SOME REMARKS ON A REPRESENTATION OF A GROUP

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1. In the course of the study of the crossed product of rings of operators, it has been shown in [4] that an arbitrary countable group admits a faithful representation as a group of outer automorphisms[1)] of an approximately finite factor on a separable Hilbert space.

Let G be an arbitrary countably infinite group. Let Δ be the set of all functions $\alpha(g)$ on G : $\alpha(g) = 1$ on a finite subset of G and $= 0$ elsewhere, and Δ is an additive group under the addition $[\alpha + \beta](g) = \alpha(g) + \beta(g) \pmod 2$, $0(g) = 0$ $(g \in G)$. Let Δ' be the set of all functions $\varphi(\gamma)$ on Δ : $\varphi(\gamma) = 1$ on a finite subset of Δ and $= 0$ elsewhere. Δ' is an additive group under the addition $[\varphi + \psi](\gamma) = \varphi(\gamma) + \psi(\gamma) \pmod 2$ and $0(\gamma) = 0$ $(\gamma \in \Delta)$. For every $\alpha \in \Delta$, $\varphi \to \varphi^{\alpha}$: $\varphi^{\alpha}(\gamma) = \varphi(\gamma + \alpha)$ is an automorphism of Δ'. Defining the product $(\varphi, \alpha)(\psi, \beta) = (\varphi^{\beta} + \psi, \alpha + \beta)$, we have a locally finite countably infinite group $\mathfrak{G}$ of all elements $(\varphi, \alpha) \in (\Delta', \Delta)$ with the identity $(0, 0)$ and $(\varphi, \alpha)^{-1} = (\varphi^{\alpha}, \alpha)$. Let $\mathbf{H}$ be the Hilbert space $l_2(\mathfrak{G})$, and for each $(\varphi, \alpha) \in \mathfrak{G}$ let $V_{(\varphi,\alpha)}$ be the unitary operator on $\mathbf{H}$ defined $[V_{(\varphi,\alpha)} f]((\psi, \beta)) = f((\psi, \beta)(\varphi, \alpha))$. Then the ring of operators $\mathbf{M}$ generated by all $V_{(\varphi,\alpha)}$ is an approximately finite factor. Next, define an operator $T_g(T_g')$ on $\Delta(\Delta')$: $[T_g\alpha](g') = \alpha(gg')$ for all $\alpha \in \Delta$, $g, g' \in G$; $[T_g'\varphi](\gamma) = \varphi(Tg^{-1}\gamma)$ for all $\varphi \in G$, and $g \to T_g(T_g')$ is an anti-isomorphism of G into a group of automorphisms of Δ (Δ'). For each $g \in G$, we define a unitary operator U_g in $\mathbf{H}$ by $[U_g f]((\varphi, \alpha)) = f((T_g'\varphi, T_g\alpha))$, and $g \to U_g$ is a faithful unitary representation of G and for each $g \in G(\neq e)$ $V_{(\varphi,\alpha)} \to U_g^{-1} V_{(\varphi,\alpha)} U_g = V_{(T_g'\varphi, T_g\alpha)}$ defines an outer automorphism of $\mathbf{M}$.

The purpose of this paper is to discuss some algebraic properties of this representation.

2. From here to the end of this paper, G is a countably infinite group and $\mathbf{M}$ is an approximately finite factor in $\mathbf{H} = l_2(\mathfrak{G})$ associated with G as in § 1. By the fixed algebra of H, a subgroup of G, we mean the subalgebra of $\mathbf{M}$ composed of all elements of $\mathbf{M}$ which are simultaneously fixed under all members of H (see [1]). In this section we shall determine the fixed algebras of some

1) An automorphism of a ring means a *-automorphism.

normal subgroups of G. We denote by φ_0 the element in Δ' which takes value 1 only at $0 \in \Delta$. Then it is easily seen that $T_g'\varphi_0 = \varphi_0$ for all $g \in G$.

LEMMA 1. *Let H be an infinite normal subgroup of G and $\alpha_1, \ldots\ldots, \alpha_n$ distinct non-zero elements in Δ. Then there exist an $h \in H$ and an i_0 $(1 \leqq i_0 \leqq n)$ such that $\alpha_1, \ldots\ldots, \alpha_n, T_h\alpha_{i_0}$ are all distinct.*

PROOF. By hypothesis, $G = H \times K$ for a group K. The set $F = \bigcup_{i=1}^{n} \{(h,k) \in G : \alpha_i((h,k)) = 1\}$ is a finite set. Let $F_1 = \{h \in H : (h,k) \in F \text{ for a } k \in K\}$. We can choose a $g_0 \in H - F_1$. Picking up an $(h_0, k_0) \in F$ and $i_0(1 \leqq i_0 \leqq n)$ such as $\alpha_{i_0}((h_0, k_0)) = 1$, we get $T_{h_0g_0^{-1}}\alpha_{i_0} \neq \alpha_j$ for all $j = 1, \ldots\ldots, n$ because $\alpha_j((g_0, k_0)) = 0$ for all $j = 1, \ldots\ldots, n$ and $[T_{h_0g_0^{-1}}\alpha_{i_0}]((g_0, k_0)) = \alpha_{i_0}((h_0, k_0)) = 1$. Hence $\alpha_1, \ldots\ldots, \alpha_n, T_{h_0g_0^{-1}}\alpha_{i_0}$ are distinct each other.

LEMMA 2. *Under the same condition as in Lemma 1, let $\varphi_1, \ldots\ldots, \varphi_n$ be distinct elements in Δ' each of which is neither 0 nor φ_0. Then there exist an $h \in H$ and $i_0(1 \leqq i_0 \leqq n)$ such that $\varphi_1, \ldots\ldots, \varphi_n, T_h'\varphi_{i_0}$ are distinct each other.*

PROOF. As in the proof of Lemma 1, let $G = H \times K$. The set $\Delta_0 = \bigcup_{j=1}^{n} \{\alpha \in \Delta : \varphi_j(\alpha) = 1\}$ is finite, and so the set $F = \bigcup_{\alpha \in \Delta_0} \{(h,k) \in G : \alpha((h,k)) = 1\}$ is also finite and non-empty. Let $F_1 = \{h \in H : (h,k) \in F \text{ for a } k \in K\}$. We can choose a $g_0 \in H - F_1$. Pick up an $i_0(1 \leqq i_0 \leqq n)$, $\alpha_0 \in \Delta_0$ and $(h_0, k_0) \in F$ such as $\varphi_{i_0}(\alpha_0) = 1$, $\alpha_0((h_0, k_0)) = 1$. As $[T_{h_0g_0^{-1}}\alpha_0]((g_0, k_0)) = \alpha_0((h_0, k_0)) = 1$, $T_{h_0g_0^{-1}}\alpha_0$ does not belong to Δ_0. Thus $\varphi_j(T_{h_0g_0^{-1}}\alpha_0) = 0$ for every $j = 1, \ldots$, n, and so $T'_{h_0g_0^{-1}}\varphi_{i_0} \neq \varphi_j$ for all $j = 1, \ldots\ldots, n$ since $[T'_{h_0g_0^{-1}}\varphi_{i_0}](T_{h_0g_0^{-1}}\alpha_0) = \varphi_{i_0}(\alpha_0) = 1$. Therefore $\varphi_1, \ldots\ldots, \varphi_n$ and $T'_{h_0g_0^{-1}}\varphi_{i_0}$ are distinct each other.

THEOREM 1. *The fixed algebra in $\mathbf{M}$ of any infinite normal subgroup H of G is the set of all elements of the form $\lambda I + \mu V_{(\varphi_0, 0)}$ (λ, μ are scalars).*

PROOF. It is obvious that all $\lambda I + \mu V_{(\varphi_0, 0)}$ belong to the fixed algebra of H as $T_h'\varphi_0 = \varphi_0$ for all $h \in H$. Let A be an element in $\mathbf{M}$ which is fixed under all $g \in H$. According to Lemma 5.3.6 in [2], there exists a unique family of scalars $\{\lambda_{(\varphi,\alpha)}\}_{(\varphi,\alpha)\in\mathfrak{G}}$ such that

$$(1) \quad A = \sum_{(\varphi,\alpha)\in\mathfrak{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}$$

where $\sum$ is taken in the sense of metric convergence in $\mathbf{M}$. Thus we have

$$(2)\quad \sum_{(\varphi,\alpha)\in\mathfrak{G}}\lambda_{(\varphi,\alpha)}V_{(\varphi,\alpha)}=\sum_{(\varphi,\alpha)\in\mathfrak{G}}\lambda_{(\varphi,\alpha)}V_{(T_g'\varphi,T_g\alpha)}\qquad \text{for all } g\in H.$$

By the uniqueness of the expression (1) we obtain

$$(3)\quad \lambda_{(\varphi,\alpha)}=\lambda_{(T_g'\varphi,T_g\alpha)}\qquad \text{for all } (\varphi,\alpha)\in\mathfrak{G} \text{ and } g\in H.$$

Now suppose that there exists a $(\psi,\beta)\in\mathfrak{G}$ which is neither $(0,0)$ nor $(\varphi_0,0)$ such as $\lambda_{(\psi,\beta)}\neq 0$. If $\beta\neq 0$, by Lemma 1 we can find an infinite sequence $\{g_n\}$ in H such that all $T_{g_n}\beta$ are distinct elements in Δ. If $\beta=0$, there exists an infinite sequence $\{h_n\}$ in H such that $T'_{h_n}\psi$ are all distinct elements in Δ' by Lemma 2. Hence there is an infinite sequence $\{k_n\}$ in H and $(T'_{k_n}\psi, T_{k_n}\beta)$'s are all distinct elements in $\mathfrak{G}$. Thus, by (3) we get $\sum_{n=1}^{\infty}|\lambda(T'_{k_n}\psi, T_{k_n}\beta)|^2=\infty$ which contradicts $\sum_{(\varphi,\alpha)\in\mathfrak{G}}|\lambda_{(\varphi,\alpha)}|^2<\infty$. Therefore $\lambda_{(\psi,\beta)}=0$ except for $(0,0)$, $(\varphi_0,0)$ and $A=\lambda_{(0,0)}I+\lambda_{(\varphi_0,0)}V_{(\varphi_0,0)}$

Next we shall determine the fixed algebra in $\mathbf{M}$ of a finite normal subgroup of G.

LEMMA 3. *Let H be a finite normal subgroup of G. Then there exist infinitely many distinct $\alpha\in\Delta$ (resp. $\varphi\in\Delta'$) such that $T_g\alpha=\alpha$ (resp. $T_g'\varphi=\varphi$) for all $g\in H$.*

PROOF. By the assumption, $G=H\times K$ for an infinite group K. Picking up an infinite sequence of distinct elements $\{k_n\}$ in K, we define $\alpha_n\in\Delta$ for each n by $\alpha_n((h,k_n))=1$ for all $h\in H$, and 0 otherwise. $\{\alpha_n\}$ is an infinite sequence of distinct elements in Δ, and it is easily seen that $T_g\alpha_n=\alpha_n$ for all $g\in H$ (n is arbitrary). Using these $\{\alpha_n\}$ we define $\varphi_n\in\Delta'$ for each n by $\varphi_n(\alpha)=1$ if $\alpha=\alpha_n$, and 0 otherwise. Then the sequence $\{\varphi_n\}$ satisfies the requirement as $T_g\alpha_n=\alpha_n$ for all $g\in H$ (n is arbitrary).

LEMMA 4. *Let H be as in Lemma 3. Then for any finite set $\mathfrak{F}$ in $\mathfrak{G}$, there exists a finite subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ containing $\mathfrak{F}$ such that $(T_g'\varphi, T_g\alpha)\in\mathfrak{G}_0$ for all $g\in H$ and $(\varphi,\alpha)\in\mathfrak{G}_0$.*

PROOF. Since $\mathfrak{G}$ is locally finite as shown in [4], $\mathfrak{F}$ generates a finite subgroup $\mathfrak{F}_1$ of $\mathfrak{G}$. Then the set $\mathfrak{F}_2=\{(T_g'\varphi,T_g\alpha)\colon(\varphi,\alpha)\in\mathfrak{F}_1,\ g\in H\}$ is a finite subset because of the finiteness of H, and so generates a finite subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ again by the local finiteness of $\mathfrak{G}$. It is obvious that $\mathfrak{G}_0$ satisfies our requirement.

Successive application of Lemma 4 leads to the following lemma, which will be used in the next section and we omit the proof.

LEMMA 5. *Let G_0 be a locally finite subgroup of a group G and $\{G_n\}$ a non-decreasing sequence of finite subgroups of G_0 such as $\bigcup_{n=1}^{\infty} G_n = G_0$. Then there exists an increasing sequence $\{\mathfrak{G}_n\}$ of finite subgroups of $\mathfrak{G}$ with the following properties.*

(i) *For each n, $(T_g'\varphi, T_g\alpha) \in \mathfrak{G}_n$* if *$(\varphi, \alpha) \in \mathfrak{G}_n$ and $g \in G_n$.*

(ii) $\bigcup_{n=1}^{\infty} \mathfrak{G}_n = \mathfrak{G}$.

Then we have the following result.

THEOREM 2. *The fixed algebra in $\mathbf{M}$ of any finite normal subgroup H of G is the approximately finite factor.*

PROOF. Applying Lemma 5 to H and $\mathfrak{G}$, we can find an increasing sequence $\{\mathfrak{G}_n\}$ of finite subgroups of $\mathfrak{G}$ such that $\mathfrak{G}$ such that $\mathfrak{G} = \bigcup_{n=1}^{\infty} \mathfrak{G}_n$ and for each $g \in H$, $(T_g'\varphi, T_g\alpha) \in \mathfrak{G}_n$ if $(\varphi, \alpha) \in \mathfrak{G}_n$. Then for every n, the subalgebra $\mathbf{N}_n$ generated by all $A = \sum_{(\varphi,\alpha)\in\mathfrak{G}_n} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}$ where for each $(\varphi, \alpha) \in \mathfrak{G}_n$, $\lambda_{(\varphi,\alpha)} = \lambda_{(T_g'\varphi, T_g\alpha)}$ for all $g \in H$, is of finite order in the sense of Definition 4.5.1 in [2], and each $A \in \mathbf{N}_n$ is fixed under all $g \in H$. An element $A = \sum_{(\varphi,\alpha)\in\mathfrak{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}$ in $\mathbf{M}$ is in the fixed algebra $\mathbf{N}$ of H if and only if $\lambda_{(\varphi,\alpha)} = \lambda_{(T_g'\varphi, T_g\alpha)}$ for all $g \in H$ ((φ, α) is arbitrary). Hence, as $\mathfrak{G} = \bigcup_{n=1}^{\infty} \mathfrak{G}_n$, the fixed algebra $\mathbf{N}$ of H is generated by the increasing sequence $\{\mathbf{N}_n\}$. Since $\mathfrak{G}$ is infinite, $\mathbf{N}$ is not of finite order and hence it is sufficient to show that $\mathbf{N}$ is a factor, by Definition 4.6.1 in [2]. Let $A = \sum_{(\varphi,\alpha)\in\mathfrak{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}$ be an element in $\mathbf{N}$ which commutes with all elements in $\mathbf{N}$. If $(T_g'\psi, T_g\beta) = (\psi, \beta)$ for all $g \in H$, $AV_{(\psi,\beta)} = V_{(\psi,\beta)}A$. Hence for such (ψ, β) we have

$$\sum_{(\varphi,\alpha)\in\mathfrak{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)(\psi,\beta)} = \sum_{(\varphi,\alpha)\in\mathfrak{G}} \lambda_{(\varphi,\alpha)} V_{(\psi,\beta)(\varphi,\alpha)}$$

snd so

$$\lambda_{(\varphi,\alpha)} = \lambda_{(\psi,\beta)^{-1}(\varphi,\alpha)(\psi,\beta)} \qquad \text{for all } (\varphi, \alpha) \in \mathfrak{G}$$

We have to show that $\lambda_{(\varphi,\alpha)} = 0$ for all $(\varphi, \alpha) \neq (0, 0)$. To do this we assume that there exists a $(\varphi, \alpha) \neq (0, 0)$ such as $\lambda_{(\varphi,\alpha)} \neq 0$.

Case 1. $\varphi \neq 0$. By Lemma 3 we can find an infinite sequence of distinct elements $\{\alpha_n\}$ in Δ such that $T_g\alpha_n = \alpha_n$ for all $g \in H$. Then, as easily seen,

$\{\varphi^{\alpha_n}\}$ contains an infinitely many distinct elements of Δ'(cf. footnote 2) in [4]). On the other hand, $(0,\beta)^{-1}(\varphi,\alpha)(0,\beta)=(\varphi^{\beta},\alpha)$. Hence $\sum_{(\psi,\beta)\in\mathfrak{G}}|\lambda_{(\psi,\beta)}|^2=\infty$ which is a contradiction. Thus $\lambda_{(\varphi,\alpha)}=0$ in this case.

Case 2. $\varphi=0$: Let $\{\varphi_n\}$ be a sequence of infinitely many distinct elements of Δ' constructed in the proof of Lemma 3. Then $\{\varphi_n^{\alpha}+\varphi_n\}$ contains an infinitely many distinct elements (see footnote 3) in [4]). As $(\psi,0)^{-1}(0,\alpha)(\psi,0)=(\psi^{\alpha}+\psi,0)$ and $T_g'\varphi_n=\varphi_n$ for all $g\in H$, we have $\sum_{(\psi,\beta)\in\mathfrak{G}}|\lambda_{(\psi,\beta)}|^2=\infty$, which is a contradiction, and $\lambda_{(\varphi,\alpha)}=0$.

Therefore $A=\lambda_{(0,0)}V_{(0,0)}$ and $\mathbf{N}$ is the approximately finite factor.

Following [1], we say that an automorphism of a ring of operators is *freely acting* if any non-zero projection contains a non-zero projection which is not fixed under the automorphism. This concept can be seen as a relaxation of the notion of ergodicity, where an automorphism of a ring of operators is *ergodic* if it does not leave any non-trivial projection fixed. It is obvious that an ergodic automorphism is freely acting. As an immediate consequence of Theorem 2 in [3] we have the following fact.

LEMMA 6. *Every outer automorphism of a finite factor is freely acting.*[2)]

Hence from Theorem 1 (or Theorem 2) we get

COROLLARY. *There is an automorphism of the approximately finite factor which is freely acting and not ergodic.*

In fact, in Theorem 1, any $g\in G$, is freely acting by Lemma 6, and the projection $\frac{1}{2}1+\frac{1}{2}V_{(\varphi_0,0)}$ is left invariant under g. Hence g is not ergodic.

3. In this section we shall specialize the group G and pursue the crossed product of $\mathbf{M}$ by G.

THEOREM 3. *If G is a locally finite group, the crossed product of the approximately finite factor* $\mathbf{M}$ *by G is also approximately finite.*

PROOF. By definition, the crossed product $(\mathbf{M},G)$ is generated by operators $\widetilde{V}_{(\varphi,\alpha)}\ ((\varphi,\alpha)\in\mathfrak{G})$ and $\widetilde{U}_g(g\in G)$ on the Hilbert space $\mathbf{H}\otimes l_2(G)$ defined by

$$\widetilde{V}_{(\varphi,\alpha)}\Big(\sum_{h\in G}x_h\otimes\varepsilon_h\Big)=\sum_{h\in G}V_{(\varphi,\alpha)}x_h\otimes\varepsilon_h,$$

2) This fact has been informed the author by Mr. N. Suzuki.

$$\widetilde{U}_g\Big(\sum_{h\in G} x_h \otimes \varepsilon_h\Big) = \sum_{h\in G} U_g x_h \otimes \varepsilon_{gh}.$$

By Lemma 5, there exist the increasing sequences $\{\mathfrak{G}_n\}$ and $\{G_n\}$ of finite subgroups of $\mathfrak{G}$ and G respectively which satisfy (i) and (ii). Denote by $\mathbf{P}_n$ the ring of operators generated by $\widetilde{V}_{(\varphi,\alpha)}$, $\widetilde{U}_g$ with $(\varphi, \alpha) \in \mathfrak{G}_n$, $g \in G_n$ for each n. It is obvious that $\mathbf{P}_1 \subseteqq \mathbf{P}_2 \subseteqq \dots \subseteqq \mathbf{P}_n \subseteqq \cdots$, and every $\mathbf{P}_n$ is of finite order by the property (i). Since $\bigcup_{n=1}^{\infty} \mathfrak{G}_n = \mathfrak{G}$ and $\bigcup_{n=1}^{\infty} G_n = G$, $(\mathbf{M}, G)$ is generated by $\{\mathbf{P}_n\}$. Hence the crossed product $(\mathbf{M}, G)$ is the approximately finite factor, since the crossed product $(\mathbf{M}, G)$ is a factor of type II_1 by Theorem 4 in [5].

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