SOME RESULTS ON THE DIRECT PRODUCT OF W^*-ALGEBRAS

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Introduction. In connection with the papers [4] and [5], the following question arises: *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite factors, and let* G *and* H *be the groups of* $*$*-automorphisms of* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then, is it true that the fixed algebra of* $G \times H$[1] *in* $\mathbf{M} \otimes \mathbf{N}$ *is the direct product of the fixed algebra of* G *in* $\mathbf{M}$ *and that of* H *in* $\mathbf{N}$? The above question motivates the preparation of this paper, but our investigation will be done from the standpoint of the general theory of the direct product of W^*-algebras, and the main result will be stated in Theorem 1 in § 1. In § 2, as the applications of Theorem 1, two results will be proved. Theorem 2 is a structure theorem on the direct product of maximal abelian W^*-subalgebras, and Theorem 3 gives the affirmative answer to the question of the fixed algebra.

1. Throughout our discussion, we mean by $R(A_\alpha)$ the W^*-algebra generated by the family of operators A_α and $R(\mathbf{M}, \mathbf{N})$ the one generated by the W^*-algebras $\mathbf{M}, \mathbf{N}$.

The following theorem is the main result of this paper.

THEOREM 1. *Let* $\mathbf{M}, \mathbf{P}$ *and* $\mathbf{N}, \mathbf{Q}$ *be* W^**-algebras on some Hilbert space* $\mathbf{H}$ *and* $\mathbf{K}$ *respectively, and satisfy the condition*

(1) $\quad ((\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}'))' = (\mathbf{M} \cap \mathbf{P}')' \otimes (\mathbf{N} \cap \mathbf{Q}')';$

then we have

(2) $\quad \mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' = (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}').$

This theorem shows that if the, so-called, commutation theorem holds for a W^*-algebra $(\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}')$ we get the conclusion (2). Hence, for example, if $\mathbf{A}$ and $\mathbf{B}$ are maximal abelian W^*-subalgebras of $\mathbf{M}$ and $\mathbf{N}$ respectively we have the relation (2) for $\mathbf{A} \otimes \mathbf{B}$ because, in this case, $((\mathbf{M} \cap \mathbf{A}') \otimes (\mathbf{N} \cap \mathbf{B}'))' = (\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$. Therefore we know that the direct product of maximal abelian W^*-subalgebras $\mathbf{A}$ and $\mathbf{B}$ is also a maximal abelian W^*-subalgebra of $\mathbf{M} \otimes \mathbf{N}$. If $\mathbf{M}$ and $\mathbf{N}$ are finite W^*-algebras, their W^*-subalgebras are also of finite type, and hence the commutation theorem always holds for these W^*-algebras. Therefore we have the conclusion

1) For the definition of $G \times H$, see Lemma 2 in [4].

$$\mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' = (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}')$$

for any W^*-algebras $\mathbf{P}$ and $\mathbf{Q}$ if $\mathbf{M}$ and $\mathbf{N}$ are finite W^*-algebras.

PROOF OF THE THEOREM. It is obvious that

$$\mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' \supseteqq (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}').$$

On the other hand, we have, by the assumption (1)

$$\begin{aligned}(\mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})')' &= R((\mathbf{M} \otimes \mathbf{N})', \mathbf{P} \otimes \mathbf{Q}) \supseteqq R(\mathbf{M}' \otimes \mathbf{N}', \mathbf{P} \otimes \mathbf{Q}) \\ &= R(R(\mathbf{M}' \otimes I, \mathbf{P} \otimes I), R(I \otimes \mathbf{N}', I \otimes \mathbf{Q})) \\ &= R(\mathbf{M}', \mathbf{P}) \otimes R(\mathbf{N}', \mathbf{Q}) = (\mathbf{M} \cap \mathbf{P}')' \otimes (\mathbf{N} \cap \mathbf{Q}')' \\ &= ((\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}'))'.\end{aligned}$$

Thus

$$\mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' \subseteqq (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}'),$$

and the proof is completed.

2. In this section, we shall apply Theorem 1 to the analysis of the direct product of maximal abelian W^*-subalgebras and the discussion of the fixed algebra, stated in the introduction. Through this section, our investigation will be restricted to the finite factors of type II.

Let $\mathbf{M}$ be a finite factor of type II and $\mathbf{A}$ a maximal abelian W^*-subalgebra of $\mathbf{M}$. Let $\mathbf{P}$ be a W^*-subalgebra generated by the unitary operators U of $\mathbf{M}$ satisfying $UAU^* = \mathbf{A}$. Then, according to [1] we have the following definition.

DEFINITION 1. $\mathbf{A}$ is called *regular* if $\mathbf{P} = \mathbf{M}$, *singular* if $\mathbf{P} = \mathbf{A}$, and *semi-regular* if $\mathbf{P}$ is a factor or equivalently $\mathbf{M} \cap \mathbf{P}' = (\lambda I)$.

Holding the fact that $\mathbf{A} \otimes \mathbf{B}$ is a maximal abelian W^*-subalgebra of $\mathbf{M} \otimes \mathbf{N}$ if $\mathbf{A}$ and $\mathbf{B}$ are maximal abelian W^*-subalgebras of $\mathbf{M}$ and $\mathbf{N}$ respectively, it is natural, in the context of Definition 1, to consider the regularity, singularity and semi-regularity of the direct product of maximal abelian W^*-subalgebras of finite factors of type II. The following theorem gives a partial answer for this question.

THEOREM 2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite factors of type* II, *and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be maximal abelian* W^**-subalgebras of* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then the following statements hold*:

(1) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are both regular*, $\mathbf{A} \otimes \mathbf{B}$ *is regular.*

(2) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are both semi-regular*, $\mathbf{A} \otimes \mathbf{B}$ *is semi-regular.*

PROOF. Let $\mathbf{P}$ (resp. $\mathbf{Q}$) be a W^*-subalgebra of $\mathbf{M}$ (resp. $\mathbf{N}$) generated by the unitary operators $U \in \mathbf{M}$ (resp. $V \in \mathbf{N}$) such as $UAU^* = \mathbf{A}$ (resp. $V\mathbf{B}V^*$

$= \mathbf{B}$), and let $\mathbf{R}$ be a W^*-subalgebra of $\mathbf{M} \otimes \mathbf{N}$ generated by the unitary operators $W \in \mathbf{M} \otimes \mathbf{N}$ satisfying $W(\mathbf{A} \otimes \mathbf{B})W^* = \mathbf{A} \otimes \mathbf{B}$.

If $\mathbf{A}$ and $\mathbf{B}$ are both regular, $\mathbf{P} = \mathbf{M}$ and $\mathbf{Q} = \mathbf{N}$. Thus we have

$$\mathbf{M} \otimes \mathbf{N} \supseteqq \mathbf{R} \supseteqq \mathbf{P} \otimes \mathbf{Q} = \mathbf{M} \otimes \mathbf{N}, \qquad \mathbf{R} = \mathbf{M} \otimes \mathbf{N},$$

and $\mathbf{A} \otimes \mathbf{B}$ is regular.

Next, suppose that $\mathbf{A}$ and $\mathbf{B}$ are both semi-regular. As $\mathbf{R} \supseteqq \mathbf{P} \otimes \mathbf{Q}$, we have $\mathbf{R}' \subseteqq (\mathbf{P} \otimes \mathbf{Q})'$ and, by Theorem 1,

$$\mathbf{M} \otimes \mathbf{N} \cap \mathbf{R}' \subseteqq \mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' = (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q})' = \lambda(I \otimes I),$$

thus $\mathbf{A} \otimes \mathbf{B}$ is semi-regular.

The rest of this section will be devoted to solve the question of the fixed algebra.

Let G be a group of *-automorphisms of a W^*-algebra $\mathbf{M}$. After [3] we obtain the following definition.

DEFINITION 2. By the *fixed algebra* of G in $\mathbf{M}$, we mean the subalgebra $\mathbf{P} = [A \in \mathbf{M} \mid A^\alpha = A$ for all $\alpha \in G]$, where A^α is the image of A under a *-automorphism α.

THEOREM 3. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite factors with the invariant* $C = 1$, *and let* G *and* H *be groups of* **-automorphisms of* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then the fixed algebra of* $G \times H$ *in* $\mathbf{M} \otimes \mathbf{N}$ *is the direct product of the fixed algebra of* G *in* $\mathbf{M}$ *and that of* H *in* $\mathbf{N}$.

PROOF. Let $\mathbf{H}$ and $\mathbf{K}$ be the underlying Hilbert spaces of $\mathbf{M}$ and $\mathbf{N}$ respectively. Then G (resp. H) admits a faithful unitary representation $\alpha \in G \rightarrow U_\alpha$ on $\mathbf{H}$ (resp. $\beta \in H \rightarrow V_\beta$ on $\mathbf{K}$) such that $U_\alpha^* A U_\alpha = A^\alpha$ for all $A \in \mathbf{M}$ (resp. $V_\beta^* B V_\beta = B^\beta$ for all $B \in \mathbf{N}$)[2]. Put

$\mathbf{P} = [A \in \mathbf{M} \mid A^\alpha = A$ for all $\alpha \in G]$, $\mathbf{Q} = [B \in \mathbf{N} \mid B^\beta = B$ for all $\beta \in H]$.
It is easily seen that $\mathbf{P} = \mathbf{M} \cap R(U_\alpha \mid \alpha \in G)'$, $\mathbf{Q} = \mathbf{N} \cap R(V_\beta \mid \beta \in H)'$. Hence, by the result in § 1, we have

$$\begin{aligned} \mathbf{P} \otimes \mathbf{Q} &= (\mathbf{M} \cap R(U_\alpha \mid \alpha \in G)') \otimes (\mathbf{N} \cap R(V_\beta \mid \beta \in H)') \\ &= \mathbf{M} \otimes \mathbf{N} \cap (R(U_\alpha \mid \alpha \in G) \otimes R(V_\beta \mid \beta \in H))' \\ &= \mathbf{M} \otimes \mathbf{N} \cap R(U_\alpha \otimes V_\beta \mid (\alpha, \beta) \in G \times H)'. \end{aligned}$$

Obviously, $(\alpha, \beta) \in G \times H \rightarrow U_\alpha \otimes V_\beta$ on $\mathbf{H} \otimes \mathbf{K}$ is a faithful unitary representation of $G \times H$ satisfying $(U_\alpha \otimes V_\beta)^* C (U_\alpha \otimes V_\beta) = C^{(\alpha,\beta)}$ for all $C \in \mathbf{M} \otimes \mathbf{N}$. Hence $\mathbf{P} \otimes \mathbf{Q}$ is the fixed algebra of $G \times H$ in $\mathbf{M} \otimes \mathbf{N}$.

2) This fact is found in Lemma 1 in [6]

As an immediate consequence of Theorem 3, we have:

COROLLARY. *In Theorem* 3, *if* G *and* H *are both ergodic, that is the fixed algebra of* G *in* $\mathbf{M}$ *and that of* H *in* $\mathbf{N}$ *are both* $\{\lambda I\}$, $G \times H$ *is also ergodic.*

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