(x) \in L(0, \pi)$.

PROOF. We may assume that $h_2(x) \geqq 0$ in $(0, \pi)$.

To show that the condition is sufficient, we observe

$$|b_n| = \frac{2}{\pi}\left|\int^{\pi} h_2(x)\sin nx\,dx\right|$$

$$\leqq \frac{2}{\pi}\int_0^{\pi/n} h_2(x)\sin nx\,dx$$

$$\leqq \frac{2n}{\pi}\int_0^{\pi/n} xh_2(x)dx$$

$$\leqq B_{24}\, nH_2\left(\frac{\pi}{n}\right),$$

where
$$H_2(x) = \int_0^x th_2(t)dt,$$

and applying (1), we have

$$\sum_{n=2}^{\infty} n^{-1+p\gamma+p}L(n)|b_n|^p \leqq B_{24}^p \sum_{n=2}^{\infty} n^{-1+p\gamma+p}\left\{nL^{\frac{1}{p}}(n)H_2\left(\frac{\pi}{n}\right)\right\}^p$$

$$\leqq B_{25}\sum_{n=2}^{\infty}\int_{\pi/n}^{\pi/(n-1)} x^{-1-p\gamma-p}\left\{\frac{L^{\frac{1}{p}}(1/x)H_2(x)}{x}\right\}^p dx$$

$$= B_{25}\int_0^{\pi} x^{-1-p\gamma-p}\left\{\frac{L^{\frac{1}{p}}(1/x)H_2(x)}{x}\right\}^p dx$$

$$\leqq B_{26}\int_0^{\pi} x^{-1-p\gamma}h_2^p(x)L\left(\frac{1}{x}\right)dx$$

$$< \infty \qquad (\gamma > -2).$$

Since $\tilde{Q}_n \geqq B_{27}h_2(\pi/n)\left(\tilde{Q}_n = \sum_{\nu=1}^{n} |b_n|\right)$, we get the necessity part of Theorem by the same way as in Theorem 2 (see [5]).

This proves the Theorem.

REFERENCES

[1] ADAMOVIĆ, Généralisation de deux théorèmes de Zygmund-B. SZ. Nagy. Publ. Inst. Math. Acad. Serbe Sci, XII, (1958), 81–100.

[2] S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ, Sur l'intégrabilité de certaines séries trigonométriques, Publ. Inst. Math. Acad. Serbe Sci. VIII, (1955), 67–84.

[3] CHEN YUNG-MING, On a maximal theorem of Hardy and Littlewood and theorems concerning Fourier constants, Math. Zeitschr, 69(1958), 418–422.

[4] T. M. FLETT, A note on some inequalities, Proc. Glasgow Math. 4(1958), 7–15.

[5] G. SUNOUCHI, Integrability of trigonometric series, J. Math. (Tokyo Metropolitan Univ.), 1(1953), 99–103.

[6] A. ZYGMUND, Trigonometrical series, Warsawa, 1935.

TÔHOKU UNIVERSITY.