ON SOME STRUCTURES WHICH ARE SIMILAR TO THE QUATERNION STRUCTURE

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In the following we study some properties of differentiable manifolds of class C^∞ which are endowed with three fields of (non trivial) mixed tensors of class C^∞: ϕ_i^h, ψ_i^h and κ_i^h satisfying the following relations:

$$\phi_i^a\phi_a^h = \varepsilon_1\delta_i^h,$$

$$\psi_i^a\psi_a^h = \varepsilon_2\delta_i^h,$$

$$\kappa_i^a\kappa_a^h = \varepsilon_3\delta_i^h,$$

$$\psi_k^a\phi_a^i = \varepsilon\,\phi_k^a\psi_a^i = -\,\varepsilon_3\kappa_k^i,$$

$$\kappa_k^a\psi_a^i = \varepsilon\,\psi_k^a\kappa_a^i = -\,\varepsilon_1\phi_k^i,$$

$$\phi_k^a\kappa_a^i = \varepsilon\kappa_k^a\phi_a^i = -\,\varepsilon_2\psi_k^i,$$

where δ_i^h denotes the Kronecker delta and $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$; $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$. The above system contains essentially the following four cases:

Case I. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$; $\varepsilon = -1$. This is the case of the well-known quaternion structure.

Case II. $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = 1$; $\varepsilon = 1$.

Case III. $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon = -1$. This case is called by Libermann the quaternion structure of the second kind, and is also called the complex-product structure by T. Nagano.

Case IV. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$; $\varepsilon = 1$. All of these structures were studied by Ehresmann and Libermann [3]*, the case I was also studied by Obata [7, 8] and Wakakuwa [10], the case III was also studied by Nagano [4].

In § 1, following T. Nagano [4], we define an almost complex structure and two almost product structures on the tangent bundle $T(M)$ of n dimensional affinely connected manifold M and show that if M is itself an almost complex or almost product manifold, then $T(M)$ turns out to have structures mentioned above, under some conditions for the defined almost complex structure to be integrable is also obtained.

In § 2 we study the affine or metric connections which make the given

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*) Numbers in brackets refer to the references at the end of the paper.

three tensor fields of the structure simultaneously covariant constant.

In § 3 we study the homogeneous groups of holonomy with respect to the connection studied in § 2 for cases II, III and IV, by making use of suitable bases in the tangent space.

In § 4 we study if the largest connected group of affine transformations with respect to the connection in § 2 preserves the structure endowed in the manifold.

1. Some structures on tangent bundles. In the sequel we assume that Latin indices i, j, k,......vary from 1 to $2n$, Greek indices α, β, γ, vary from 1 to n, and α^* is $\alpha + n$, so α^*, β^*, γ^*,......vary from $n+1$ to $2n$.

1. 1. Let u^α be the local coordinates of a point in M, then the local coordinates of an element (i. e. a tangent vector of M at a point $u \in M$) of the tangent bundle $T(M)$ of M are $(u^i) = (u^\alpha, u^{\alpha^*}) = (u^\alpha, v^\alpha)$, where v^α are components of a tangent vector with respect to the natural frame at (u^α), i. e. the frame constituted by the vectors $\frac{\partial}{\partial u^\alpha}$. Corresponding to a coordinate transformation $u'^\alpha = u'^\alpha(u^1,\ldots\ldots, u^n)$ we have a coordinate transformation in $T(M)$: $u'^\alpha = u'^\alpha(u^1,\ldots\ldots, u^n)$, $u'^{\alpha^*} = v'^\alpha = \frac{\partial u'^\alpha}{\partial u^\beta} v^\beta = \frac{\partial u'^\alpha}{\partial u^\beta} u^{\beta^*}$ which is called the extended coordinate transformation of $u'^\alpha = u'^\alpha(u^1,\ldots\ldots, u^n)$. Then we can define tensors of $T(M)$ by using the transformation matrix of the extended coordinate transformation, i. e.

$$\left(\frac{\partial u'^i}{\partial u^j}\right) = \begin{pmatrix} \dfrac{\partial u'^\alpha}{\partial u^\beta} & \dfrac{\partial u'^\alpha}{\partial u^{\beta^*}} \\ \dfrac{\partial u'^{\alpha^*}}{\partial u^\beta} & \dfrac{\partial u'^{\alpha^*}}{\partial u^{\beta^*}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u'^\alpha}{\partial u^\beta} & 0 \\ \dfrac{\partial^2 u'^\alpha}{\partial u^\beta \partial u^\gamma} v^\gamma & \dfrac{\partial u'^\alpha}{\partial u^\beta} \end{pmatrix}.$$

Now we assume that M is an affinely connected manifold having the connection parameters $\Gamma_{\beta\gamma}^\alpha$. It is known that if ξ^α's are components of a contravariant vector fields of M, then $(0, \xi^\alpha)$ and $(\xi^\alpha, -\Gamma_{\beta\gamma}^\alpha \xi^\beta v^\gamma)$ are both contravariant vector fields of $T(M)$ [9].

By the way, we would like to show that these vector fields are respectively a fundamental vector field and a horizontal vector field in $T(M)$, whose linear connection is uniquely determined by $\Gamma_{\beta\gamma}^\alpha$ of M.

Let $P(M)$ be the principal fibre bundle consisting of all frames of M, with the base manifold M and structural group $GL(n, R)$. With respect to the local coordinates u^α, every frame of M is represented as a set (l_α), $l_\alpha = X_\alpha^\beta \frac{\partial}{\partial u^\beta}$, so $(u^\alpha, X_\alpha^\beta)$ may be considered as local coordinate system in $P(M)$. Let (Y_β^α) be the inverse matrix of (X^{α}_{β}) and put $\omega^{\alpha}_{\beta}=\Gamma^{\alpha}_{\gamma\beta}du^{\gamma}$, then the distribution Q_x defining the connection in $P(M)$ is the annihilator of $\theta^{\gamma}_{\alpha}=Y^{\gamma}_{\beta}(dX^{\beta}_{\alpha}+\omega^{\beta}_{\delta}X^{\delta}_{\alpha})$ [1]. Hence any horizontal vector in $P(M)$ may be represented as $\xi^{\alpha}\dfrac{\partial}{\partial u^{\alpha}}-\Gamma^{\beta}_{\gamma\delta}\xi^{\gamma}X^{\delta}_{\alpha}\dfrac{\partial}{\partial X^{\beta}_{\alpha}}$ or $(\xi^{\alpha},\ -\Gamma^{\beta}_{\gamma\delta}\xi^{\gamma}X^{\delta}_{\alpha})$. As $T(M)$ may be regarded as an associated fibre bundle of $P(M)$ with n-dimensional vector space F as standard fibre, corresponding to the connection in $P(M)$, there is a uniquely determined connection in $T(M)$. Let $x\in P(M)$ has the local coordinates $(u^{\alpha},\ X^{\beta}_{\alpha})$, the fibre F_u through (u^{α}) is identified with the tangent space T_u at this point by identifying the element $x\cdot\xi_{\alpha}$ (for definition, see [6]) with the vector $l_{\alpha}=X^{\beta}_{\alpha}\dfrac{\partial}{\partial u^{\beta}}$ (where ξ^{α} is a fixed base of the standard fibre F) in T_u. Then the distribution defining the connection in $T(M)$ is the image of the horizontal subspace Q_x under the differential ϕ' of the mapping ϕ: $x\rightarrow x\cdot\xi_0$ for a certain fixed $\xi_0=\alpha^{\beta}\xi_{\beta}$, i. e. $(u^{\alpha},\ X^{\beta}_{\alpha})\rightarrow(u^{\alpha},\ \alpha^{\beta}X^{\gamma}_{\beta})=(u^{\alpha},\ v^{\gamma})$. Since $\phi'(\xi^{\alpha},\ \eta^{\beta}_{\alpha})=(\xi^{\alpha},\ v^{\beta}Y^{\gamma}_{\beta}\eta^{\alpha}_{\gamma})$, where $(\xi^{\alpha},\ \eta^{\beta}_{\alpha})$ are components of tangent vector in $P(M)$ with respect to local coordinates, we have $\phi'(\xi^{\alpha},\ -\Gamma^{\beta}_{\gamma\delta}\xi^{\gamma}X^{\delta}_{\alpha})=(\xi^{\alpha},\ -\Gamma^{\alpha}_{\gamma\beta}\xi^{\beta}v^{\gamma})$. Thus $(\xi^{\alpha},\ -\Gamma^{\alpha}_{\beta\gamma}\xi^{\beta}v^{\gamma})$ is a horizontal vector field in $T(M)$. It is evident that $(0,\ \xi^{\alpha})$ is a fundamental vector field in $T(M)$ as its projection in T_u vanishes.

1. 2. Now consider a linear mapping ψ on each tangent space of $T(M)$, whose matrix $(\psi_j{}^i)$ with respect to local coordinates is given as follows:

$$\text{(1. 1)}\qquad \begin{cases} \psi_{\beta}{}^{\alpha}=\Gamma^{\alpha}_{\beta\rho}v^{\rho}, & \psi_{\beta^*}{}^{\alpha}=\delta^{\alpha}_{\beta},\\ \psi_{\beta}{}^{\alpha^*}=-\delta^{\alpha}_{\beta}-\Gamma^{\alpha}_{\rho\lambda}\Gamma^{\rho}_{\beta\sigma}v^{\lambda}v^{\sigma}, & \psi_{\beta^*}{}^{\alpha^*}=-\Gamma^{\alpha}_{\beta\rho}v\,. \end{cases}$$

It can be easily seen that

$$\psi_k{}^a\psi_a{}^i=-\delta^i_k.$$

The considered linear mapping tranforms the vectors in the tangent space of $T(M)$ in the following manner: Let $\underset{(\delta)}{\xi^{\alpha}}$ be n linearly independent vector fields spanning the tangent space of M at each point of a suitable coordinate neighborhood U, then the $2n$ vector fields $(0,\ \underset{(\delta)}{\xi^{\alpha}})$ and $(\underset{(\delta)}{\xi^{\alpha}},\ -\Gamma^{\alpha}_{\beta\gamma}\underset{(\delta)}{\xi^{\beta}}v^{\gamma})$ are linearly independent and span the tangent space of $T(M)$ at each point of $\pi^{-1}(U)$. Then

$$\text{(1. 2)}\qquad \psi:\begin{cases} (0,\ \underset{(\delta)}{\xi^{\alpha}})\longrightarrow(\underset{(\delta)}{\xi^{\alpha}},\ -\Gamma^{\alpha}_{\beta\gamma}\underset{(\delta)}{\xi^{\beta}}v^{\gamma}),\\ (\underset{(\delta)}{\xi^{\alpha}},\ -\Gamma^{\alpha}_{\beta\gamma}\underset{(\delta)}{\xi^{\beta}}v^{\gamma})\longrightarrow(0,\ -\underset{(\delta)}{\xi^{\alpha}}); \end{cases}$$

where $(0,\ \underset{(\delta)}{\xi^{\alpha}})\rightarrow(\underset{(\delta)}{\xi^{\alpha}},\ -\Gamma^{\alpha}_{\beta\gamma}\underset{(\delta)}{\xi^{\beta}}v^{\gamma})$ is an isomorphism from the vertical subspace onto the horizontal subspace of the tangent space of $T(M)$, and is the inverse of the product of the following two isomorphisms:

$$\pi' : (\underset{(\delta)}{\xi^\alpha}, \ -\Gamma_{\beta\gamma}^{\alpha} \underset{(\delta)}{\xi^\beta} v^\gamma) \longrightarrow (\underset{(\delta)}{\xi^\alpha}, \ 0),$$

$$(\underset{(\delta)}{\xi^\alpha}, \ 0) \longrightarrow (0, \ \underset{(\delta)}{\xi^\alpha}),$$

the former of these isomorphisms is the one induced by the differential of the natural projection π, and the latter is the one which defines the soldering property of $T(M)$.

It is proved by Eckmann and Frölicher that for an analytic almost complex structure $\psi_j{}^i$ to be integrable, it is necessary and sufficient that [11]

$$(1.3) \qquad t_{kl}{}^i = \psi_l{}^t\left(\frac{\partial \psi_t{}^i}{\partial u^k} - \frac{\partial \psi_k{}^i}{\partial u^t}\right) - \psi_k{}^t\left(\frac{\partial \psi_t{}^i}{\partial u^l} - \frac{\partial \psi_l{}^i}{\partial u^t}\right) = 0.$$

For our specified case, we can get $t_{kl}{}^i$ by straightforward calculation: As $u^\alpha = u^\alpha$, $u^{\alpha^*} = v^\alpha$, we have

$$\frac{\partial \psi_\beta{}^\alpha}{\partial u^\gamma} = \frac{\partial \Gamma_{\beta\rho}^{\alpha}}{\partial u^\gamma} v^\rho, \quad \frac{\partial \psi_\beta{}^\alpha}{\partial u^{\gamma^*}} = \Gamma_{\beta\gamma}^{\alpha}, \quad \frac{\partial \psi_{\beta^*}^{\alpha}}{\partial u^\gamma} = 0, \quad \frac{\partial \psi_{\beta^*}^{\alpha}}{\partial u^{\gamma^*}} = 0,$$

$$\frac{\partial \psi_\beta{}^{\alpha^*}}{\partial u^\gamma} = -\left(\frac{\partial \Gamma_{\rho\lambda}^{\alpha}}{\partial u^\gamma}\Gamma_{\beta\sigma}^{\rho} + \Gamma_{\rho\lambda}^{\alpha}\frac{\partial \Gamma_{\beta\sigma}^{\rho}}{\partial u^\gamma}\right) v^\lambda v^\sigma, \quad \frac{\partial \psi_\beta{}^{\alpha^*}}{\partial u^{\gamma^*}} = -\left(\Gamma_{\rho\gamma}^{\alpha}\Gamma_{\beta\lambda}^{\rho} + \Gamma_{\rho\lambda}^{\alpha}\Gamma_{\beta\gamma}^{\rho}\right) v^\lambda,$$

$$\frac{\partial \psi_{\beta^*}{}^{\alpha^*}}{\partial u^\gamma} = -\frac{\partial \Gamma_{\beta\rho}^{\alpha}}{\partial u^\gamma} v^\rho, \quad \frac{\partial \psi_{\beta^*}{}^{\alpha^*}}{\partial u^{\gamma^*}} = -\Gamma_{\beta\gamma}^{\alpha}.$$

Putting these relations in the expression (1.3) of Nijenhuis tensor $t_{kl}{}^i$, we have

$$(1.4) \quad \begin{cases} t_{\beta\gamma}^{\alpha} = (R_{\rho\beta\tau}^{\alpha}\Gamma_{\gamma\sigma}^{\tau} - R_{\rho\gamma\tau}^{\alpha}\Gamma_{\beta\gamma}^{\tau} + 2\,T_{\tau\nu}^{\ \alpha}\Gamma_{\gamma\sigma}^{\tau}\Gamma_{\beta\rho}^{\nu})\, v^\rho v^\sigma + 2\,T_{\beta\gamma}^{\alpha}, \\ t_{\beta^*\gamma}^{\ \alpha} = (R_{\rho\beta\gamma}^{\alpha} + 2\,T_{\tau\beta}^{\alpha}\Gamma_{\gamma\rho}^{\tau})\, v^\rho, \\ t_{\beta\gamma^*}^{\alpha} = (R_{\rho\beta\gamma}^{\alpha} + 2\,T_{\gamma\tau}^{\alpha}\Gamma_{\beta\rho}^{\tau})\, v^\rho, \\ t_{\beta^*\gamma^*}^{\ \alpha} = 2\,T_{\gamma\beta}^{\alpha}, \\ t_{\beta\gamma}{}^{\alpha^*} = (R_{\epsilon\beta\gamma}^{\alpha} + 2\,T_{\rho\beta}^{\alpha}\Gamma_{\gamma\epsilon}^{\rho} + 2\,T_{\gamma\rho}^{\alpha}\Gamma_{\beta\epsilon}^{\rho} + 2\,T_{\gamma\beta}^{\rho}\Gamma_{\rho\epsilon}^{\alpha})\, v^\epsilon \\ \qquad + [\{R_{\sigma\tau\beta}^{\rho}\Gamma_{\gamma\epsilon}^{\tau} - R_{\sigma\tau\gamma}^{\rho}\Gamma_{\beta\epsilon}^{\tau} + 2\,T_{\nu\tau}^{\rho}\Gamma_{\beta\sigma}^{\nu}\Gamma_{\gamma\epsilon}^{\tau}\}\Gamma_{\rho\lambda}^{\alpha} + R_{\lambda\tau\rho}^{\alpha}\Gamma_{\beta\sigma}^{\rho}\Gamma_{\gamma\epsilon}^{\tau}]\, v^\lambda v^\sigma v^\epsilon, \\ t_{\beta^*\gamma}{}^{\alpha^*} = (R_{\sigma\gamma\beta}^{\rho}\Gamma_{\rho\lambda}^{\alpha} + R_{\lambda\tau\beta}^{\alpha}\Gamma_{\gamma\sigma}^{\tau} + 2\,T_{\beta\rho}^{\alpha}\Gamma_{\tau\lambda}^{\rho}\Gamma_{\gamma\sigma}^{\tau})\, v^\lambda v^\sigma - 2\,T_{\beta\gamma}^{\alpha}, \\ t_{\beta\gamma^*}{}^{\alpha^*} = (R_{\sigma\gamma\beta}^{\rho}\Gamma_{\rho\lambda}^{\alpha} + R_{\lambda\gamma\rho}^{\alpha}\Gamma_{\beta\sigma}^{\rho} + 2\,T_{\tau\gamma}^{\rho}\Gamma_{\beta\sigma}^{\tau}\Gamma_{\rho\lambda}^{\alpha})\, v^\lambda v^\sigma - 2\,T_{\beta\gamma}^{\alpha}, \\ t_{\beta^*\gamma^*}{}^{\alpha^*} = (R_{\lambda\gamma\beta}^{\alpha} + 2\,T_{\beta\gamma}^{\rho}\Gamma_{\rho\lambda}^{\alpha})\, v^\lambda. \end{cases}$$

Where $R_{\rho\beta\gamma}^{\alpha}$ and $T_{\beta\gamma}^{\alpha}$ are respectively the curvature tensor and torsion tensor of $\Gamma_{\beta\gamma}^{\alpha}$ in affinely connected manifold M.

From the theorem of Eckmann and Frölicher and the above expressions (1.4), we have

THEOREM 1. 1. *The almost complex structure* $(\psi_j{}^i)$ *defined above* (1. 1) *in the tangent bundle* $T(M)$ *of an affinely connected manifold* M *of class* C^ω *is integrable, if and only if* M *is locally flat.*

Suppose that M is a Riemannian manifold. If we substitute $\Gamma_{\beta\gamma}^{\alpha}$ by the Riemann-Christoffel symbol $\{{}_{\beta}{}^{\alpha}{}_{\gamma}\}$ of the Riemann metric $g_{\alpha\beta}$ of M in the above treatment, we get the corresponding results for the case of Riemannian manifold.

Let G_{ij} be the Riemann metric defined by Prof. S. Sasaki [9] in the tangent bundle of Riemannian manifold, i. e.

$$(1.5)\qquad \left\{\begin{aligned} G_{\alpha\beta} &= g_{\alpha\beta} + g_{\rho\sigma}\{{}_{\mu}{}^{\rho}{}_{\alpha}\}\{{}_{\nu}{}^{\sigma}{}_{\beta}\}v^{\mu}v^{\nu},\\ G_{\alpha\beta^*} &= g_{\rho\beta}\{{}_{\lambda}{}^{\rho}{}_{\alpha}\}v^{\lambda},\\ G_{\alpha^*\beta^*} &= g_{\alpha\beta}, \end{aligned}\right.$$

then we have

$$(1.6)\qquad \psi_j{}^s\psi_k{}^t G_{st} = G_{jk}.$$

That is, G_{ij} is the Hermitian metric in the almost complex manifold $T(M)$ defined above.

If we put

$$(1.7)\qquad \psi_{ij} = \psi_i{}^r G_{rj},$$

then we have

$$(1.8)\qquad \left\{\begin{aligned} &\psi_{\alpha\beta} = (g_{\rho\beta}\{{}_{\lambda}{}^{\rho}{}_{\alpha}\} - g_{\alpha\rho}\{{}_{\beta}{}^{\rho}{}_{\lambda}\})v^{\lambda},\ \psi_{\alpha\beta^*} = -g_{\alpha\beta},\\ &\psi_{\alpha^*\beta} = g_{\alpha\beta},\ \psi_{\alpha^*\beta^*} = 0. \end{aligned}\right.$$

Let ψ_{ijk} be defined by

$$(1.9)\qquad \psi_{ijk} = \left(\frac{\partial\psi_{ij}}{\partial u^k} + \frac{\partial\psi_{jk}}{\partial u^i} + \frac{\partial\psi_{ki}}{\partial u^j}\right),$$

then by making use of the relation

$$\frac{\partial g_{\rho\beta}}{\partial u^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial u^\beta} = g_{\nu\beta}\{{}_{\rho}{}^{\nu}{}_{\gamma}\} - g_{\nu\gamma}\{{}_{\rho}{}^{\nu}{}_{\beta}\},$$

we have the following expressions:

$$(1.10)\quad \left\{\begin{aligned} &\psi_{\alpha\beta\gamma} = -\{g_{\alpha\rho}R^{\rho}_{\lambda\gamma\beta} + g_{\beta\rho}R^{\rho}_{\lambda\alpha\gamma} + g_{\gamma\rho}R^{\rho}_{\lambda\beta\alpha}\}v^{\lambda},\\ &\psi_{\alpha\beta^*\gamma} = \psi_{\alpha\beta\gamma^*} = \psi_{\alpha\beta^*\gamma^*} = \psi_{\alpha^*\beta\gamma} = \psi_{\alpha^*\beta^*\gamma} = \psi_{\alpha^*\beta\gamma^*} = \psi_{\alpha^*\beta^*\gamma^*} = 0. \end{aligned}\right.$$

Since it is known [11] that a pseudo Kählerian manifold is characterized by skew symmetric tensor ψ_{ij} and symmetric tensor G_{ij} satisfying

$$(1.11)\qquad \psi_j{}^s\psi_k{}^t G_{st} = G_{jk},\ t_{kl}{}^i = 0,\ \psi_{ijk} = 0,$$

we have by the theorem of Newlander and Nirenberg [5] from the expressions of $t_{kl}{}^{i}$ (1.4) and ψ_{ijk} (1.10) the following:

THEOREM 1. 2. *For Riemannian manifold M having metric $g_{\alpha\beta}$, $T(M)$ is Kählerian with respect to the structure* (1.1) *and the metric* (1.5) *if and only if M is locally flat* [4].

1. 3. In the tangent bundle $T(M)$ of an affinely connected manifold M, we can also define the following two almost product structures $\phi_j{}^i$ and $\kappa_j{}^i$:

$$
(1.12)\quad \begin{cases} \phi_\beta{}^\alpha = -\Gamma_{\beta\rho}^\alpha v^\rho,\ \phi_{\beta^*}^\alpha = -\delta_\beta^\alpha,\ \phi_\beta{}^{\alpha^*} = -\delta_\beta^\alpha + \Gamma_{\rho\lambda}^\alpha \Gamma_{\beta\sigma}^\rho v^\lambda v^\sigma,\ \phi_{\beta^*}^{\alpha^*} = \Gamma_{\beta\rho}^\alpha v^\rho; \\ \kappa_\beta{}^\alpha = -\delta_\beta^\alpha,\ \kappa_{\beta^*}^\alpha = 0,\ \kappa_\beta{}^{\alpha^*} = 2\Gamma_{\beta\rho}^\alpha v^\rho,\ \kappa_{\beta^*}^{\alpha^*} = \delta_\beta^\alpha. \end{cases}
$$

We can easily see that $\kappa_j{}^i$ is induced by the linear transformation κ on the tangent space of $T(M)$ which has the vertical and horizontal subspace as the proper subspaces corresponding respectively to the proper value 1 and -1. Moreover, it is easily shown that

$$
(1.13)\quad \begin{cases} -\phi_j{}^i = \kappa_j{}^a \psi_a{}^i = -\psi_j{}^a \kappa_a{}^i, \\ \phi_k{}^a \psi_a{}^i = -\psi_k{}^a \phi_a{}^i = \kappa_k{}^i,\ \kappa_k{}^a \phi_a{}^i = -\phi_k{}^a \kappa_a{}^i = -\psi_k{}^i. \end{cases}
$$

Thus we have

THEOREM 1. 3. *In the tangent bundle $T(M)$ of an affinely connected manifold M, $(\psi_j{}^i, \phi_j{}^i, \kappa_j{}^i)$ defines a complex-product structure* [4].

In the remaining part of this section we assume that M is a differentiable manifold of dimension $n = 2m$ with an almost complex or almost product structure $\varphi_\beta{}^\alpha$:

$$
(1.14)\quad \varphi_\beta{}^\gamma \varphi_\gamma{}^\alpha = \varepsilon_1 \delta_\beta^\alpha,
$$

where $\varepsilon_1 = 1$ in case of almost product structure and $\varepsilon_1 = -1$ in case of almost complex structure.

The extended tensor field $\bar{\varphi}_j{}^i$ of φ_β^α defined by

$$
(1.15)\quad \bar{\varphi}_\beta{}^\alpha = \varphi_\beta{}^\alpha,\ \bar{\varphi}_{\beta^*}^\alpha = 0,\ \bar{\varphi}_\beta{}^{\alpha^*} = \frac{\partial \varphi_\beta^\alpha}{\partial u^\gamma} v^\gamma,\ \bar{\varphi}_{\beta^*}^{\alpha^*} = \varphi_\beta^\alpha
$$

gives rise to almost complex or almost product structure in the tangent bundle $T(M)$ of M, i. e. $\bar{\varphi}_k{}^a \bar{\varphi}_a{}^i = \varepsilon_1 \delta_k^i$. On the other hand, if M is an affinely connected manifold having the connection parameters $\Gamma_{\beta\gamma}^\alpha$, we have already defined an almost complex structure $\psi_j{}^i$ (1.1). Then for the structures $\bar{\varphi}_j{}^i$ and $\psi_j{}^i$ we have the following relations:

$$
\begin{cases} \psi_\beta{}^k \bar{\varphi}_k{}^\alpha - \bar{\varphi}_\beta{}^k \psi_k{}^\alpha = -\left(\dfrac{\partial \varphi_\beta^\alpha}{\partial u^\rho} + \varphi_\beta{}^\gamma \Gamma_{\gamma\rho}^\alpha - \varphi_\gamma{}^\alpha \Gamma_{\beta\rho}^\gamma \right) v^\rho, \\ \psi_{\beta^*}{}^k \bar{\varphi}_k{}^\alpha - \bar{\varphi}_{\beta^*}{}^k \psi_k{}^\alpha = 0, \end{cases}
$$

$$(1.16)\quad \begin{cases} \psi_\beta{}^k\overline{\varphi}_k{}^{\alpha^*} - \overline{\varphi}_\beta{}^k\psi_k{}^{\alpha^*} = -\Gamma_{\lambda\rho}^{\alpha}v^\rho v^\sigma\left(\dfrac{\partial\varphi_\beta^\lambda}{\partial u^\sigma} + \varphi_\beta{}^\gamma\Gamma_{\gamma\sigma}^\lambda - \varphi_\lambda{}^\gamma\Gamma_{\beta\sigma}^\gamma\right) \\ \qquad\qquad - \Gamma_{\beta\rho}^{\lambda}v^\rho v^\sigma\left(\dfrac{\partial\varphi_\lambda^\alpha}{\partial u^\sigma} + \varphi_\gamma{}^\lambda\Gamma_{\gamma\sigma}^\alpha - \varphi_\gamma{}^\alpha\Gamma_{\lambda\sigma}^\gamma\right), \\ \psi_{\beta^*}{}^k\overline{\varphi}_k{}^{\alpha^*} - \overline{\varphi}_{\beta^*}{}^k\psi_k{}^{\alpha^*} = \left(\dfrac{\partial\varphi_\beta^\alpha}{\partial u^\rho} + \varphi_\beta{}^\gamma\Gamma_{\gamma\rho}^\alpha - \varphi_\gamma{}^\alpha\Gamma_{\beta\rho}^\gamma\right)v^\rho. \end{cases}$$

So, if $\Gamma_{\beta\gamma}^\alpha$ is a φ-connection, i. e. $\varphi_{\beta,\gamma}^{\alpha} = 0$, then we get

$$(1.17)\qquad \psi_k{}^a\overline{\varphi}_a{}^t = \overline{\varphi}_k{}^a\psi_a{}^t.$$

Put $\psi_k{}^a\overline{\varphi}_a{}^t = \varepsilon_1\kappa_k{}^t$, then we have

$$(1.18)\quad \begin{cases} \kappa_k{}^a\kappa_a{}^t = -\varepsilon_1\delta_k^t, \\ \kappa_j{}^a\overline{\varphi}_a{}^t = \overline{\varphi}_j{}^a\kappa_a{}^t = \psi_j{}^t,\ \kappa_j{}^a\psi_a{}^t = \psi_j{}^a\kappa_a{}^t = -\varepsilon_1\overline{\varphi}_j{}^t. \end{cases}$$

Thus we have

THEOREM 1. 4. *If $\Gamma_{\beta\gamma}^\alpha$ is a φ-connection, no matter of $\varepsilon_1 = +1$ or -1, the pair $(\overline{\varphi}_j{}^t, \psi_j{}^t)$ defines a structure of case II in the tangent bundle $T(M)$ of an affinely connected manifold M. The converse also holds good.*

Finally for the extended tensor field $\overline{\varphi}_j{}^t$ and the almost product structure $\kappa_j{}^t$ in $T(M)$ defined above (1.12), we have the following relations:

$$(1.19)\quad \begin{cases} \overline{\varphi}_j{}^a\kappa_a{}^\alpha = \kappa_\gamma{}^a\overline{\varphi}_a{}^\alpha = -\varphi_\gamma^\alpha,\ \overline{\varphi}_{\gamma^*}{}^a\kappa_a{}^\alpha = \kappa_{\gamma^*}{}^a\overline{\varphi}_a{}^\alpha = 0, \\ \overline{\varphi}_\gamma{}^a\kappa_a{}^{\alpha^*} = \left(\dfrac{\partial\varphi_\gamma^\alpha}{\partial u^\rho} + 2\varphi_\gamma{}^\beta\Gamma_{\beta\gamma}^\alpha\right)v^\rho,\ \kappa_\gamma{}^a\overline{\varphi}_a{}^{\alpha^*} = \left(-\dfrac{\partial\varphi_\gamma^\alpha}{\partial u^\rho} + 2\varphi_\lambda{}^\alpha\Gamma_{\gamma\rho}^\lambda\right)v^\rho, \\ \overline{\varphi}_{\gamma^*}{}^a\kappa_a{}^{\alpha^*} = \kappa_{\gamma^*}{}^a\overline{\varphi}_a{}^{\alpha^*} = \varphi_\gamma^\alpha. \end{cases}$$

Hence we have

$$(1.20)\quad \begin{cases} \overline{\varphi}^a{}_\gamma\kappa_a{}^\alpha - \kappa_\gamma{}^a\overline{\varphi}_a{}^\alpha = \overline{\varphi}_{\gamma^*}{}^a\kappa_a{}^\alpha - \kappa_{\gamma^*}{}^a\overline{\varphi}_a{}^\alpha = \overline{\varphi}_{\gamma^*}{}^a\kappa_a{}^{\alpha^*} - \kappa_{\gamma^*}{}^a\overline{\varphi}_a{}^{\alpha^*} = 0, \\ \overline{\varphi}_\gamma{}^a\kappa_a{}^{\alpha^*} - \kappa_\gamma{}^a\overline{\varphi}_a{}^{\alpha^*} = 2\,\varphi_{\gamma,\rho}^\alpha v^\rho. \end{cases}$$

If $\Gamma_{\beta\gamma}^\alpha$ is a φ-connection, then we get

$$(1.21)\qquad \overline{\varphi}_k{}^a\kappa_a{}^t = \kappa_k{}^a\overline{\varphi}_a{}^t.$$

Put $\overline{\varphi}_k{}^a\kappa_a{}^t = -\varepsilon_1\eta_k{}^t$, then we have

$$(1.22)\quad \begin{cases} \eta_k{}^a\eta_a{}^t = \varepsilon_1\delta_k^t, \\ \kappa_k{}^a\eta_a{}^t = \eta_k{}^a\kappa_a{}^t = -\varepsilon_1\overline{\varphi}_k{}^t,\ \overline{\varphi}_k{}^a\eta_a{}^t = \eta_k{}^a\overline{\varphi}_a{}^t = -\kappa_k{}^t. \end{cases}$$

Thus we have

THEOREM 1. 5. *If $\Gamma_{\beta\gamma}^\alpha$ is a φ-connection, then in case $\varepsilon_1 = 1$, the pair $(\overline{\varphi}_j{}^t, \kappa_j{}^t)$ defines a structure of case IV and in case $\varepsilon_1 = -1$, the pair $(\overline{\varphi}_j{}^t,$*

$\kappa_j^{\,i}$) defines a structure of case II in the tangent bundle $T(M)$. The converse also holds good.

The same things also hold, if we replace $\kappa_j^{\,i}$ by $\phi_j^{\,i}$ of (1.12) in the above theorem.

2. Affine and metric (ϕ, ψ)-connection. Suppose the manifold has the structure $(\phi_i^{\,h}, \psi_i^{\,h}, \kappa_i^{\,h})$. Starting with any affine connection given on the manifold, we shall obtain an affine connection, with respect to which $\phi_i^{\,h}$, $\psi_i^{\,h}$ (and consequently $\kappa_i^{\,h}$ also, because $\kappa_k^{\,i} = -\varepsilon_3 \psi_k^{\,a} \phi_a^{\,i}$) are simultaneously covariant constant. Such a connection is called a (ϕ, ψ)-connection.

2. 1. To obtain a (ϕ, ψ)-connection we have to make use of affine connections which make a given almost product structure covariant constant and affine connections which make a given almost complex structure covariant constant. Although the former were completely determined by Walker, Willmore, Yano [12] and Fukami [2], and the latter were completely determined by Obata [8], we would like to note here that by merely slight modification of the method of Obata [8], one can get results which are applicable at a time both to the case of almsot complex structure and the case of almost product structure. For example, we have the following:

THEOREM 2. 1. *In an almost complex ($\varepsilon_1 = -1$) or almost product ($\varepsilon_1 = 1$) manifold with the structure $\phi_i^{\,h}$ ($\phi_i^{\,a}\phi_a^{\,h} = \varepsilon_1 \delta_i^{\,h}$), an affine connection $\Gamma_{ji}^{\,h}$ is a ϕ-connection if and only if there exists an affine connection $\overset{0}{\Gamma}{}_{ji}^{\,h}$ such that $\Gamma_{ji}^{\,h} = \Phi \overset{0}{\Gamma}{}_{ji}^{\,h}$. Moreover, $\Phi(\overset{0}{\Gamma}{}_{ji}^{\,h} + A_{ji}^{\,h}) = \Phi \overset{0}{\Gamma}{}_{ji}^{\,h} + \frac{1}{2}(A_{ji}^{\,h} + \varepsilon_1 \phi_i^{\,b} A_{jb}^{\,a} \phi_a^{\,h})$ for any tensor $A_{ji}^{\,h}$.*

In the statement of this theorem, ϕ-connection is by definition a connection which makes the structure $\phi_i^{\,h}$ covariant constant, and the operation Φ is defined as follows:

$$\Phi \overset{0}{\Gamma}{}_{ji}^{\,h} = \overset{0}{\Gamma}{}_{ji}^{\,h} + \frac{1}{2}\varepsilon_1 (\nabla_j \phi_i^{\,a}) \phi_a^{\,h}, \tag{2.2}$$

where ∇_j denotes the covariant derivative with respect to $\overset{0}{\Gamma}{}_{ji}^{\,h}$.

COROLLARY 2. 2. *In a manifold with an almost complex or an almost product structure $\phi_i^{\,h}$ there always exists a ϕ-connection, which is expressed in the form* (2. 2) *starting from an arbitrary connection $\overset{0}{\Gamma}{}_{ji}^{\,h}$ on the manifold.*

We define also

$$(2.3)\qquad \begin{cases} \Psi\overset{0}{\Gamma}{}_{ji}{}^{h} = \overset{0}{\Gamma}{}_{ji}{}^{h} + \dfrac{1}{2}\varepsilon_2(\overset{0}{\nabla}_j\psi_i{}^a)\psi_a{}^h, \\ K\overset{0}{\Gamma}{}_{ji}{}^{h} = \overset{0}{\Gamma}{}_{ji}{}^{h} + \dfrac{1}{2}\varepsilon_3(\overset{0}{\nabla}_j\kappa_i{}^a)\kappa_a{}^h, \end{cases}$$

for any given affine connection $\Gamma_{ji}{}^h$ in the manifold. Then we have [8]

THEOREM 2. 3. *In a manifold with a structure defined by* $(\phi_i{}^h, \psi_i{}^h, \kappa_i{}^h)$, *an affine connection* $\Gamma_{ji}{}^h$ *is a* (ϕ, ψ)*-connection if and only if there exists an affine connection* $\overset{0}{\Gamma}{}_{ji}{}^h$ *such that* $\overset{0}{\Gamma}{}_{ji}{}^h = \Phi\Psi\overset{0}{\Gamma}{}_{ji}{}^h$.

PROOF. We can prove this theorem by quite the same way as in the case of quaternion structure. By use of the later part of Theorem 2. 1, we have

$$(2.4)\qquad \begin{cases} \Phi\Psi\overset{0}{\Gamma}{}_{ji}{}^{h} = \Phi\left(\overset{0}{\Gamma}{}_{ji}{}^{h} + \dfrac{1}{2}\varepsilon_2(\overset{0}{\nabla}_j\psi_i{}^a)\psi_a{}^h\right) \\ = \overset{0}{\Gamma}{}_{ji}{}^{h} + \dfrac{1}{2}\varepsilon_1(\overset{0}{\nabla}_j\phi_i{}^a)\phi_a{}^h + \dfrac{1}{4}\varepsilon_2(\nabla_j\psi_i{}^a)\psi_a{}^h + \dfrac{1}{4}\varepsilon_1\varepsilon_2\phi_i{}^b((\overset{0}{\nabla}_j\psi_b{}^d)\psi_d{}^a)\phi_a{}^h. \end{cases}$$

On the other hand, from $\phi_i{}^b\psi_b{}^d = -\varepsilon\varepsilon_3\kappa_i{}^d$, we have

$$\phi_i{}^b(\overset{0}{\nabla}_j\psi_b{}^d) + \psi_b{}^d(\overset{0}{\nabla}_j\phi_i{}^b) = -\varepsilon\varepsilon_3(\nabla_j\kappa_i{}^d),$$

hence

$$\phi_i{}^b(\nabla_j\psi_b{}^d)\psi_d{}^a\phi_a{}^h = -\varepsilon_2(\overset{0}{\nabla}_j\phi_i{}^a)\phi_a{}^h + \varepsilon(\overset{0}{\nabla}_j\kappa_i{}^a)\kappa_a{}^h.$$

Putting this in the above expression of $\Phi\Psi\overset{0}{\Gamma}{}_{ji}{}^h$, we have

$$(2.5)\qquad \Phi\Psi\overset{0}{\Gamma}{}_{ji}{}^{h} = \overset{0}{\Gamma}{}_{ji}{}^{h} + \frac{1}{4}\{\varepsilon_1(\overset{0}{\nabla}_j\phi_i{}^a)\phi_a{}^h + \varepsilon_2(\overset{0}{\nabla}_j\psi_i{}^a)\psi_a{}^h + \varepsilon_3(\overset{0}{\nabla}_j\kappa_i{}^a)\kappa_a{}^h\}.$$

Since the last term is symmetric in $\phi_i{}^h$, $\psi_i{}^h$ and $\kappa_i{}^h$, we find

$$(2.6)\qquad \Phi\Psi = \Psi\Phi = \Psi K = K\Psi = K\Phi = \Phi K.$$

From which it is evident that $\Phi\Psi\overset{0}{\Gamma}{}_{ji}{}^h = \Psi\Phi\overset{0}{\Gamma}{}_{ji}{}^h$ is a (ϕ, ψ)-connection by the former part of Theorem 2. 1.

COROLLARY 2. 4. *In a manifold with a structure defined by* $(\phi_i{}^h, \psi_i{}^h, \kappa_i{}^h)$, *there always exists a* (ϕ, ψ)*-connection, which is expressed in the form* (2. 4) *starting from an arbitrary affine connection* $\overset{0}{\Gamma}{}_{ji}{}^h$ *on the manifold.*

2. 2. We consider an almost complex or almost product manifold defined by the structure $\phi_i{}^h(\phi_i{}^a\phi_a{}^h = \varepsilon_1\delta_i{}^h)$. If a Riemannian metric g_{ih} on the manifold satisfies

$$g_{ih} = \phi_i^{\ b}\phi_h^{\ a} g_{ba}, \tag{2.7}$$

the metric g_{ih} is called a Hermitian metric [8, 11] or an almost product metric [2] in each case, but we call it equally as a metric associated with the structure $\phi_i^{\ h}$.

Given an arbitrary Riemannian metric γ_{ih} in the manifold, it is easily seen that $g_{ih} = \frac{1}{2}(\gamma_{ih} + \phi_i^{\ a}\phi_h^{\ b}\gamma_{ba})$ is a metric associated with the structure $\phi_i^{\ h}$. We have also the following

THEOREM 2. 5. *In a manifold with a structure defined by* $(\phi_i^{\ h}, \psi_i^{\ h}, \kappa_i^{\ h})$, *there always exists a metric associated with all of* $\phi_i^{\ h}$, $\psi_i^{\ h}$ *and* $\kappa_i^{\ h}$.

PROOF. Let γ_{ih} be an arbitrary Riemannian metric in the manifold. Then it is evident that the following g_{ih} is a Riemannian metric:

$$g_{ih} = \frac{1}{4}(\gamma_{ih} + \phi_i^{\ a}\phi_h^{\ b}\gamma_{ab} + \psi_i^{\ a}\psi_h^{\ b}\gamma_{ab} + \kappa_i^{\ a}\kappa_h^{\ b}\gamma_{ab}). \tag{2.8}$$

It is also easily seen that g_{ih} is associated with all of $\phi_i^{\ h}$, $\psi_i^{\ h}$ and $\kappa_i^{\ h}$.

In a differentiable manifold with a metric tensor g_{ih}, an affine connection $\Gamma_{ji}^{\ h}$ is said to be metric (with respect to g_{ih}) if $\nabla_j g_{ih} = 0$. We define an operator Λ by

$$\Lambda\overset{0}{\Gamma}{}_{ji}^{\ h} = \overset{0}{\Gamma}{}_{ji}^{\ h} + \frac{1}{2} g^{ha}\nabla_j g_{ia}, \tag{2.9}$$

where g^{ha} is the inverse of g_{ia}. It is known that an affine connection $\Gamma_{ji}^{\ h}$ is metric if and only if there is an affine connection $\overset{0}{\Gamma}{}_{ji}^{\ h}$ such that $\Gamma_{ji}^{\ h} = \Lambda\overset{0}{\Gamma}{}_{ji}^{\ h}$. It is also known that Λ commutes with each of Φ, Ψ and K. Noting that $\Phi\Psi = \Psi\Phi = \Psi K = K\Psi = K\Phi = \Phi K$ (2. 5), we have

THEOREM 2. 6. *In a manifold with a structure* $(\phi_i^{\ h}, \psi_i^{\ h}, \kappa_i^{\ h})$ *and a metric associated with all of* $\phi_i^{\ h}$, $\psi_i^{\ h}$ *and* $\kappa_i^{\ h}$, *there always exists a metric* (ϕ, ψ)-*connection. Such a connection and only such one is written in the form* $\Phi\Psi\Lambda\overset{0}{\Gamma}{}_{ji}^{\ h} = \Psi\Phi\Lambda\overset{0}{\Gamma}{}_{ji}^{\ h} = \cdots\cdots$ *for some affine connection* $\overset{0}{\Gamma}{}_{ji}^{\ h}$.

3. Groups of holonomy. In this section we study the homogeneous groups of homolony for the case II, III and IV. Group of holonomy of case I has been studied by Obata [7] and Wakakuwa [10].

We consider in the manifold a (ϕ, ψ)-connection, i.e. the connection which makes all of the three tensor fields $\phi_i^{\ h}$, $\psi_i^{\ h}$, $\kappa_i^{\ h}$ covariant constant. These tensor fields are left invariant by the group of holonomy with respect to the considered

(ϕ, ψ)-connection. Let P be any point of the manifold, then the linear transformations $\mathfrak{J}$, $\mathfrak{F}$ and K induced respectively by the fields $\phi_i^{\ h}$, $\psi_i^{\ h}$ and $\kappa_i^{\ h}$ in the tangent space at P commute with each element of the group of holonomy $\mathfrak{H}_P$. By choosing base in the tangent space adequately, we can put the matrices representing the elements of $\mathfrak{H}_P$ in considerably simple forms.

3. 1. We consider first the case III. Since $\mathfrak{J}^2 = -E$ (E denotes the identity transformation), $\mathfrak{F}^2 = E$ and $-\mathfrak{J}\mathfrak{F} = \mathfrak{F}\mathfrak{J} = -\kappa$, if a is any proper vector corresponding to the proper value 1 of $\mathfrak{F}$, then we have $\mathfrak{F}(\mathfrak{J}a) = -\mathfrak{J}(\mathfrak{F}a) = -\mathfrak{J}a$, which shows that $\mathfrak{J}a$ is a proper vector corresponding to the proper value -1 of $\mathfrak{F}$. Hence the dimensions of the positive subspace [3] (that is the subspace of the tangent space consisting of all the proper vectors corresponding to the proper value $+1$) and the negative subspace (similarly defined) are equal, thus the manifold must be even-dimensional, say $2n$. Let $(e_1, \ldots\ldots, e_n)$ be a basis of positive subspace of $\mathfrak{F}$, then $(\mathfrak{J}e_1, \ldots\ldots, \mathfrak{J}e_n)$ is a basis of negative subspace of $\mathfrak{F}$. With respect to the basis $(e_1, \ldots\ldots, e_n, \mathfrak{J}e_1, \ldots\ldots, \mathfrak{J}e_n)$ of the tangent space at P, the transformations $\mathfrak{J}$, $\mathfrak{F}$ and K are respectively represented as follows:

$$(3.\ 1)\qquad \mathfrak{J} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, \qquad \mathfrak{F} = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix},$$

where E_n denotes the $n \times n$ unit matrix. Since the elements of $\mathfrak{H}_P$ commute with each of the above three transformations, they are represented as follows with respect to the above basis:

$$(3.\ 2)\qquad \begin{pmatrix} A_n & 0 \\ 0 & A_n \end{pmatrix},$$

where A_n is any $n \times n$ matrix. Conversely if, with respect to suitable basis of tangent space, the elements of the group of holonomy can be represented as the above from, then the elements of the group of holonomy commute with each of $\mathfrak{J}$, $\mathfrak{F}$ and K. Hence there correspond in the manifold three tensor fields $\phi_i^{\ h}$, $\psi_i^{\ h}$, $\kappa_i^{\ h}$ and the manifold has a structure of case III. Thus we have

THEOREM 3. 1. *A manifold is of the case III and its connection is a* (ϕ, ψ)*-connection, if the elements of its group of holonomy can be represented as form* (3. 2) *refered to suitable bases. Conversely if the manifold is of the case III, then the elements of the group of holonomy with respect to* (ϕ, ψ)*-connection can be represented as form* (3. 2) *refered to suitable bases.*

3. 2. We consider a manifold of case II or case IV. In such a manifold there is a tensor field $\kappa_i^{\ h}$ such that $\kappa_i^{\ a}\kappa_a^{\ h} = \delta_i^h$. Let K be the linear transformation induced by this field in the tangent space at P. We choose a basis of the tangent space such that the former p vectors form a basis of the positive subspace and the latter $q = n - p$ vectors form a basis of the negative subspace of K. With respect to this basis, K may be represented as follows:

$$K = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}. \tag{3.3}$$

Since $\mathfrak{J}$ and $\mathfrak{F}$ both commute with K, they can be represented as:

$$\begin{pmatrix} A_p & 0 \\ 0 & B_q \end{pmatrix}. \tag{3.4}$$

In case II, we have $\mathfrak{J}^2 = -E$, $\mathfrak{F}^2 = -E$, hence

$$A_p^2 = -E, \quad B_q^2 = -E,$$

which show that if we restrict $\mathfrak{J}$(or $\mathfrak{F}$) to the positive subspace T_P^+ or negative subspace T_P^- of K, it may be seen as transformation induced by an almost complex structure. Thus both p and q are even, say $p = 2l$, $q = 2m$ and M is even-dimensional. As T_P^+ and T_P^- are complementary to each other, the complexification $(T_P)^C$ of the tangent space T_P is the direct sum of the complexification $(T_P^+)^C$, $(T_P^-)^C$ of T_P^+ and T_P^-. Then by choosing suitable bases [7] in $(T_P^+)^C$ and $(T_P^-)^C$, we can represent the restriction of K in T_P^+ and T_P^- respectively as follows:

$$\begin{pmatrix} 0 & E_l \\ -E_l & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}.$$

With respect to such base in $(T_P)^C$, $\mathfrak{F}$ and $\mathfrak{F} = K\mathfrak{J}$ can be represented respectively as

$$\mathfrak{J} = \begin{pmatrix} 0 & E_l & 0 & 0 \\ -E_l & 0 & 0 & 0 \\ 0 & 0 & 0 & E_m \\ 0 & 0 & -E_m & 0 \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} 0 & E_l & 0 & 0 \\ -E_l & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_m \\ 0 & 0 & E_m & 0 \end{pmatrix}. \tag{3.5}$$

Since each element of the group of holonomy commutes with $\mathfrak{J}$, $\mathfrak{F}$ and K in (3.5) and (3.3), the elements of the group of holonomy are represented as

$$\begin{pmatrix} A_p & -B_p & 0 & 0 \\ B_p & A_p & 0 & 0 \\ 0 & 0 & C_q & -D_q \\ 0 & 0 & D_q & C_q \end{pmatrix}. \tag{3.6}$$

Conversely if the elements of the group of holonomy can be represented by matrices of the above form, then they commute with $\mathfrak{J}$, $\mathfrak{F}$ and K. Thus there exist in the manifold three tensor fields $\phi_i^{\ h}$, $\psi_i^{\ h}$ and $\kappa_i^{\ h}$ corresponding respectively to $\mathfrak{J}$, $\mathfrak{F}$ and K in (3.5) and (3.3). Thus we have

THEOREM 3. 2. *A manifold is of case II and its connection is a (ϕ, ψ)-connection, if the elements of its group of holonomy can be represented as form* (3.6) *refered to suitable bases. Conversely if the manifold is of case II, then the elements of group of holonomy with respect to (ϕ, ψ)-connection can be represented as form* (3.6) *refered to suitable bases.*

3. 3. Finally we consider the case IV in which the three tensor fields $\phi_i^{\ h}$, $\psi_i^{\ h}$ and $\kappa_i^{\ h}$ are linearly independent. Since $\mathfrak{J}^2 = \mathfrak{F}^2 = E$, $\mathfrak{J}$ and $\mathfrak{F}$ can be represented in the form (3.4) in which

$$A_p^2 = E_p, \quad B_q^2 = E_q.$$

Thus if we restrict $\mathfrak{J}$ (or $\mathfrak{F}$) in the positive subspace T_P^+ or negative subspace T_P^- of K, it can be seen as a transformation $\mathfrak{L}_p$ or a transformation $\mathfrak{L}_q$ induced in T_P^+ and T_P^- by almost product structures. By suitable choice of bases in T_P^+ and T_P^-, $\mathfrak{L}_p$ and $\mathfrak{L}_q$ can be respectively represented as

$$\begin{pmatrix} E_r & 0 \\ 0 & -E_s \end{pmatrix}, \quad \begin{pmatrix} E_t & 0 \\ 0 & -E_u \end{pmatrix}, \quad r+s=p,\ t+u=q.$$

Hence with respect to such base in tangent space $\mathfrak{J}$ and $\mathfrak{F} = K\mathfrak{J}$ can be represented

$$(3.7) \qquad \mathfrak{J} = \begin{pmatrix} E_r & 0 & 0 & 0 \\ 0 & -E_s & 0 & 0 \\ 0 & 0 & E_t & 0 \\ 0 & 0 & 0 & -E_u \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} E_r & 0 & 0 & 0 \\ 0 & -E_s & 0 & 0 \\ 0 & 0 & -E_t & 0 \\ 0 & 0 & 0 & E_u \end{pmatrix}.$$

The elements of the group of holonomy commute with $\mathfrak{J}$, $\mathfrak{F}$ and K in (3.7) and (3.3), so they can be represented as follows:

$$(3.8) \qquad \begin{pmatrix} A_r & 0 & 0 & 0 \\ 0 & B_s & 0 & 0 \\ 0 & 0 & C_t & 0 \\ 0 & 0 & 0 & D_u \end{pmatrix}.$$

Conversely if the elements of the group of holonomy can be represented as the form (3.8). then they commute with $\mathfrak{J}$, $\mathfrak{F}$ and K of (3.7) and (3.3). Hence in the manifold there exist three tensor fields $\phi_i^{\ h}$, $\psi_i^{\ h}$ and $\kappa_i^{\ h}$ of case IV corresponding to $\mathfrak{J}$, $\mathfrak{F}$ and K of (3.7) and (3.3). Thus we have

THEOREM 3. 3. *A manifold is of case IV with three linearly independent tensor fields and its connection is a (ϕ, ψ)-connection, if the elements of its group of holonomy can be represented as form* (3. 8) *refered to suitable bases. Conversely if the manifold is of case IV with three linearly independent tensor fields, then the elements of the group of holonomy with respect to (ϕ, ψ)-connection can be represented as* (3. 8) *refered to suitable bases.*

4. Transformations preserving the structures. Obata [7] has proved that in an irreducible almost complex manifold of dimension $2n$, if the largest connected group of affine transformations does not preserve the almost complex structure, then n is even and the homogeneous holonomy group is contained in the real representation of the quaternionian linear group. In this section we study analogous things for almost product manifolds but only some partial results are obtained.

4. 1. Let φ be a differentiable homeomorphism or a transformation of an n-dimensional differentiable manifold M onto itself. We denote by the same letter φ the differential of φ, its extension to the tensor spaces and also that to the algebra of tensor fields.

For any geometric object Ω, if $\Omega = \varphi\Omega$, we say that Ω is invariant under φ or that φ preserves Ω. If Γ is an affine connection on the manifold, and $\Gamma = \varphi\Gamma$, then φ is called an affine transformation (with respect to the given affine connection Γ).

Let $A(M)$ be the group of all affine transformations of M onto itself, and $A_0(M)$ be the connected component of the identity in $A(M)$. It is known [7] that the set $P(r, s)$ of all parallel tensor fields of type (r, s) on M is a finite dimensional vector space over R, and that $A(M)$ acts on $P(r, s)$ as a group of automorphisms. Moreover, the homomorphism ρ of $A(M)$ into $GL(h, R)$ (where $h = \dim P(r, s)$) defined by $\rho(\varphi)\cdot\xi = \varphi\cdot\xi$ for any $\xi \in P(r, s)$ is continuous.

Let M be of dimension $2n$ and has an almost product structure ϕ, which induces in the tangent space T_P a transformation F whose positive and negative subspaces T_P^+, T_P^- are both assumed to be of dimension n. On M we consider an affine connection which makes the given structure covariant constant. Let $\mathfrak{H}_P$ be the homogeneous holonomy group of M at the point $P \in M$. We choose a basis of T_P such that its former n vectors constitute a basis of T_P^+ and its latter n vectors constitute a base of T_P^- (thus each vector of the basis is a proper vector of F), then F has the following form relative to this basis:

$$F = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}, \tag{4. 1}$$

E_n being the unit matrix of degree n. Since F is invariant under $\mathfrak{H}_P$ (because

ϕ is covariant constant), the matrix F commutes with any elements of $\mathfrak{H}_P$, so that any element A of $\mathfrak{H}_P$ has the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{4.2}$$

relative to the above basis. A_1, A_2 being matrices of degree n. This means that $\mathfrak{H}_P$ is a subgroup of the direct product $GL(n, R) \times GL(n, R)$.

Let $P(1, 1)$ be the vector space spanned by all parallel tensor fields of type $(1, 1)$ on M and $P^+(1, 1)$, $P^-(1, 1)$, be respectively the subset of all the elements κ of $P(1, 1)$ such that $\kappa_i{}^a\kappa_a{}^h = \delta_i^h$ or $\kappa_i{}^a\kappa_a{}^h = -\delta_i^h$. Then any element φ of $A(M)$ transforms linearly $P(1, 1)$, $P^+(1, 1)$ and $P^-(1, 1)$ respectively onto itself. Assigning $\kappa \in P(1, 1)$ to the value K of κ at P, $P(1, 1)$ is isomorphic with the subspace of the tensor space of type $(1, 1)$ over T_P consisting of all tensors invariant under $\mathfrak{H}_P$, i. e. $P(1, 1)$ is isomorphic with the commutator algebra $\mathfrak{K}$ of $\mathfrak{H}_P$. It is obvious that $P^+(1, 1)$ and $P^-(1, 1)$ are respectively isomorphic with the subset $\mathfrak{K}^+$ or $\mathfrak{K}^-$ consisting of the commutator K such that $K^2 = E$ or $K^2 = -E$.

We consider first the manifold whose group of holonomy $\mathfrak{H}_P$ consists of the elements (4.2) in which A_1, A_2 are both irreducible on complex field, and contains a subgroup $\overline{\mathfrak{H}}_P \neq \{\text{identity}\}$ consisting of elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \tag{4.3}$$

in which A is irreducible on complex field.

Let

$$\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$$

be any element in the commutator algebra of $\mathfrak{H}_P$, then

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

which gives

$$\begin{cases} A_1K_1 = K_1A_1, & A_2K_4 = K_4A_2, \\ A_1K_2 = K_2A_2, & A_2K_3 = K_3A_1. \end{cases} \tag{4.4}$$

As $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$, the last two formulas hold also for $A_1 = A_2 = A$, i. e.

$$AK_2 = K_2A, \quad AK_3 = K_3A.$$

Since in each element (4. 2) of $\mathfrak{H}_P$, A_1 and A_2 are irreducible on complex field, by Schur's lemma we have

$$(4.5)\qquad \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 E_n & \alpha_2 E_n \\ \alpha_3 E_n & \alpha_4 E_n \end{pmatrix},$$

where α_1, α_2, α_3, α_4 are real numbers.

If $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ as a proper subgroup, then $\mathfrak{H}_P$ actually contains elements (4. 2) in which $A_1 \neq A_2$, then from the last two formulas of (4. 4) we have $\alpha_2 = \alpha_3 = 0$. Thus in this case, any element in the commutator $\mathfrak{K}$ of $\mathfrak{H}_P$ can be written as

$$(4.6)\qquad \begin{pmatrix} \alpha_1 E_n & 0 \\ 0 & \alpha_4 E_n \end{pmatrix}.$$

Hence the following two elements form a basis of $\mathfrak{K}$:

$$(4.7)\qquad E = \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \qquad F = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}.$$

Let $K = aE + bF \in \mathfrak{K}^+$, then we have $a^2 + b^2 = 1$, $ab = 0$. It follows that $\mathfrak{K}^+$ contains only the following four elements: $\pm E, \pm F$. Thus $P^+(1, 1)$ consists of $\pm \delta_i^h$ and $\pm \phi_i^h$ which induce $\pm F$ in the tangent space. Since $\rho(\varphi)\cdot P^+(1, 1) \subset P^+(1, 1)$ for every $\varphi \in A(M)$, we have $\rho(\varphi)\phi = \pm\phi$ (because $\rho(\varphi)\cdot I = \pm I$ for every $\varphi \in A(M)$). ρ being continuous, we have $\rho(\varphi)\cdot\phi = \phi$ for every $\varphi \in A_0(M)$, i. e. $A_0(M)$ preserves the almost product structure ϕ.

If $\mathfrak{H}_P = \overline{\mathfrak{H}}_P$, then in (4. 5) $\alpha_2 = \alpha_3 = 0$ need not hold, and the following two elements and E, F in (4. 7) form a basis of $\mathfrak{K}$:

$$(4.8)\qquad G = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

It is evident that these four elements satisfy the following relations:

$$(4.9)\qquad \begin{cases} F^2 = E, \quad G^2 = E, \quad H^2 = -E, \\ GH = -HG = F, \quad HF = -FH = G, \quad GF = -FG = H. \end{cases}$$

Let ψ_i^h, κ_i^h be respectively the parallel tensor field deduced by parallel displacement from G and H. Then $(\phi_i^h, \psi_i^h, \kappa_i^h)$ define on the manifold M a structure of case III. Let $K = aE + bF + cG + dH$, then the conditions for K to belong to $\mathfrak{K}^+$ or $\mathfrak{K}^-$ are: $a^2 + b^2 + c^2 - d^2 = 1$ or -1, $ab = ac = ad = 0$. From these relations we conclude that $\mathfrak{K}^+$ consists of $\pm E$ and elements of the form $bF + cG + dH$ with $b^2 + c^2 - d^2 = 1$, and that $\mathfrak{K}^-$ consists of the elements of the form $bF + cG + dH$ with $b^2 + c^2 - d^2 = -1$.

Since $\rho(\varphi)\cdot P^+(1, 1) \subset P^+(1, 1)$, $\rho(\varphi)\phi = \pm I$ or $\rho(\varphi)\phi = b\phi + c\psi + d\kappa$ with $b^2 + c^2 - d^2 = 1$. But as $\rho(\varphi)(\pm I) = \pm I$ for every $\varphi \in A(M)$, so $\rho(\varphi)\phi \neq \pm I$. Thus we have

$$(4.10) \qquad \begin{cases} \rho(\varphi)\phi = a_{11}\phi + a_{21}\psi + a_{31}\kappa & \in P^+(1, 1), \\ \rho(\varphi)\psi = a_{12}\phi + a_{22}\psi + a_{32}\kappa & \in P^+(1, 1), \end{cases}$$

similarly we have

$$(4.10) \qquad \rho(\varphi)\kappa = a_{13}\phi + a_{23}\psi + a_{33}\kappa \quad \in P^-(1, 1),$$

where

$$(4.11) \qquad \begin{cases} a_{11}^2 + a_{21}^2 - a_{31}^2 = 1, \\ a_{12}^2 + a_{22}^2 - a_{32}^2 = 1, \\ a_{13}^2 + a_{23}^2 - a_{33}^2 = -1. \end{cases}$$

We identify

$$\rho(\varphi) = \begin{pmatrix} a_{11} & a_{22} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, we denote by $P^*(1, 1)$ the subspace of $P(1, 1)$ spanned by ϕ, ψ and κ. Then $\rho(\varphi)$, for every $\varphi \in A(M)$, is an automorphism of $P^*(1, 1)$. If $\xi = \xi_1\phi + \xi_2\psi + \xi_3\kappa$ and $\eta = \eta_1\phi + \eta_2\psi + \eta_3\kappa$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in R$, then we have

$$\xi\cdot\eta = (\xi_1\eta_1 + \xi_2\eta_2 - \xi_3\eta_3)I + (\xi_2\eta_3 - \xi_3\eta_2)\phi + (\xi_3\eta_1 - \xi_1\eta_3)\psi - (\xi_1\eta_2 - \xi_2\eta_1)\kappa.$$

We call $\xi_1\eta_1 + \xi_2\eta_2 - \xi_3\eta_3$ the scalar product of ξ and η, and denote it as (ξ, η). And $(\xi_2\eta_3 - \xi_3\eta_2)\phi + (\xi_3\eta_1 - \xi_1\eta_3)\psi - (\xi_1\eta_2 - \xi_2\eta_1)\kappa$ is called the vector product of ξ and η, and is denoted by $\xi \times \eta$. Then the above relation is written

$$\xi\cdot\eta = (\xi, \eta)I + \xi \times \eta.$$

Since $\rho(\varphi)$ is a homomorphism, $\rho(\varphi)(\xi\cdot\eta) = (\rho(\varphi)\cdot\xi)\cdot(\rho(\varphi)\cdot\eta)$ for every $\varphi \in A(M)$. So we have

$$(\rho(\varphi)\xi, \rho(\varphi)\eta) = (\xi, \eta),$$
$$(\rho(\varphi)\cdot\xi) \times (\rho(\varphi)\cdot\eta) = \rho(\varphi)\cdot(\xi \times \eta).$$

Thus the linear transformation $\rho(\varphi)$ leaves invariant the inner and vector products in $P^*(1. 1)$, so that $\rho(\varphi)$ is a Lorenz transformation. And we have

THEOREM 4. 1. *Let M be an almost product manifold. Suppose that 1^0 the elements of its holonomy group $\mathfrak{H}_P$ are represented as (4. 2) in which A_1,*

A_2 are $n \times n$ matrices and irreducible on the complex field and that 2^0 $\mathfrak{H}_P$ *contains a subgroup* $\overline{\mathfrak{H}}_P \neq \{identity\}$ *which consists only elements of the form* (4. 3) *where A is irreducible on the complex field. Then the almost product structure is preserved by $A_0(M)$ except for the case when M is a manifold with a structure of case III. In this case $A(M)$ acts on the vector space spanned by ϕ, ψ, κ as a group of Lorenz transformations.*

4. 2. We consider next the manifold whose group of holonomy $\mathfrak{H}_P$ consists of the elements of the form (4. 2) in which A_1, A_2 are both irreducible on real field but reducible on complex field. Moreover, we assume that $\mathfrak{H}_P$ contains a subgroup $\overline{\mathfrak{H}}_P \neq \{\text{identity}\}$ which consists of elements of form (4. 3) in which A is irreducible on real field but reducible on complex field. Let T_P^1 and T_P^2 be the n-dimensional subspaces of the tangent space T_P at P which are left invariant by the group of holonomy $\mathfrak{H}_P$. The complexification $(T_P)^C$ of T_P is the direct sum of the respective complexification $(T_P^1)^C$, $(T_P^2)^C$ of T_P^1 and T_P^2. It is known ([7] appendix) that if we choose suitable bases in $(T_P^1)^C$ and $(T_P^2)^C$, then the elements of the group of holonomy restricted to $(T_P^1)^C$ or $(T_P^1)^C$ are respectively represented by the matrices of the form

$$\begin{pmatrix} B & 0 \\ 0 & \overline{B} \end{pmatrix},$$

where B is a complex matrix of degree $m(n = 2m)$, and $\overline{B}$ is the conjugate complex matrix of B. With respect to such base in $(T_P)^C$ the elements of the group of holonomy are represented as

$$\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & \overline{B}_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & \overline{B}_2 \end{pmatrix}, \tag{4. 12}$$

in which the complex matrices B_1 and B_2 are both irreducible on complex field. Any real linear transformation of T_P may be represented as

$$\begin{pmatrix} K_1 & K_2 & H_1 & H_2 \\ \overline{K}_2 & \overline{K}_1 & \overline{H}_2 & \overline{H}_1 \\ L_1 & L_2 & M_1 & M_2 \\ \overline{L}_2 & \overline{L}_1 & \overline{M}_2 & \overline{M}_1 \end{pmatrix}. \tag{4. 13}$$

The conditions for the matrix (4. 13) to commute with every element of the group of holonomy are as follows:

$$
\left\{
\begin{aligned}
&K_1B_1 = B_1K_1, \quad K_2\overline{B}_1 = B_1K_2,\\
&H_1B_2 = B_1H_1, \quad H_2\overline{B}_2 = B_1H_2,\\
&L_1B_1 = B_2L_1, \quad L_2\overline{B}_1 = B_2L_2,\\
&M_1B_2 = B_2M_1, \quad M_2\overline{B}_2 = B_2M_2.
\end{aligned}
\right. \tag{4.14}
$$

As the group $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ $(B_1 = B_2 = B)$ as a subgroup, we have also the following

$$
\left\{
\begin{aligned}
&H_1B = BH_1, \quad H_2\overline{B} = BH_2,\\
&L_1B = BL_1, \quad L_2\overline{B} = BL_2.
\end{aligned}
\right. \tag{4.15}
$$

Since B_1, B_2 are irreducible on the complex field, if $\mathfrak{H}_P = \overline{\mathfrak{H}}_P$ the commutator of the group of holonomy must be of the following form [7] with respect to a suitable base in $(T_P)^C$:

$$
\begin{pmatrix}
\alpha_1E_m & \beta_1J_l & \alpha_2E_m & \beta_2J_l\\
\overline{\beta}_1J_l & \overline{\alpha}_1E_m & \overline{\beta}_2J_l & \overline{\alpha}_2E_m\\
\alpha_3E_m & \beta_3J_l & \alpha_4E_m & \beta_4J_l\\
\overline{\beta}_3J_l & \overline{\alpha}_3E_m & \overline{\beta}_4J_l & \overline{\alpha}_4E_m
\end{pmatrix}, \tag{4.16}
$$

where β's may be zero. If some $\beta \neq 0$, then $m = 2l$ and J_l is of the following form:

$$
J_l = \begin{pmatrix} 0 & -E_l\\ E_l & 0\end{pmatrix}.
$$

But if $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ as a proper subgroup, then $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0$. Because, as $H_1 = \alpha_2E_m$ satisfies $H_1B_2 = B_1H_1$ for $B_1 \neq B_2$, it follows that $\alpha_2 = 0$. Similarly $\alpha_3 = 0$. Now if $\beta_2 \neq 0$, then det $H_2 \neq 0$ and we have $\overline{B}_2 = H_2^{-1}B_1H_2 = \overline{B}_1$ which contradict to the assumption that $\overline{\mathfrak{H}}_P$ is a proper subgroup of $\mathfrak{H}_P$ Similarly $\beta_3 = 0$. Thus if $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ as a proper subgroup, the elements of the commutator algebra of $\mathfrak{H}_P$ are of the form:

$$
\begin{pmatrix}
\alpha_1E_m & \beta_1J_l & 0 & 0\\
\overline{\beta}_1J_l & \overline{\alpha}_1E_m & 0 & 0\\
0 & 0 & \alpha_4E_m & \beta_4J_l\\
0 & 0 & \overline{\beta}_4J_l & \overline{\alpha}_4E_m
\end{pmatrix}. \tag{4.17}
$$

If $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ as a proper subgroup and both factor G_1, G_2 of $\mathfrak{H}_P = G_1 \times G_2$ are not the subgroup of real representation of quaternionian linear group, then the elements of the commutator algebra of $\mathfrak{H}_P$ are of the form

$$(4.18)\qquad \begin{pmatrix} \alpha_1 E_m & 0 & 0 & 0 \\ 0 & \bar{\alpha}_1 E_m & 0 & 0 \\ 0 & 0 & \alpha_4 E_m & 0 \\ 0 & 0 & 0 & \bar{\alpha}_4 E_m \end{pmatrix}.$$

We can choose adequate real base in $(T_P)^C$, such that the following items hold:

(i) If $\mathfrak{H}_P$ contains $\bar{\mathfrak{H}}_P$ as proper subgroup and is a direct product of two subgroups of $GL(n, R)$, none of which is the subgroup of the real representation of quaternionian linear group, then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ with } \begin{pmatrix} E_m & 0 \\ 0 & E_m \end{pmatrix}, \quad \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix},$$

that is, the following four elements:

$$(4.19)\qquad \begin{aligned} E &= \begin{pmatrix} E_m & 0 & 0 & 0 \\ 0 & E_m & 0 & 0 \\ 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & E_m \end{pmatrix}, \quad F = \begin{pmatrix} E_m & 0 & 0 & 0 \\ 0 & E_m & 0 & 0 \\ 0 & 0 & -E_m & 0 \\ 0 & 0 & 0 & -E_m \end{pmatrix}, \\ G &= \begin{pmatrix} 0 & E_m & 0 & 0 \\ -E_m & 0 & 0 & 0 \\ 0 & 0 & 0 & E_m \\ 0 & 0 & -E_m & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & E_m & 0 & 0 \\ -E_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_m \\ 0 & 0 & E_m & 0 \end{pmatrix}. \end{aligned}$$

(ii) If $\mathfrak{H}_P$ contains $\bar{\mathfrak{H}}_P$ as proper subgroup and is a direct product of two groups which are both the subgroup of real representation of quaternionian linear group, then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of $(m = 2l)$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ with } \begin{pmatrix} E_m & 0 \\ 0 & E_m \end{pmatrix}, \quad \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}, \quad \begin{pmatrix} J_l & 0 \\ 0 & -J_l \end{pmatrix}, \quad \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix},$$

that is the four elements in (4.19) and the following four elements:

$$(4.20)\qquad I = \begin{pmatrix} J_l & 0 & 0 & 0 \\ 0 & -J_l & 0 & 0 \\ 0 & 0 & J_l & 0 \\ 0 & 0 & 0 & -J_l \end{pmatrix}, \quad J = \begin{pmatrix} J_l & 0 & 0 & 0 \\ 0 & -J_l & 0 & 0 \\ 0 & 0 & -J_l & 0 \\ 0 & 0 & 0 & J_l \end{pmatrix},$$

$$K=\begin{pmatrix}0 & J_l & 0 & 0\\ J_l & 0 & 0 & 0\\ 0 & 0 & 0 & J_l\\ 0 & 0 & J_l & 0\end{pmatrix},\quad L=\begin{pmatrix}0 & J_l & 0 & 0\\ J_l & 0 & 0 & 0\\ 0 & 0 & 0 & -J_l\\ 0 & 0 & -J_l & 0\end{pmatrix}.$$

(iii) If $\mathfrak{H}_P=\overline{\mathfrak{H}}_P$, then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\ \text{with}\ \begin{pmatrix}E_m & 0\\ 0 & E_m\end{pmatrix},\ \begin{pmatrix}0 & -E_m\\ E_m & 0\end{pmatrix},\ \begin{pmatrix}J_l & 0\\ 0 & -J_l\end{pmatrix},\ \begin{pmatrix}0 & J_l\\ J_l & 0\end{pmatrix}.$$

In case (ii) the eight elements in (4.19) and (4.20) satisfy the following relations:

$$\begin{aligned}
&E=F^2=-G^2=-H^2=-I^2=-J^2=-K^2=-L^2,\\
&F=FE=-GH=-HG=-IJ=-JI=-KL=-LK,\\
&G=FH=GE=HF=-IK=-JL=KI=LJ,\\
&H=FG=GF=HE=-IL=-JK=KJ=LI,\\
&I\ =FJ=GK=HL=IE=JF=-KG=-LH,\\
&J\ =FI=GL=HK=IF=JE=-KH=-LG,\\
&K=FL=-GI=-HJ=IG=JH=KE=LF,\\
&L\ =FK=-GJ=-HJ=IH=JG=KF=LE.
\end{aligned}\tag{4.21}$$

Let $R=aE+bF+cG+dH+eI+fJ+gK+hL$, then the conditions for R to belong to $\mathfrak{R}^+$ or $\mathfrak{R}^-$ are as follows:

$$\left\{\begin{aligned}
&(a+b)^2-(c+d)^2-(e+f)^2-(g+h)^2=1 \text{ or } -1,\\
&(a+b)(c+d)=0,\ (a+b)(e+f)=0,\ (a+b)(g+h)=0,\\
&ab-cd-ef-gh=0,\\
&ac+bd=0,\ ae+bf=0,\ ag+bh=0.
\end{aligned}\right.\tag{4.22}$$

Solving these equations,we get for case (ii) the following results: $\mathfrak{R}^+$ consists of $\pm E$ and $\pm F$. $\mathfrak{R}^-$ consists of $cG+dH+eI+fJ+gK+hL$ with $cd+ef+gh=0$ and $c^2+d^2+e^2+f^2+g^2+h^2=1$. Then it follows that in such a manifold the almost product structure ϕ (which induces F on the tangent space) is left invariant by $A_0(M)$, but none of the almost complex structures is left invariant.

Let $S=aE+bF+cG+dH$, then the conditions for S to belong to $\mathfrak{R}^+$ or $\mathfrak{R}^-$ are as follows:

$$\left\{\begin{aligned}
&(a+b)^2-(c+d)^2=1 \text{ or } -1,\\
&(a+b)(c+d)=0,\ ab-cd=0,\ ad+bc=0.
\end{aligned}\right.\tag{4.23}$$

Solving these equations we get for case (i) the following results:

$\mathfrak{R}^+$ consists of $\pm E$, $\pm F$. $\mathfrak{R}^-$ consists of $\pm G$, $\pm H$. It follows that in this case the manifold has a structure of case II and all the structures ϕ, ψ, κ are left invariant by $A_0(M)$. Thus we have

THEOREM 4. 2. *Assume that the group of holonomy $\mathfrak{H}_P$ of the manifold is the direct product of two subgroups of $GL(n, R)$, each of which is irreducible on real field but reducible on complex field. Moreover we assume that $\mathfrak{H}_P$ contains a subgroup $\bar{\mathfrak{H}}_P$ whose elements are all of the form* (4. 3) *in which A is inreducible in real field but reducible in complex field. Then if none of the factors of the direct product is the subgroup of real represeniation of quaternionian linear group, the manifold has a structure $(\phi_i^h, \psi_i^h, \kappa_i^h)$ of case II and all of ϕ, ψ and κ are preserved by $A_0(M)$. If both of the factors of the direct product are subgroups of real representation of quaternionian linear group, then the manifold has a structure defined by seven tensor fields deduced from elements in* (4. 21) *by parallel displacement. In such a manifold the almost product structure ϕ (deduced from F) is left invariant, but almost complex structures are not.*

4. 3. Finally we consider the manifold whose group of holonomy is the direct product of four subgroups of $GL(m, R)$, each of which is irreducible on complex field. Then the elements of the group of holonomy $\mathfrak{H}_P$ can be represented as

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{4. 24}$$

where A, B, C, D are matrices of degree m. Moreover, we assume that $\mathfrak{H}_P$ contains a subgroup $\bar{\mathfrak{H}}_P \neq \{\text{identity}\}$ whose elements are all of the form:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}; \quad (A: \text{irreducible on complex field}). \tag{4. 25}$$

The condition for the matrix:

$$\begin{pmatrix} P_1 & Q_1 & P_2 & Q_2 \\ R_1 & S_1 & R_2 & S_2 \\ P_3 & Q_3 & P_4 & Q_4 \\ R_3 & S_3 & R_4 & S_4 \end{pmatrix} \tag{4. 26}$$

to be a commutator of the group $\mathfrak{H}_P$ are as follows:

$$\begin{aligned} &AP_1 = P_1A,\ AQ_1 = Q_1B,\ AP_2 = P_2C,\ AQ_2 = Q_2D, \\ &BR_1 = R_1A,\ BS_1 = S_1B,\ BR_2 = R_2C,\ BS_2 = S_2D, \\ &CP_3 = P_3A,\ CQ_3 = Q_3B,\ CP_4 = P_4C,\ CQ_4 = Q_4D, \\ &DR_3 = R_3A,\ DS_3 = S_3B,\ DR_4 = R_4C,\ DS_4 = S_4D. \end{aligned} \tag{4.27}$$

As $\mathfrak{H}_P$ contains $\overline{\mathfrak{H}}_P$ as subgroup, the above relations also hold good when we put $A = B = C = D$ in them. Since each factor of the direct product of $\mathfrak{H}_P$ is irreducible on complex field we conclude that the commutators of $\mathfrak{H}_P = \overline{\mathfrak{H}}_P$ are of the following form:

$$\begin{pmatrix} \alpha_1E_m & \beta_1E_m & \alpha_2E_m & \beta_2E_m \\ \gamma_1E_m & \delta_1E_m & \gamma_2E_m & \delta_2E_m \\ \alpha_3E_m & \beta_3E_m & \alpha_4E_m & \beta_4E_m \\ \gamma_3E_m & \delta_3E_m & \gamma_4E_m & \delta_4E_m \end{pmatrix}. \tag{4.28}$$

If the group $\mathfrak{H}_P$ contains merely the elements of the following form (in this case we denote $\mathfrak{H}_P = \widetilde{\mathfrak{H}}_P$):

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix};\ (A,\ B\text{: irreducible on complex field}), \tag{4.29}$$

then the commutators of $\mathfrak{H}_P = \widetilde{\mathfrak{H}}_P$ can be represented as

$$\begin{pmatrix} \alpha_1E_m & 0 & \alpha_2E_m & 0 \\ 0 & \delta_1E_m & 0 & \delta_2E_m \\ \alpha_3E_m & 0 & \alpha_4E_m & 0 \\ 0 & \delta_3E_m & 0 & \delta_4E_m \end{pmatrix}. \tag{4.30}$$

If $\mathfrak{H}_P$ actually contains elements of the form (4.24) in which any two of A, B, C, D are distinct from each other, then the commutators of $\mathfrak{H}_P$ are of the following form:

$$\begin{pmatrix} \alpha_1E_m & 0 & 0 & 0 \\ 0 & \delta_1E_m & 0 & 0 \\ 0 & 0 & \alpha_4E_m & 0 \\ 0 & 0 & 0 & \delta_4E_m \end{pmatrix}. \tag{4.31}$$

Then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\ \text{with}\ \begin{pmatrix}E_m & 0\\0 & E_m\end{pmatrix},\ \begin{pmatrix}E_m & 0\\0 & -E_m\end{pmatrix},$$

that is, the following four elements:

$$(4.32)\qquad \begin{aligned} E &= \begin{pmatrix}E_m & 0 & 0 & 0\\0 & E_m & 0 & 0\\0 & 0 & E_m & 0\\0 & 0 & 0 & E_m\end{pmatrix}, & F &= \begin{pmatrix}E_m & 0 & 0 & 0\\0 & E_m & 0 & 0\\0 & 0 & -E_m & 0\\0 & 0 & 0 & -E_m\end{pmatrix},\\ G &= \begin{pmatrix}E_m & 0 & 0 & 0\\0 & -E_m & 0 & 0\\0 & 0 & E_m & 0\\0 & 0 & 0 & -E_m\end{pmatrix}, & H &= \begin{pmatrix}E_m & 0 & 0 & 0\\0 & -E_m & 0 & 0\\0 & 0 & -E_m & 0\\0 & 0 & 0 & E_m\end{pmatrix}.\end{aligned}$$

In case when $\mathfrak{H}_P$ contains merely the elements of the form (4. 29), i. e. when $\mathfrak{H}_P = \widetilde{\mathfrak{H}}_P$, then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\ \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\ \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\ \text{with}\ \begin{pmatrix}E_m & 0\\0 & E_m\end{pmatrix},\ \begin{pmatrix}E_m & 0\\0 & -E_m\end{pmatrix},$$

that is, the four elements in (4. 23) and the following elements:

$$(4.33)\qquad \begin{aligned} \widetilde{E} &= \begin{pmatrix}0 & 0 & E_m & 0\\0 & 0 & 0 & E_m\\E_m & 0 & 0 & 0\\0 & E_m & 0 & 0\end{pmatrix}, & \widetilde{F} &= \begin{pmatrix}0 & 0 & -E_m & 0\\0 & 0 & 0 & -E_m\\E_m & 0 & 0 & 0\\0 & E_m & 0 & 0\end{pmatrix},\\ \widetilde{G} &= \begin{pmatrix}0 & 0 & E_m & 0\\0 & 0 & 0 & -E_m\\E_m & 0 & 0 & 0\\0 & -E_m & 0 & 0\end{pmatrix}, & \widetilde{H} &= \begin{pmatrix}0 & 0 & -E_m & 0\\0 & 0 & 0 & E_m\\E_m & 0 & 0 & 0\\0 & -E_m & 0 & 0\end{pmatrix}.\end{aligned}$$

If $\mathfrak{H}_P = \overline{\mathfrak{H}}_P$, then the commutator algebra of $\mathfrak{H}_P$ is spanned by the Kronecker products of

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\ \text{with}\begin{pmatrix}E_m & 0\\0 & E_m\end{pmatrix},\begin{pmatrix}E_m & 0\\0 & -E_m\end{pmatrix},\begin{pmatrix}0 & E_m\\E_m & 0\end{pmatrix},\begin{pmatrix}0 & E_m\\-E_m & 0\end{pmatrix}.$$

The eight elements in (4. 23) and (4. 33) satisfy the following relations:

$$\begin{cases} E = F^2 = G^2 = H^2 = \widetilde{E}^2 = -\widetilde{F}^2 = \widetilde{G}^2 = -\widetilde{H}^2,\\ F = FE = GH = HG = \widetilde{E}\widetilde{F} = -\widetilde{F}\widetilde{E} = \widetilde{G}\widetilde{H} = -\widetilde{H}\widetilde{G},\end{cases}$$

$$(4.34)\quad \begin{cases} G = FH = GE = HF = \widetilde{E}\widetilde{G} = -\widetilde{F}\widetilde{H} = \widetilde{G}\widetilde{E} = -\widetilde{H}\widetilde{F}, \\ H = FG = GF = HE = \widetilde{E}\widetilde{H} = -\widetilde{F}\widetilde{G} = \widetilde{G}\widetilde{F} = -\widetilde{H}\widetilde{E}, \\ \widetilde{E} = -F\widetilde{F} = G\widetilde{G} = -H\widetilde{H} = \widetilde{E}E = \widetilde{F}F = \widetilde{G}G = \widetilde{H}H, \\ \widetilde{F} = -F\widetilde{E} = G\widetilde{H} = -H\widetilde{G} = \widetilde{E}F = \widetilde{F}E = \widetilde{G}H = \widetilde{H}G, \\ \widetilde{G} = -F\widetilde{H} = G\widetilde{E} = -H\widetilde{F} = \widetilde{F}G = \widetilde{F}H = \widetilde{G}E = \widetilde{H}F, \\ \widetilde{H} = -F\widetilde{G} = G\widetilde{F} = -H\widetilde{E} = \widetilde{E}H = \widetilde{F}G = \widetilde{G}F = \widetilde{H}E. \end{cases}$$

Let $R = aE + bF + cG + dH + \widetilde{a}\widetilde{E} + \widetilde{b}\widetilde{F} + \widetilde{c}\widetilde{G} + \widetilde{d}\widetilde{H}$, then the conditions for R to belong to $\mathfrak{R}^+$ or $\mathfrak{R}^-$ are as follows:

$$(4.35)\quad \begin{cases} (a+c)^2 + (b+d)^2 + (\widetilde{a}+\widetilde{c})^2 - (\widetilde{b}+\widetilde{d})^2 = 1 \text{ or } -1, \\ (a+c)(b+d) = 0,\ (a+c)(\widetilde{a}+\widetilde{c}) = 0,\ (a+c)(\widetilde{b}+\widetilde{d}) = 0, \\ ac + bd + \widetilde{a}\,\widetilde{c} - \widetilde{b}\widetilde{d} = 0, \\ ab + cd = 0,\ a\widetilde{a} + c\widetilde{c} = 0,\ a\widetilde{b} + c\widetilde{d} = 0. \end{cases}$$

By solving these equations we have the following results:

$\mathfrak{R}^+$ consists of $\pm E$, $\pm G$, $bF + dH + \widetilde{a}\widetilde{E} + \widetilde{c}\widetilde{G} + \widetilde{b}\widetilde{F} + \widetilde{d}\widetilde{H}$ with $(b+c)^2 + (\widetilde{a}+\widetilde{c})^2 - (\widetilde{b}+\widetilde{d})^2 = 1$, $bd + \widetilde{a}\,\widetilde{c} - \widetilde{b}\widetilde{d} = 0$ and $\pm\frac{1}{2}E \pm \frac{1}{2}G + \widetilde{b}(\widetilde{F} \pm \widetilde{H}) + \widetilde{a}(\widetilde{E} \pm \widetilde{G}) + \widetilde{b}(F \pm H)$ with $b^2 + \widetilde{a}^2 - \widetilde{b}^2 = \frac{1}{4}$. $\mathfrak{R}^-$ consists of $bF + dH + \widetilde{a}\widetilde{E} + \widetilde{c}\widetilde{G} + \widetilde{b}\widetilde{F} + \widetilde{d}\widetilde{H}$ with $(b+d)^2 + (\widetilde{a}+\widetilde{c})^2 - (\widetilde{b}+\widetilde{d})^2 = -1$, $bd + \widetilde{a}\,\widetilde{c} - \widetilde{b}\widetilde{d} = 0$. It follows that if $\mathfrak{H}_P = \widetilde{\mathfrak{H}}_P$ the manifold has seven structures deduced from (4.34) by parallel displacements, but none of the almost product structures or almost complex structures is preserved by $A_0(M)$ except the one induced by G.

Let $S = aE + bF + cG + dH$, then the conditions for S to belong to $\mathfrak{R}^+$ are as follows:

$$(4.36)\quad \begin{cases} (a+c)^2 + (b+d)^2 = 1,\ (a+c)(b+d) = 0, \\ ac + bd = 0,\ ab + cd = 0. \end{cases}$$

By solving these equations we conclude that $\mathfrak{R}^+$ consists of $\pm E$, $\pm F$, $\pm G$, $\pm H$, $\pm\frac{1}{2}(E+G) \pm \frac{1}{2}(F-H)$ and $\pm\frac{1}{2}(F+H) \pm \frac{1}{2}(E-G)$. Thus if $\mathfrak{H}_P$ actually contains elements of the form (4.24) in which any two of A, B, C, D are distinct from each other, then the manifold has a structure of case IV defined by ϕ, ψ, κ (induces respectively F, G, H in T_P) and all of these structures are left invariant by $A_0(M)$. Thus we have

THEOREM 4.3. *Assume that the group of holonomy $\mathfrak{H}_P$ of the manifold is the direct product of four subgroups of $GL(m, R)$ each of which is irreducible on complex field. Moreover we assume that $\mathfrak{H}_P$ contains a subgroup $\bar{\mathfrak{H}}_P$ whose elements are all of the form* (4.25). *If $\mathfrak{H}_P$ contains elements of the form* (4.24) *in which any two of A, B, C, D are distinct from each other, then the manifold has a structure of case IV and all of ϕ, ψ and κ are left invariant by $A_0(M)$. If $\mathfrak{H}_P = \widetilde{\mathfrak{H}}_P$, then the manifold has seven structures which induce in T_P the transformations in* (4.34), *but none of the structures is left invariant by $A_0(M)$ except κ.*

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