A NOTE ON A MAXIMAL FUNCTION

THOMAS MUIRHEAD FLETT

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1. Let f be integrable in $[-\pi, \pi]$ and be periodic with period 2π, and let $u(\rho, \theta)$ be the Poisson integral of f. In a recent paper [12], E. Stein has introduced the function

$$M_\lambda(\theta) = \sup_{0 \leqq \rho < 1} \left\{ \delta^{\lambda-1} \int_{\delta \leqq |t| \leqq \pi} \frac{|u(\rho, \theta + t)|^2}{|t|^\lambda} dt \right\}^{\frac{1}{2}},$$

where $\delta = 1 - \rho$. This function provides an estimate of the behaviour of $u(\rho, \psi)$ as the point (ρ, ψ) tends to the point $(1, \theta)$ in a "tangential" manner. Stein has proved that

(i) if f belongs to $L^p(-\pi, \pi)$, where $1 \leqq p < 2$, and if $0 < \mu < 1$ and $\lambda = 2/p$, then[1]

$$\left\{ \int_{-\pi}^{\pi} M_\lambda^{\mu p}(\theta) d\theta \right\}^{1/\mu} \leqq A(p, \mu) \int_{-\pi}^{\pi} |f(\theta)|^p d\theta, \tag{1.1}$$

(ii) if $|f|^p \log^+ |f|$ is integrable in $[-\pi, \pi]$, where $1 \leqq p < 2$, and if $\lambda = 2/p$, then

$$\int_{-\pi}^{\pi} M_\lambda^p(\theta) d\theta \leqq A(p) \int_{-\pi}^{\pi} |f(\theta)|^p \log^+ |f(\theta)| d\theta + A(p), \tag{1.2}$$

(iii) if f belongs to $L^p(-\pi, \pi)$, where $p > 1$, and if $\lambda > \sup(1, 2/p)$, then

$$\int_{-\pi}^{\pi} M_\lambda^p(\theta) d\theta \leqq A(p, \lambda) \int_{-\pi}^{\pi} |f(\theta)|^p d\theta. \tag{1.3}$$

In [4], I have considered the analogous function

$$L_{k,\lambda}(|\varphi|; \theta) = \sup_{0 \leqq \rho < 1} \left\{ \delta^{\lambda-1} \int_{-\pi}^{\pi} \frac{|\varphi(\rho e^{i\theta + it})|^k}{|1 - \rho e^{it}|^\lambda} dt \right\}^{1/k},$$

where φ is regular in $|z| < 1$, and have proved that if φ belongs to H^k, where $k > 0$, and if $\lambda > 1$, then

$$\int_{-\pi}^{\pi} L_{k,\lambda}^k(|\varphi|; \theta) d\theta \leqq A(k, \lambda) \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^k d\theta. \tag{1.4}$$

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1) We use $A(b, c, \ldots)$ to denote a positive constant depending only on $b, c, \ldots$, not necessarily the same on any two occurences ; A by itself will denote a positive absolute constant.

It is an easy corsequence of Parseval's theorem that, if the Taylor series of $\varphi(z)$ is $\sum_0^\infty c_m z^m$, and if $s_n(\theta) = \sum_0^n c_m e^{mi\theta}$, then

$$L_{2,2}(|\varphi|;\theta) = 2\pi \sup_{0\leqq\rho<1}\left\{(1-\rho)\sum_0^\infty |s_n(\theta)|^2\rho^{2n}\right\}^{\frac{1}{2}}.$$

In this form the function $L_{2,2}$ has been used by Chow [1] and Sunouchi [13]. In particular, Sunouchi has proved that if φ belongs to H and $0<\mu<1$, then

$$\left\{\int_{-\pi}^{\pi} L_{2,2}^{\mu}(|\varphi|;\theta)d\theta\right\}^{1/\mu} \leqq A(\mu)\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|d\theta, \tag{1. 5}$$

and that if φ belongs to $H\log^+ H$, then

$$\int_{-\pi}^{\pi} L_{2,2}(|\varphi|;\theta)d\theta \leqq A\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|\log^+|\varphi(e^{i\theta})|d\theta + A. \tag{1. 6}$$

These are, of course, the analogues of the case $p=1$ of (1. 1) and (1. 2) for the fonction $L_{2,2}(|\varphi|;\theta)$. Results concerning the analogue of the function $L_{2,2}$ in which $s_n(\theta)$ is replaced by the n-th partial sum of the Fourier series of an integrable f also occur implicitly in work of Marcinkiewicz [9] and Zygmund [14].

In this note I consider the function

$$L_{k,\lambda}(w;\theta) = \sup_{0\leqq\rho<1}\left\{\delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{w^k(\rho,\theta+t)}{|1-\rho e^{it}|^{\lambda}}dt\right\}^{1/k}$$

for a function $w(\rho,\theta)$ non-negative and subharmonic in the circle $\rho<1$, and obtain results which contain the inequalities (1. 1) – (1. 6) as particular cases. More precisely, we show that results of the form of (1. 3) and (1. 4) are relatively simple consequences of the Hardy-Littlewood maximal theorem. The results of the form of (1. 1), (1. 2), (1. 5), and (1. 6) are more difficult, and essentially we follow the method used by Stein. There is, however, a key result here, namely that which deals with the function $L_{k,\lambda}(u;\theta)$ for general $k>1$ in the case in which u is the Poisson integral of a non-negative f such that either f or $f\log^+ f$ is integrable (Theorem 2), and by reducing everything to this key result we are able to avoid many of the minor complications of Stein's argument. Finally, in §8 we use our results for $L_{k,\lambda}$ to give a very direct proof of a theorem on the Cesàro means of power series.

2. For any $w(\rho,\theta)$ defined and non-negative in the unit circle (and measurable in θ for each ρ), we write

$$L_{k,\lambda}(w;\theta)=\sup_{0\leqq\rho<1}\left\{\delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{w^k(\rho,\theta+t)}{|1-\rho e^{it}|^{\lambda}}dt\right\}^{1/k}, \tag{2. 1}$$

where $\delta=1-\rho$.

Our first result concerning the function $L_{k,\lambda}$ is of the type of (1. 3) and (1. 4).

THEOREM 1. *Let $f(\theta)$ be non-negative and periodic with period 2π, and be of class $L^p(-\pi,\pi)$, where $p>1$. Let also $u(\rho,\theta)$ be the Poisson integral of $f(\theta)$ (so that $u(\rho,\theta)\geqq 0$). Then, if $\lambda>k\geqq 1$,*

$$\int_{-\pi}^{\pi}L_{k,\lambda}^p(u;\theta)d\theta\leqq A(k,p,\lambda)\int_{-\pi}^{\pi}f^p(\theta)d\theta. \tag{2. 2}$$

The proof of Theorem 1 depends on a number of known inequalities which we state in the form of lemmas. In the case of Lemma 1, we actually state more than is required for the proof of Theorem 1; the additional results are used later.

LEMMA 1. *Let $f(\theta)$ be non-negative and integrable in $[-\pi,\pi]$ and be periodic with period 2π, and let*

$$f^*(\theta)=\sup_{0<|h|\leqq\pi}\left\{\frac{1}{h}\int_0^h f(\theta+t)dt\right\}. \tag{2. 3}$$

If $0<\mu<1$, then

$$\left\{\int_{-\pi}^{\pi}f^{*\mu}(\theta)\,d\theta\right\}^{1/\mu}\leqq A(\mu)\int_{-\pi}^{\pi}f(\theta)d\theta. \tag{2. 4}$$

Also

$$\int_{-\pi}^{\pi}f^*(\theta)d\theta\leqq A\int_{-\pi}^{\pi}f(\theta)\,\log^+f(\theta)d\theta+A, \tag{2. 5}$$

and, for $p>1$,

$$\int_{-\pi}^{\pi}f^{*p}(\theta)d\theta\leqq A(p)\int_{-\pi}^{\pi}f^p(\theta)d\theta, \tag{2. 6}$$

whenever the integrals on the right are finite.

These are the Hardy-Littlewood "Real Max" inequalities [5].

LEMMA 2. *Let $f(\theta)$ be non-negative and integrable in $[-\pi,\pi]$ and be periodic with period 2π, let $f^*(\theta)$ be defined by* (2. 3), *and let*

$$I_\omega(\theta)=\delta^{\omega-1}\int_{-\pi}^{\pi}\frac{f(\theta+t)}{|1-\rho e^{it}|^{\omega}}dt,$$

where $\delta = 1 - \rho$ *and* $\omega > 1$. *Then*

$$I_\omega(\theta) \leqq A(\omega) f^*(\theta).$$

LEMMA 3. *Let* $f(\theta)$ *be non-negative and integrable in* $[-\pi, \pi]$ *and be periodic with period* 2π, *let* $u(\rho, \theta)$ *be the Poisson integral of* $f(\theta)$ (*so that* $u(\rho, \theta) \geqq 0$), *and let* $f^*(\theta)$ *be defined by* (2. 3). *Then*

$$u(\rho, \theta + t) \leqq A f^*(\theta) |1 - \rho e^{it}| / \delta,$$

where $\delta = 1 - \rho$.

Both of these lemmas are due essentially to Hardy and Littlewood [6] (see also [2], Lemma 4).

LEMMA 4. *If* $P(\rho, t)$ *is the Poisson kernel, and if* $1 < a \leqq 2$, *then*

$$\int_{-\pi}^{\pi} \frac{P(\rho, s - t)}{|1 - \rho e^{it}|^a} dt \leqq \frac{A(a)}{|1 - \rho e^{is}|^a}.$$

This is a particular case of Lemma 2 of [3].

Consider now the proof of Theorem 1. We prove that if $u(\rho, \theta)$ is the Poisson integral of an integrable f, and if $\lambda > k \geqq 1$, then

$$L_{k,\lambda}(u; \theta) \leqq A(k, \lambda) f^*(\theta),$$

where f^* is defined by (2. 3). The inequality (2. 2) follows immediately from this and (2. 6).

Since $L_{k,\lambda}$ is a decreasing function of λ, we may suppose that $\lambda \leqq k + 1$. By Lemma 3,

$$\delta^{\lambda-1} \int_{-\pi}^{\pi} \frac{u^k(\rho, \theta + t)}{|1 - \rho e^{it}|^\lambda} dt \leqq A(k) f^{*k-1}(\theta)\, \delta^{\lambda-k} \int_{-\pi}^{\pi} \frac{u(\rho, \theta + t)}{|1 - \rho e^{it}|^{1+\lambda-k}} dt.$$

Also

$$\begin{aligned} \delta^{\lambda-k} \int_{-\pi}^{\pi} \frac{u(\rho, \theta + t)}{|1 - \rho e^{it}|^{1+\lambda-k}} dt &= \frac{\delta^{\lambda-k}}{2\pi} \int_{-\pi}^{\pi} f(\theta + s) ds \int_{-\pi}^{\pi} \frac{P(\rho, s - t)}{|1 - \rho e^{it}|^{1+\lambda-k}} dt \\ &\leqq A(k, \lambda) \delta^{\lambda-k} \int_{-\pi}^{\pi} \frac{f(\theta + s)}{|1 - \rho e^{is}|^{1+\lambda-k}} ds \\ &\leqq A(k, \lambda) f^*(\theta), \end{aligned}$$

by Lemmas 4 and 2, and this is the required result.

3. The key result for the function $L_{k,\lambda}$ of the type of (1. 1), (1. 2), (1. 5), and (1. 6) is as follows.

THEOREM 2. *Let* $f(\theta)$ *be non-negative and integrable in* $[-\pi, \pi]$ *and* *be periodic with period* 2π, *and let* $u(\rho, \theta)$ *be the Poisson integral of* $f(\theta)$. *Then*

(i) *if* $0<\mu<1$ *and* $\lambda \geqq k>1$,

$$\left\{\int_{-\pi}^{\pi} L_{k,\lambda}^{\mu}(u;\theta)d\theta\right\}^{1/\mu} \leqq A(k, \lambda, \mu)\int_{-\pi}^{\pi} f(\theta)d\theta, \tag{3. 1}$$

(ii) *if* $\lambda \geqq k>1$, *and if* $f \log^+ f$ *is integrable*,

$$\int_{-\pi}^{\pi} L_{k,\lambda}(u;\theta)d\theta \leqq A(k, \lambda)\int_{-\pi}^{\pi} f(\theta)\log^+ f(\theta)d\theta + A(k, \lambda). \tag{3. 2}$$

Since $L_{k,\lambda}$ is a decreasing function of λ, it is enough to prove these results when $\lambda = k$. Further, if the results (3. 1) and (3. 2) hold for any given k, then they hold also for all larger k. To prove this, we observe that if $r>k$, then

$$L_{r,r}(u;\theta) \leqq A(k, r) f^{*(r-k)/r}(\theta) L_{k,k}^{k/r}(u;\theta)$$

(this follows immediately from the definition (2. 1) and Lemma 3). Thus if $0<\mu \leqq 1$,

$$\int_{-\pi}^{\pi} L_{r,r}^{\mu} d\theta \leqq A(k, r)\int_{-\pi}^{\pi} f^{*\mu(r-k)/r} L_{k,k}^{\mu k/r} d\theta$$

$$\leqq A(k, r)\left\{\int_{-\pi}^{\pi} f^{*\mu} d\theta\right\}^{(r-k)/r}\left\{\int_{-\pi}^{\pi} L_{k,k}^{\mu} d\theta\right\}^{k/r},$$

by Hölder's inequality with indices $r/(r-k)$ and r/k. Using the inequalities (2. 4) and (2. 5) for f^* and the inequalities (3. 1) and (3. 2) for $L_{k,k}$, we obtain the inequalities (3. 1) and (3. 2) for $L_{r,r}$, and this proves the statement.

In the proof of Theorem 2, we may now suppose that k is as near 1 as we please. More precisely, we suppose that $1<k \leqq 4/3$.

We prove next that if $1<k \leqq 4/3$, then

$$L_{k,k}(u;\theta) \leqq A(k) \sup_{0 \leqq \rho<1}\left\{\delta^{k-1}\int_{-\pi}^{\pi} \frac{f(\theta+s)u^{k-1}(\rho^2, \theta+s)}{|1-\rho e^{is}|^k} ds\right\}^{1/k}. \tag{3. 3}$$

We have

$$\delta^{k-1}\int_{-\pi}^{\pi} \frac{u^k(\rho, \theta+t)}{|1-\rho e^{it}|^k} dt = \frac{\delta^{k-1}}{2\pi}\int_{-\pi}^{\pi} f(\theta+s)ds\int_{-\pi}^{\pi} \frac{u^{k-1}(\rho, \theta+t)}{|1-\rho e^{it}|^k} P(\rho, s-t)dt. \tag{3. 4}$$

By Hölder's inequality with indices $1/(k-1)$ and $1/(2-k)$, and by Lemma 4,

$$\int_{-\pi}^{\pi} \frac{u^{k-1}(\rho, \theta+t)}{|1-\rho e^{it}|^k} P(\rho, s-t)dt \tag{3. 5}$$

$$\leqq \left\{ \int_{-\pi}^{\pi} u(\rho, \theta + t) P(\rho, s - t) dt \right\}^{k-1} \left\{ \int_{-\pi}^{\pi} \frac{P(\rho, s - t)}{|1 - \rho e^{it}|^{k/(2-k)}} dt \right\}^{2-k}$$

$$\leqq \frac{A(k) u^{k-1}(\rho^2, \theta + s)}{|1 - \rho e^{is}|^k}$$

(since $k/(2 - k) \leqq 2$), and (3. 3) is an immediate consequence of (3. 4) and (3. 5).

4. Suppose now that $1 < k \leqq 4/3$, and for any $y > 0$ let D_y be the set of θ in $-\pi < \theta < \pi$ such that $L_{k,k}(u; \theta) > y$. Then we have

(4. 1)
$$|D_y| \leqq \frac{A(k)}{y} \int_{-\pi}^{\pi} f(\theta) d\theta.$$

Further, there exists a constant c, depending only on k, such that

(4. 2)
$$|D_y| \leqq \frac{A(k)}{y} \int_{f(\theta) > cy} f(\theta) d\theta.$$

The inequalities (3. 1) and (3. 2) follow almost immediately from (4. 1) and (4. 2), respectively. The arguments used in the deduction of (3. 1) and (3. 2), and also those used in the deduction of (4. 2) from (4. 1), are identical to those used by Stein [12], and we therefore omit them.

Consider then the proof of (4. 1). Here, too, the argument used is similar to that of Stein, but the restriction to the particular case considered in Theorem 2 enables us to make certain simplifications, and we therefore give the proof of (4. 1) in full. We require two further lemmas.

LEMMA 5. *Let $f(\theta)$ be non-negative and integrable in $[-\pi, \pi]$ and be periodic with period 2π, let $f^*(\theta)$ be defined by* (2. 3), *and for any $z > 0$ let E_z be the set of θ in $-\pi < \theta < \pi$ for which $f^*(\theta) > z$. Then E_z is open, and*

$$|E_z| \leqq \frac{2}{z} \int_{-\pi}^{\pi} f(\theta) d\theta.$$

This lemma is due to F. Riesz [11].

LEMMA 6. *Let Q be an open set situated in $[-\pi, \pi]$, and let P be its complement relative to this interval. Let P' be the set of points θ congruent modulo 2π to points of P, and for any θ let $\Delta(\theta)$ denote the distance of θ from P'. Let also f be non-negative and integrable over $[-\pi, \pi]$ and be periodic with period 2π, and let*

$$N(\theta) = \int_{-\pi}^{\pi} \frac{\Delta^{k-1}(\theta + t)}{|t|^k} f(\theta + t) dt,$$

where $k > 1$. *Then* $N(\theta)$ *is finite p. p., and*

$$\int_P N(\theta)d\theta \leqq A(k)\int_{-\pi}^{\pi} f(\theta)d\theta.$$

This lemma is due to Marcinkiewicz and Zygmund (see e. g. Stein [12]).

Now let z be a positive number, to be chosen later, and let E_z be the set of θ in $-\pi < \theta < \pi$ such that $f^*(\theta) > z$. By Lemma 5, E_z is open. Let P be the complement of E_z relative to the interval $[-\pi, \pi]$, and let $\Delta(\theta)$ be the distance function of Lemma 6 corresponding to this set P. Since $f^*(\theta) \leqq z$ when θ belongs to P, we have

$$L_{k,k}^k(u;\theta) \leqq B_1 z^k + B_2 z^{k-1}\int_{-\pi}^{\pi} \frac{\Delta^{k-1}(\theta+s)}{|s|^k} f(\theta+s)ds \tag{4. 3}$$

for any θ of P, where B_1 and B_2 are constants depending only on k. For, by Lemma 3,

$$\begin{aligned} u^{k-1}(\rho^2, \theta+s) &\leqq A(k)z^{k-1}\frac{|1-\rho e^{i\Delta(\theta+s)}|^{k-1}}{\delta^{k-1}} \\ &\leqq A(k)z^{k-1}\left\{1+\frac{\Delta^{k-1}(\theta+s)}{\delta^{k-1}}\right\} \end{aligned}$$

whenever θ belongs to P, so that, by (3. 3),

$$\begin{aligned} L_{k,k}^k(u;\theta) \leqq A(k)z^{k-1} &\sup_{0\leqq\rho<1}\left\{\delta^{k-1}\int_{-\pi}^{\pi}\frac{f(\theta+s)}{|1-\rho e^{is}|^k}ds\right\} \\ &+ A(k)z^{k-1}\int_{-\pi}^{\pi}\frac{\Delta^{k-1}(\theta+s)}{|s|^k}f(\theta+s)ds, \end{aligned} \tag{4. 4}$$

and (4. 3) follows from (4. 4) and Lemma 2, since $f^*(\theta) \leqq z$ in P.

So far z is at our disposal. Choose now $B_1 z^k = \frac{1}{2}y^k$. By (4. 3), the set $D_y \cap P$ is contained in the set of θ in which

$$B_2 z^{k-1}\int_{-\pi}^{\pi}\frac{\Delta^{k-1}(\theta+s)}{|s|^k}f(\theta+s)ds > \frac{1}{2}y^k,$$

i. e. is contained in the set of θ in which $N(\theta) > B_3 y$, where $N(\theta)$ is the function of Lemma 6, and B_3 is an $A(k)$. But, by Lemma 6,

$$B_3 y|D_y \cap P| \leqq \int_{D_y\cap P} N d\theta \leqq \int_P N d\theta \leqq A(k)\int_{-\pi}^{\pi} f\,d\theta.$$

Since also

$$|E_z| \leqq \frac{2}{z}\int_{-\pi}^{\pi} f d\theta = \frac{A(k)}{y}\int_{-\pi}^{\pi} f d\theta$$

(by Lemma 5), and

$$|D_y| = |D_y \cap P| + |D_y \cap E_z| \leqq |D_y \cap P| + |E_z|,$$

this completes the proof of (4. 1), and so of Theorem 2.

5. We pass next to subharmonic functions. Here we have

THEOREM 3. *Let $w(\rho, \theta)$ be non-negative and subharmonic in the unit circle. Then*

(i) *if* $0 < \mu < 1$ *and* $\lambda \geqq k > 1$,

$$\left\{\int_{-\pi}^{\pi} L_{k,\lambda}^{\mu}(w;\,\theta)d\theta\right\}^{1/\mu} \leqq A(k,\,\lambda,\,\mu)\lim_{R\to 1-}\int_{-\pi}^{\pi} w(R,\,\theta)d\theta, \tag{5. 1}$$

(ii) *if* $\lambda \geqq k > 1$,

$$\int_{-\pi}^{\pi} L_{k,\lambda}(w;\,\theta)d\theta \leqq A(k,\,\lambda)\lim_{R\to 1-}\int_{-\pi}^{\pi} w(R,\,\theta)\,\log^{+}w(R,\,\theta)d\theta + A(k,\,\lambda), \tag{5. 2}$$

(iii) *if* $p > 1$ *and* $\lambda > k \geqq 1$,

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(w;\,\theta)d\theta \leqq A(k,\,p,\,\lambda)\lim_{R\to 1-}\int_{-\pi}^{\pi} w^{p}(R,\,\theta)d\theta, \tag{5. 3}$$

provided in each case that the limit on the right is finite.

We require a further lemma.

LEMMA 7. *Let $w(\rho, \theta)$ be subharmonic in the unit circle and satisfy the relation*

$$\int_{-\pi}^{\pi} w(\rho,\,\theta)d\theta \leqq C \tag{5. 4}$$

for $\rho < 1$, and for $0 < \rho \leqq R < 1$ let $u_R(\rho, \theta)$ be the Poisson integral of the values of $w(\rho, \theta)$ on the circle $\rho = R$. Then as $R \to 1-$ the function $u_R(\rho, \theta)$ converges in the circle $\rho < 1$, and uniformly in any circle $\rho \leqq r < 1$, to a (finite) harmonic function $u^*(\rho, \theta)$ such that

$$w(\rho,\,\theta) \leqq u^*(\rho,\,\theta)$$

in $\rho < 1$.

This lemma is proved implicitly by Littlewood [7].

Consider now the proof of the theorem. Since w, $w \log^+ w$, and w^p $(p > 1)$

are subharmonic[2], the integrals on the right of (5. 1), (5. 2), and (5. 3) are increasing functions of R[3], so that the limits on the right of these inequalities always exist. Moreover, if these limits are finite, then the condition (5. 4) is satisfied, and we may apply the results of Lemma 7.

We observe next that if the conditions of Lemma 7 are satisfied, then, for any fixed $\rho < 1$, $u_R(R\rho, \theta) \to u^*(\rho, \theta)$ as $R \to 1-$, uniformly in θ; this is a simple consequence of the uniformity of the convergence of $u_R(\rho, \theta)$ to $u^*(\rho, \theta)$ and of the (uniform) continuity of $u^*(\rho, \theta)$. Hence for any fixed $\rho < 1$

$$(5.\ 5)\qquad \delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{w^k(\rho,\ \theta+t)}{|1-\rho e^{it}|^\lambda}dt \leqq \delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{u^{*k}(\rho,\ \theta+t)}{|1-\rho e^{it}|^\lambda}dt$$
$$= \delta^{\lambda-1}\int_{-\pi}^{\pi}\lim_{R\to1-}\left\{\frac{u_R^k(R\rho,\ \theta+t)}{|1-\rho e^{it}|^\lambda}\right\}dt$$
$$= \lim_{R\to1-}\left\{\delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{u_R^k(R\rho,\ \theta+t)}{|1-\rho e^{it}|^\lambda}dt\right\}.$$

Now for any function $F(R,\ \rho)$ defined in the square $0 \leqq R < 1,\ 0 \leqq \rho < 1$ and such that $\lim_{R\to1-} F(R,\ \rho)$ exists for each ρ, we have

$$(5.\ 6)\qquad \sup_{0\leqq\rho<1}\{\lim_{R\to1-} F(R,\ \rho)\} \leqq \liminf_{R\to1-}\{\sup_{0\leqq\rho<1} F(R,\ \rho)\}.$$

Writing $v_R(\rho, \theta) = u_R(R\rho, \theta)$, we thus obtain from (5. 5) and (5. 6) the relation

$$L^l_{k,\lambda}(w;\ \theta) \leqq \liminf_{R\to1-} L^l_{k,\lambda}(v_R;\ \theta)$$

for any $l > 0$ and so, by the extension of Fatou's lemma which involves limits inferior on both sides of the inequality,

$$(5.\ 7)\qquad \int_{-\pi}^{\pi} L^l_{k,\lambda}(w;\ \theta)d\theta \leqq \liminf_{R\to1-}\int_{-\pi}^{\pi} L^l_{k,\lambda}(v_R;\ \theta)d\theta.$$

Applying now the results of Theorems 1 and 2 to the function $v_R(\rho, \theta) = u_R(R\rho,\ \theta)$ in the circle $\rho < 1$, making $R \to 1-$, and using (5. 7), we obtain immediately the results of Theorem 3.

6. It is not difficult now to extend the results of Theorem 3 to other indices.

THEOREM 4. *Let $w(\rho,\ \theta)$ be non-negative and subharmonic in the unit circle. Then*

(i) *if* $1 \leqq p < k$, $0 < \mu < 1$, *and* $\lambda \geqq k/p$,

2) See, for example, Rado [10], § 3. 13.
3) See, for example, Rado [10], § 2. 4.

$$\left\{\int_{-\pi}^{\pi} L_{k,\lambda}^{\mu p}(w;\, \theta)d\theta\right\}^{1/\mu} \leqq A(k,\, p,\, \lambda,\, \mu) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta)d\theta,$$

(ii) *if* $1 \leqq p < k$ *and* $\lambda \geqq k/p$,

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(w;\, \theta)d\theta \leqq A(k,\, p,\, \lambda) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta) \log^+ w(R,\, \theta)d\theta + A(k, p,\, \lambda),$$

(iii) *if* $k \geqq 1$, $p > 1$, *and* $\lambda > \sup\,(1,\, k/p)$,

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(w;\, \theta)d\theta \leqq A(k,\, p,\, \lambda) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta)d\theta,$$

provided in each case that the limit on the right is finite.

We observe first that w^p is subharmonic whenever w is subharmonic and $p \geqq 1$, and that

$$L_{r,\lambda}(w^p;\, \theta) = L_{pr,\lambda}^{p}(w;\, \theta).$$

Hence, by the inequalities (5. 1) and (5. 2) applied to w^p, with r in place of k,

$$\left\{\int_{-\pi}^{\pi} L_{pr,\lambda}^{\mu p}(w;\, \theta)d\theta\right\}^{1/\mu} \leqq A(r,\, \lambda,\, \mu) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta)d\theta$$

and

$$\int_{-\pi}^{\pi} L_{pr,\lambda}^{p}(w;\, \theta)d\theta \leqq A(r, \lambda) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta) \log^+ w(R,\, \theta)d\theta + A(r,\, \lambda)$$

for $\lambda \geqq r > 1$. If in these we write $k = pr$, we obtain immediately the results (i) and (ii) of Theorem 4.

To obtain the result of (iii), we apply the inequality (5. 3), with r in place of k and q in place of p, to the function w^l, where $l \geqq 1$. This gives

$$\int_{-\pi}^{\pi} L_{lr,\lambda}^{lq}(w;\, \theta)d\theta \leqq A(q, r,\, \lambda) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^{lq}(R,\, \theta)d\theta$$

for $q > 1$ and $\lambda > r \geqq 1$. Writing $lr = k$, $lq = p$, we obtain from this the inequality

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(w;\, \theta)d\theta \leqq A(k,\, p,\, q,\, \lambda) \lim_{R\to 1-} \int_{-\pi}^{\pi} w^p(R,\, \theta)d\theta \tag{6. 1}$$

for $k \geqq 1$, $p > 1$, $q > 1$, and $\lambda > qk/p \geqq 1$. Here q is at our disposal subject only to the condition $q > 1$. If now $k/p < 1$, we choose $q = p/k$, and then (6. 1) holds for $\lambda > 1$. On the other hand, if $\lambda > k/p \geqq 1$, we choose q so that both $q > 1$ and $\lambda > qk/p$, and then again (6. 1) holds. Thus (6. 1) holds for $\lambda > \sup\,(1,\, k/p)$, and this is the required result.

7. There are similar results for harmonic functions and regular functions. These are as follows.

THEOREM 5. *Let $f(\theta)$ be integrable in $[-\pi, \pi]$ and be periodic with period 2π, and let $u(\rho, \theta)$ be the Poisson integral of $f(\theta)$. Then*

(i) *if $1 \leqq p < k$, $0 < \mu < 1$, and $\lambda \geqq k/p$, and if f belongs to $L^p(-\pi, \pi)$,*

$$\left\{\int_{-\pi}^{\pi} L_{k,\lambda}^{\mu p}(|u|; \theta)d\theta\right\}^{1/\mu} \leqq A(k, p, \lambda, \mu)\int_{-\pi}^{\pi} |f(\theta)|^p d\theta,$$

(ii) *if $1 \leqq p < k$ and $\lambda \geqq k/p$, and if $|f|^p \log^+ |f|$ is integrable,*

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(|u|; \theta)d\theta \leqq A(k, p, \lambda)\int_{-\pi}^{\pi} |f(\theta)|^p \log^+ |f(\theta)| d\theta + A(k, p, \lambda),$$

(iii) *if $k \geqq 1$, $p > 1$, and $\lambda > \sup(1, k/p)$, and if f belongs to $L^p(-\pi, \pi)$,*

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(|u|; \theta)d\theta \leqq A(k, p, \lambda)\int_{-\pi}^{\pi} |f(\theta)|^p d\theta.$$

THEOREM 6. *Let $\varphi(z)$ be regular in $|z| < 1$. Then*

(i) *if $0 < p < k$, $0 < \mu < 1$, and $\lambda \geqq k/p$, and if φ belongs to H^p,*

$$\left\{\int_{-\pi}^{\pi} L_{k,\lambda}^{\mu p}(|\varphi|; \theta)d\theta\right\}^{1/\mu} \leqq A(k, p, \lambda, \mu)\int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta,$$

(ii) *if $0 < p < k$ and $\lambda \geqq k/p$, and if φ belongs to $H^p \log^+ H$,*

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(|\varphi|; \theta)d\theta \leqq A(k, p, \lambda)\int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p \log^+ |\varphi(e^{i\theta})| d\theta + A(k, p, \lambda),$$

(iii) *if $k > 0$, $p > 0$, and $\lambda > \sup(1, k/p)$, and if φ belongs to H^p,*

$$\int_{-\pi}^{\pi} L_{k,\lambda}^{p}(|\varphi|; \theta)d\theta \leqq A(k, p, \lambda)\int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta.$$

The cases $k = 2$ of the results of Theorem 5 are slightly stronger than the results of Stein listed in § 1, since $M_\lambda(\theta) \leqq A(\lambda)L_{2,\lambda}(|u|; \theta)$.

Here Theorem 5 is an immediate consequence of Theorem 4. Alternatively, it may be deduced directly from Theorems 1 and 2 using an argument similar to that used in deducing Theorem 4 from Theorem 3, for, by Jensen's inequality, $|u|^p$ does not exceed the Poisson integral of $|f|^p$ for $p \geqq 1$.

Theorem 6 follows easily from Theorem 3 by an argument similar to that used in deducing Theorem 4 from Theorem 3 (for $|\varphi|^p$ is subharmonic for every $p > 0$).

9. Cesàro means of power series. We give finally a new proof of the following theorem of Stein, Zygmund, and others (for references, see Stein [12]).

THEOREM 7. *Let $\varphi(z)$ be regular in $|z|<1$, let $\varphi(z)=\sum_0^\infty c_n z^n$, and let $\sigma_n^\alpha(\theta)$ be the n-th (C,α) mean of the series $\sum_0^\infty c_n e^{nt\theta}$. Then*

(i) *if $0<p<1$, $0<\mu<1$ and $\alpha\geqq 1/p-1$, and if φ belongs to H^p,*

$$\left\{\int_{-\pi}^{\pi}\{\sup_n|\sigma_n^\alpha(\theta)|\}^{\mu p}d\theta\right\}^{1/\mu}\leqq A(p,\ \alpha,\ \mu)\int_{-\pi}^{\pi}|\varphi(e^{i\vartheta})|^p d\theta, \tag{8. 1}$$

(ii) *if $0<p<1$ and $\alpha\geqq 1/p-1$, and if φ belongs to $H^p\log^+H$,*

$$\int_{-\pi}^{\pi}\{\sup_n|\sigma_n^\alpha(\theta)|\}^p d\theta\leqq A(p,\alpha)\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|^p\log^+|\varphi(e^{i\theta})|d\theta+A(p,\ \alpha), \tag{8. 2}$$

(iii) *if $p>0$ and $\alpha>\sup(0,\ 1/p-1)$, and if φ belongs to H^p,*

$$\int_{-\pi}^{\pi}\{\sup_n|\sigma_n^\alpha(\theta)|\}^p d\theta\leqq A(p,\ \alpha)\int_{-\pi}^{\pi}|\varphi(e^{i\theta})|^p d\theta. \tag{8. 3}$$

It is familiar that

$$\frac{\varphi(ze^{i\theta})}{(1-z)^{\alpha+1}}=\sum_0^\infty E_n^\alpha\ \sigma_n^\alpha(\theta)z^n$$

for $|z|<1$, where

$$E_n^\alpha=\frac{(\alpha+1)\,(\ldots\ldots)\,(\alpha+n)}{n\,!}\qquad (n>0),\quad E_o^\alpha=1.$$

Hence

$$E_n^\alpha\ \sigma_n^\alpha(\theta)\rho^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varphi(\rho e^{i\theta+it})e^{-nti}}{(1-\rho e^{it})^{\alpha+1}}dt, \tag{8. 4}$$

and so

$$|E_n^\alpha||\sigma_n^\alpha(\theta)|\rho^n\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\varphi(\rho e^{i\theta+it})|}{|1-\rho e^{it}|^{\alpha+1}}dt \tag{8. 5}$$

for $\rho<1$. Multiplying both sides of (8. 5) by $(1-\rho)^\alpha$, taking $\rho=1-1/(n+1)$ on the left, and using the fact that $E_n^\alpha\sim A(\alpha)n^\alpha$ for $\alpha>-1$, we obtain

$$\begin{aligned}\sup_n|\sigma_n^\alpha(\theta)|&\leqq A(\alpha)\sup_{0\leqq\rho<1}\left\{\delta^\alpha\int_{-\pi}^{\pi}\frac{|\varphi(\rho e^{i\theta+it})|}{|1-\rho e^{it}|^{\alpha+1}}dt\right\}\\&=A(\alpha)L_{1,\alpha+1}(|\varphi|\,;\,\theta)\end{aligned}$$

for $\alpha > -1$, and the theorem follows immediately from this and Theorem 6.

9. The proof of Theorem 7 given above shows that the inequalities (8. 1)—(8. 3) do not depend on the presence of the oscillating factor e^{-nti} in the integral formula (8. 4). In this respect they are similar to the case $0 < p \leqq 1$ of the inequaltiy which replaces (8. 3) in the limiting case $\alpha = \sup\ (0,\ 1/p - 1)$, namely

$$(9.\ 1) \qquad \int_{-\pi}^{\pi} \left\{ \sup_{n \geqq 1} \left(\frac{|\sigma_n^{\alpha}(\theta)|}{\log^{\beta}(n+1)} \right) \right\}^p d\theta \leqq A(p,\ \alpha,\ \beta) \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^p d\theta.$$

which holds for $\alpha = 1/p - 1$ and $\beta = 1/p$ (see [4]). The case $p > 1$ of (9. 1), which holds for $\alpha = 0,\ \beta = 1/p$, or $\alpha = 0,\ \beta = (p-1)/p$ according as $1 < p \leqq 2$ or $p \geqq 2$ (Littlewood and Paley [8]), would thus appear to be the most delicate of all these inequalities for Cesàro means, for all the known proofs of this case require some consideration of the oscillating factor in (8. 4).

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DEPARTMENT OF PURE MATHEMATICS,
THE UNIVERSITY,
LIVERPOOL.