ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION I[1)]

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1. Introduction. In the preceding paper (Sunouchi-Watari [7]), we have determined the class of saturation for various methods of summation in the theory of Fourier series. In this note we shall prove Fourier integral analogues with necessary modification. But we need somewhat complicated arguments because of the non-existence of the Fourier transform for the class $L^p(p > 2)$ and the pointwise indetermination of the transform even if it exists.

Suppose now that a singular integral operator with kernel $k(t)$

$$(1) \qquad T_\xi(t) \equiv T_\xi(t;f) = \frac{1}{\sqrt{2\pi}\,\xi}\int_{-\infty}^{\infty} f(t+u)k\left(\frac{u}{\xi}\right)du$$

exists. In the following line, the norm means (C) or $L^p(p \geqq 1)$ norm. If there are a positive non-decreasing function $\varphi(\xi)$ and a class of functions $\mathfrak{K}$ such that

(1°) $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$ as $\xi \to 0$ implies that $f(t)$ is an invariant element of the operator $T_\xi(t;f)$,

(2°) $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$ implies $f(t) \in \mathfrak{K}$,

(3°) For every $f \in \mathfrak{K}$, we have $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$,

then it is said that the singular integral operator (1) having the kernel $k(t)$ is saturated with the order $\varphi(\xi)$ and the class $\mathfrak{K}$.

Our problem is to determine the order $\varphi(\xi)$ and the class $\mathfrak{K}$ of saturation. Recently P.L. Butzer [2] has solved the saturation problem for some singular integral operators from the general theory of semi-groups. But many popular singular integral operators don't make semi-groups. We shall give here a direct method to determine the class of saturation for the general operators of singular integrals and supply proofs of some conjectures of Butzer.

We suppose that the kernel $k(t)$ has a continuous function $K(u)$ as its Fourier transform. Moreover we suppose that for some positive functions $\varphi(\xi)$ $(\varphi(\xi) \downarrow 0$ as $\xi \to +0)$ and $\psi(u)$, we have

$$(2) \qquad \lim_{\xi \to +0} \frac{1 - K(u\xi)}{\varphi(\xi)} = c\psi(u) \qquad (c \neq 0)$$

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for all u. Roughly speaking, we shall see that this $\varphi(\xi)$ is the order of saturation and $\psi(u)$ determines the class of saturation.

2. The case $f(t) \in L^p(-\infty, \infty)$ $(1 \leqq p \leqq 2)$. Since $f(t) \in L^p(-\infty, \infty)$. $(1 \leqq p \leqq 2)$, there is a Fourier transform $F(u)$ such as

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut}\, dt \qquad (p = 1)$$

$$F(u) = \underset{a\to\infty}{\overset{(q)}{\lim}} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(t) e^{-iut}\, dt \qquad (1 < p \leqq 2)$$

where $1/p + 1/q = 1$.

By the convolution theorem,

$$T_\xi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) k(u\xi) e^{itu}\, du,$$

where this integral is assumed to exist.

First we consider the proposition (1°).

Since the formal Fourier inversion of $f(t) - T_\xi(t)$ is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{1 - K(u\xi)\right\} F(u)\, e^{iut}\, dt,$$

the Fejér's integral of this is

$$\sigma_\lambda[f(t) - T_\xi(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \{f(t) - T_\xi(t)\} \frac{2 \sin^2 \frac{\lambda}{2}(x-t)}{\lambda(x-t)^2}\, dt$$

$$= \int_{-\lambda}^{\lambda} \left(1 - \frac{|u|}{\lambda}\right) F(u) \{1 - K(u\xi)\} e^{iut}\, dt.$$

From Jensen's inequality, we have

$$\|\sigma_\lambda[T_\xi(t) - f(t)]\| \leqq \|T_\xi(t) - f(t)\|.$$

Applying Fatou's lemma and noticing (2), we get

$$\left\| \int_{-\lambda}^{\lambda} \left(1 - \frac{|u|}{\lambda}\right) F(u)\, \psi(u)\, e^{iut}\, du \right\| = o(1).$$

The uniqueness theorem of trigonometrical integral (Cooper [3]) yields

$$F(u)\,\psi(u) = 0, \qquad \text{a. e.}$$

and $\psi(u) \neq 0$ implies $F(u) = 0$, a. e. That is $f(t) = 0$, a. e.

By the same argument the hypothesis

$$\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$$

implies

$$\sup_{\lambda} \left\| \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)\, \psi(u)\, e^{iut}\, du \right\| = O(1).$$

If $p = 1$, by Cramer's theorem [4], $F(u)\psi(u)$ is representable by the Fourier-Stieltjes integral of a function of bounded variation. Especially if $\psi(u) = u^k$ ($k =$ a positive integer), then $f^{(k-1)}(t) \in BV(-\infty, \infty)$ and $\psi(u) = |u|^k$, then $\tilde{f}^{(k-1)}(t) \in BV(-\infty, \infty)$ when k is odd integer.

If $1 < p \leqq 2$, then by a theorem of Offord [6], if we put

$$g_\lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)\, \psi(u)\, e^{-iut}\, du$$

then, there is a function $g(t) \in L^p(-\infty, \infty)$ such that

$$\underset{\lambda\to\infty}{\overset{(p)}{\lim}}\; g_\lambda(t) = g(t).$$

When $\psi(u) = u^k$, or $|u|^k$, then $f^{(k)}(t) \in L^p$.

3. The case $f(t) \in L^p\,(-\infty, \infty)$, $(2 < p < \infty)$ and the case $C(-\infty, \infty)$.

In this case we will assume

$$f(t)/(1 + |t|) \in L(-\infty, \infty).$$

Of course $f(t) \in L^p(-\infty, \infty)$ implies this. Instead of the Fourier transform of $f(t)$, we may take the first transform of Bochner-Hahn (Bochner [1] and Titchmarsh [8]). We define

$$\mathfrak{F}(u) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} + \int_{1}^{\infty}\right] \frac{f(t)}{it}\, e^{-iut}\, dt + \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{f(t)(e^{-iut} - 1)}{it}\, dt.$$

Since $\mathfrak{F}(u)$ is absolutely continuous in every finite interval, we can define

$$F(u) = \mathfrak{F}'(u).$$

Then the Fejér's integral of $f(t) - T_\xi(t)$ is the same form in the ordinary case and we have

$$\left\| \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)\, \psi(u)\, e^{iut}\, du \right\| = o(1)$$

under the assumption $\|T_\xi(t) - f(t)\| = o\{\varphi(\xi)\}$ and

$$\sup_{\lambda} \left\| \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)\, \psi(u)\, e^{iut}\, du \right\| = O(1)$$

under the assumption $\|T_\xi(t) - f(t)\| = O\{\varphi(\xi)\}$.

Using the same theorems as in the above case, we can conclude respectively that

$$f(t) = 0, \quad \text{a. e.}$$

and

$$\overset{(p)}{\lim_{\lambda\to\infty}} \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \left(1 - \frac{|u|}{\lambda}\right) F(u)\,\psi(u)\,e^{itu}\,du, \ (2<p<\infty, p=\infty \text{ for the case } C)$$

exists.

If the norm is L^p-norm and $\psi(k) = u^k$ or $|u|^k$, then $f^{(k)}(t) \in L^p(2<p<\infty)$ and if the norm is C-norm and $\psi(u) = u^k$, then $f^{(k)}(t) \in L^\infty$, and if $\psi(k) = |u|^k$ where $k =$ odd integer, then $\tilde{f}^{(k)}(t) \in L^\infty$.

Thus we have complete solutions of the problems (1°) and (2°). For the proof of (3°), we can use the so-called singular integral method and there are many avaiable results.

4. Determination of the class of saturation for various singular integral operators. In this section we apply the above general theory to the special singularintegral operators. Since the proof of (3°) is routine argument, we omit it.

(a) Fejér's singular integral.
This is

$$F_\xi(t; f) = \frac{1}{2\pi\xi} \int_{-\infty}^{\infty} f(t+u) \left[\frac{\sin (u/2\xi)}{u/2\xi}\right]^2 du.$$

Since

$$k(t) = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin (t/2)}{t/2}\right]^2 \quad \text{and} \quad K(u) = 1 - |u|,$$

$\varphi(\xi) = \xi$ and $\psi(u) = |u|$ respectively. When we take the C-norm, the class of saturation is the functions $\{f(t) | \tilde{f}'(t) \in L^\infty\}$ and the order is ξ and the invariant element is o. We denote it by

$$\text{Sat } [F_\xi(t)]_C = \{f(t) | \tilde{f}'(t) \in L^\infty ; \xi ; o\}$$

Analogously we get

$$\text{Sat } [F_\xi(t)]_{L^p} = \{f(t) | f'(t) \in L^p; \ \xi; \ o\}, \ (p > 1)$$

$$\text{Sat } [F_\xi(t)]_L = \{f(t) | \tilde{f}(t) \in BV; \ \xi; \ o\}.$$

(b) Poisson-Cauchy's singular integral.

$$P_\xi(t;f) = \frac{\xi}{\pi} \int_{-\infty}^{\infty} \frac{f(t+u)}{u^2+\xi^2}\,du.$$

Since $$k(t) = \sqrt{\frac{2}{\pi}}\frac{1}{1+t^2} \quad \text{and } K(u) = e^{-|u|}$$

we have

$$\varphi(\xi) = \xi, \ \psi(u) = |u|.$$

The saturation is the same as the Fejér's integral.

(c) The generalized Gauss-Weierstrass singular integral. This is

$$T_\xi(t; f) = \frac{p}{2\Gamma(1/p)}\frac{1}{\xi}\int_{-\infty}^{\infty} f(t+u)\, e^{-|u/\xi|^p}\, du \quad (p > 0).$$

If $p=1$, this is Picard's singular integral and $p=2$, this is Gauss-Weierstrass singular integral.

Since $$k(t)= \frac{\sqrt{2\pi}\, p}{2\Gamma(1/p)}\, e^{-|t|^p}$$

we have

$$K(u)= \frac{p}{2\Gamma(1/p)}\int_{-\infty}^{\infty} e^{-|t|^p} e^{-tut} dt = \frac{p}{\Gamma(1/p)}\int_0^{\infty} e^{-t^p} \cos ut\, dt$$

$$\sim \frac{p}{\Gamma(1/p)}\left\{\frac{\Gamma(1/p)}{p} - \frac{\Gamma(3/p)}{2p}u^2\right\}$$

for small u (Hardy [5], p. 282). So

$$1 - k(u\xi) \sim Cu^2 \xi^2$$

and

$$\varphi(\xi) = \xi^2 \text{ and } \psi(u) = u^2.$$

$$\text{Sat } [T_\xi(t)]_C = \{f(t)|f''(t) \in L^\infty ;\ \xi^2 ;\ o\},$$

$$\text{Sat } [T_\xi(t)_{L^p} = \{f(t)|f''(t) \in L^p ;\ \xi^2 ;\ o\}, \quad (p > 1)$$

$$\text{Sat } [T_\xi(t)]_L = \{f[t)|f'(t) \in BV ;\ \xi^2 ;\ o\}.$$

Some of these are conjectures of Butzer [2].

LITERATURES

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Added in proof. During the present paper was printing, there have appeared following two papers, "Sur le rôle de la transformation de Fourier dans quelques problèmes d'approximation" by P. L. Butzer, Comptes Rendus Paris, 249 (1959), 2467-2469, and "Saturation sur un groupe abélien localement compact", by H.Buchwater, Comptes Rendus Paris 250 (1960), 808-810. These papers include the result in the article 2 of the present paper. But this case (L^p ($1 \leqq p \leqq 2$)-appoximation) is treated essentialy in the same way to the Fourier series case. L^p ($p > 2$)-case and C-case of the present paper are new.