NOTE ON SUSPENSION AND HOPF INVARIANT

TŌRU WADA

(Received March 1, 1960)

1. Introduction. Let A be a k-connected special complex ($k \geqq 0$), let ${}^nA$ be the n-fold suspension of A, and let ${}^nA * {}^nA$ be a join of the two copies of ${}^nA$. James [5] defined the homomorphism

$$h: \pi_{r+1}({}^{n+1}A : C_+, C_-) \to \pi_{r+1}({}^nA * {}^nA),$$

using a canonical isomorphism and a combinatorial extension of the reduced product space. And, he defined Hopf invariant $H = h \circ i$ on the special complex, where i is the inclusion homomorphism in the triad sequence.

On the sphere, the generalized suspension homomorphism agrees with the classical one, but we do not know any relation between the generalized Hopf invariant H and the classical one H_0, for instance, Toda's generalized Hopf invariant (cf. James [6]).

Concerning it, we obtain the following

THEOREM I. *On the sphere S^n, H agrees with H_0 except their signs, if $r \leqq 3n - 4$.*

James proved that

$$h: \pi_{r+1}({}^{n+1}A : C_+, C_-) \to \pi_{r+1}({}^nA * {}^nA)$$

is the isomorphism onto, if $r \leqq 3(k + n) + 1$.

Then, we have the exact sequence

$$\pi_{3(k+n)+1}({}^nA) \xrightarrow{E} \pi_{3(k+n)+2}({}^{n+1}A) \xrightarrow{H} \cdots\cdots$$
$$\cdots\cdots \xrightarrow{H} \pi_{r+2}({}^nA * {}^nA) \xrightarrow{\Delta'} \pi_r({}^nA) \xrightarrow{E} \pi_{r+1}({}^{n+1}A) \xrightarrow{H}$$
$$\pi_{r+1}({}^nA * {}^nA) \to \cdots\cdots.$$

And ${}^nA * {}^nA$ is $(2(k + n) + 2)$-connected.

Therefore, as is well known, the generalized suspension homomorphism

$$E: \pi_r({}^nA) \to \pi_{r+1}({}^{n+1}A)$$

is onto isomorphism, if $r \leqq 2(k + n)$, and onto homomorphism, if $r = 2(k + n) + 1$, and if $r \leqq 3(k + n) + 1$, then

$$\text{image } E = \text{kernel } H.$$

Moreover, we shall generalize the suspension theorem on a certain class of special complexes, i. e.

THEOREM II. *If $\pi_{k+1}(A)$ is the cyclic group of order u, then the kernel of the suspension homomorphism*

$$E: \pi_{2(k+n)+1}({}^nA) \longrightarrow \pi_{2(k+n)+2}({}^{n+1}A)$$

is the cyclic subgroup G of $\pi_{2(k+n)+1}({}^nA)$ generated by

$$\lambda = [E^n a_{k+1}, \quad E^n a_{k+1}],$$

where a_{k+1} is the generator of $\pi_{k+1}(A)$. And the order of G is given by the following table.

Table of the group G $(m=2(k+n)+3)$.

$k+n+1$ \ u	even	odd	
		there is no element of $\pi_m({}^nA)$ with Hopf invariant unity	there is an element of $\pi_m({}^nA)$ with Hopf invariant unity
∞	∞ $(H(\lambda)=\pm 2)$	2	0
even	$u/2$	2	0
odd	u	0	

As for the suspension homomorphism and Hopf invariant, the following theorem doesn't afford a relationship directly, but it is interesting itself.

THEOREM III. *Let X, Y be arcwise connected topological spaces, and let $f: X\rightarrow Y$ be a continuous onto mapping. If $f^{-1}(y)$ is contractible for each point y, then the induced homomorphisms of f*

$$f_{*}: \pi_r(X)\rightarrow \pi_r(Y)$$

$$f_{\#}: H_r(X)\rightarrow H_r(Y)$$

are onto isomorphisms for all integers $r>0$.

In appendix we apply this theorem to prove the Jame's result cited above.

2. Proof of Theorem I.

H. Toda defined Hopf invariant H_0 such that

$$\begin{aligned} H_0 = \chi_{j+1}\circ Q_0\circ w_r\circ \varphi_r\circ i:\ & \pi_r(S^n)\xrightarrow{i}\pi_r(S^n;\ E_+, E_-) \\ & \xrightarrow{\varphi_r}\pi_r(S^n_1\vee S^n_2;\ D_+, D_-)\xrightarrow{w_r}\pi_r(S^n_1\vee S^n_2;\ S^n_1, S^n_2) \\ & \xrightarrow{Q_0}\pi_{r+1}(S^n_1\times S^n_2,\ S^n_1\vee S^n_2)\xrightarrow{\chi_{r+1}}\pi_{j+1}(S^{2n}), \end{aligned}$$

where i is the injection, Q_0 is the inverse of the onto isomorphism in

Cor. 8.3 [2], χ_{r+1} is the homomorphism induced by the shrinking map, and φ_r is the homomorphism defined by G. W. Whitehead [8] p. 287, and w_r is the induced homomorphism of the map $w: (S_1^n \vee S_2^n;\ D_+, D_-) \rightarrow (S_1^n \vee S_2^n;\ S_1^n, S_2^n)$ defined by

$$w(x, t) = \begin{cases} (x,\ 2t), & \textit{if } x \in S_1^n,\ 0 \leqq t \leqq \dfrac{1}{2}, \\ (x,\ 2t-1), & \textit{if } x \in S_2^n,\ \dfrac{1}{2} \leqq t \leqq 1, \\ x^0, & \text{otherwise.} \end{cases}$$

Now, consider the sequence of homomorphism

$$\pi_r(S^n) \xrightarrow{\chi_*^{-1}} \pi_r(S^n, E) \xrightarrow{j_*} \pi_r(S^n;\ E_+, E_-) \xleftarrow{P} \pi_{r-n+1}(E_-, S^{n-1}) \xrightarrow{\partial'} \pi_{r-n}(S^{n+1}),$$

where, χ_* is the homomorphism induced by the shrinking map, j_* is the inclusion, ∂' is the boundary homomorphism, and

$$P: \pi_{r-n+1}(E_-, S^{n-1}) \otimes \pi_n(E_+, S^{n-1}) \rightarrow \pi_r(S^n;\ E_+, E_-)$$

is the homomorphism defined by

$$P(\alpha \otimes \beta) = [\alpha, \beta]_t, \qquad \alpha \in \pi_{r-n-1}(E_-, S^{n-1}),\ \beta \in \pi_n(E_+, S^{n-1}).$$

Since $r \leqq 3n-4$, P is the isomorphism (by Theorem 1.2 [4]), thus

$$h_0 = \partial' \circ P^{-1} \circ j_* \circ \chi_*^{-1}: \pi_r(S^n) \rightarrow \pi_{r-n}(S^{n-1})$$

is a homomorphism, and by Lemma 2.5 [4],

$$(2.1) \qquad E^n h_0 = \pm H_0.$$

It is easy to see that the diagram

$$\begin{array}{ccc} \pi_r(S^n) & \xrightarrow{\quad i \quad} & \pi_r(S^n;\ E_+, E_-) \\ \nwarrow \scriptstyle{\chi_*} & & \nearrow \scriptstyle{j_*} \\ & \pi_j(S^n, E) & \end{array}$$

is commutative and χ_* is the isomorphism for each r, and

$$(2.2) \qquad i = j_* \circ \chi_*^{-1}.$$

Let

$$\mathbf{P}: \pi_{r-n-1}(S^{n-1}) \to \pi_r(S^n;\ E_+, E_-)$$

be the homomorphism defined by

$$\mathbf{P}(\alpha) = \{\alpha, \iota_{n-1}\}, \quad \alpha \in \pi_{r-n-1}(S^{n-1}).$$

James proved that

$$h \circ \mathbf{P} = E^n. \tag{2.3}$$

And, by the definition of the triad Whitehead product,

$$\varepsilon P = \partial' \circ \mathbf{P}. \tag{2.4}$$

Since $r \leqq 3n-4$, we have

$$\begin{aligned} h_0 &= \partial' \circ P^{-1} \circ j_* \circ \chi_*^{-1} \\ &= \partial' \circ P^{-1} \circ i && \text{by (2.2)} \\ &= \varepsilon \mathbf{P} \circ i, && \text{by (2.4)} \end{aligned}$$

thus

$$i = \varepsilon \mathbf{P} \circ h_0. \tag{2.5}$$

From (2.5), (2.3) and (2.1) it follows that

$$\begin{aligned} H &= h \circ i \\ &= \varepsilon\ h \circ \mathbf{P} \circ h_0 \\ &= \varepsilon\ E^n \circ h_0 \\ &= \pm\, \varepsilon\ H_0. \end{aligned}$$

3. Proof of Theorem II. Since ${}^nA$ is $(k+n)$-connected, we have that

$$\pi_{k+n+2}({}^nC_+, {}^nA) \approx \pi_{k+n+2}({}^nC_-, {}^nA) \approx \pi_{k+n+1}({}^nA)$$

is the cyclic group of order u. Hence

$$\pi_{k+n+2}({}^nC_+, {}^nA) \otimes \pi_{k+n+2}({}^nC_-, {}^nA)$$

is the cyclic group of order u.

We can deform the triad $({}^{n+1}A;\ {}^{n+1}C_+, {}^{n+1}C_-)$ to the triad which is the same homotopy type as it and satisfies the condition of Blakers-Massey [1].

Thus, by the main theorem of [3], we see that

$$\pi_{2(k+n)+3}({}^{n+1}A;\ {}^{n+1}C_+, {}^{n+1}C_-)$$

is the cyclic group of order u, and its generator is

$$\{E^n a_{k+1}, E^n a_{k+1}\}.$$

In Theorem 2.4, [5], put $\alpha = \beta = E^n a_{k+1}$, then

$$\{E^n a_{k+1}, E^n a_{k+1}\} = (-1)^{(k+n+1)}\{E^n a_{k+1}, E^n a_{k+1}\}$$
$$= (-1)^{(k+n+1)}\, i\,[E^{n+1} a_{k+1}, E^{n+1} a_{k+1}].$$

Therefore,

$$i\,[E^{n+1} a_{k+1}, E^{n+1} a_{k+1}] = (-1)^{(k+n)}(1-(-1)^{(k+n+1)})\,\{E^n a_{k+1}, E^n a_{k+1}\},$$

precisely,

(3.1) $$i\,[E^{n+1} a_{k+1}, E^{n+1} a_{k+1}] = \begin{cases} 0, & k+n+1: \text{even} \\ 2\{E^n a_{k+1}\, E^n a_{k+1}\}, & k+n+1: \text{odd}. \end{cases}$$

Now

$$\text{kernel } E = \text{image } \Delta,$$

and the generator of $\pi_{2(k+n)+3}({}^{n+1}A;\ {}^{n+1}C_+, {}^{n+1}C_-)$ is $\{E^{n+1} a_{k+1}, E^{n+1} a_{k+1}\}$ and

$$\Delta\{E^{n+1} a_{k+1}, E^{n+1} a_{k+1}\} = [E^n a_{k+1}, E^n a_{k+1}].$$

Therefore it follows that kernel of E is the cyclic group generated by $[E^n a_{k+1}, E^n a_{k+1}]$.

Moreover,

(3.2) if $n+k+1$ is even, then

$$i: \pi_{2(k+n)+1}({}^nA) \to \pi_{2(k+n)+1}({}^nA;\ {}^nC_+, {}^nC_-)$$

restricted on $\{E^n a_{k+1}, E^n a_{k+1}\}$, is onto isomorphism.

For $i\,[E^n a_{k+1}, E^n a_{k+1}] = 2\{E^{n-1} a_{k+1}, E^{n-1} a_{k+1}\} \neq 0.$

Hence form (3.1) and (3.2), if $n+k+1$ is even and u is even, then $\{[E^n a_{k+1}, E^n a_{k+1}]\}$ is the cyclic group of order $\frac{u}{2}$. If $u=\infty$ then

$$H([E^n a_{k+1}, E^n a_{k+1}]) = \pm 2$$

and $\{[E^n a_{k+1}, E^n a_{k+1}])$ is the infinite cyclic group. When u is odd,

$$2\{E^{n-1} a_{k+1}, E^{n-1} a_{k+1}\}$$

is, too, the generator of $\pi_{2(k+n)+1}({}^nA;\ {}^nC_+, {}^nC_-)$, therefore, $\{[E^n a_{k+1}, E^n a_{k+1}]\}$ is the cyclic group of order u.

Next, if $k+n+1$ is odd, then

$$i\,[E^n a_{k+1}, E^n a_{k+1}] = 0,$$

therefore

$$\{[E^n a_{k+1}, E^n a_{k+1}]\} \subset E\pi_{2(k+n)}({}^{n-1}A),$$

and if $k+n+1$ is odd and u is odd, then

$$\begin{aligned}[E^n a_{k+1}, E^n a_{k+1}] &= \Delta \{E^{n+1} a_{k+1}, E^{n+1} a_{k+1}\} \\ &= \Delta \circ i \circ l[E^{n+2} a_{k+1}, E^{n+2} a_{k+1}] \\ &= 0,\end{aligned}$$

where l is an integer.
Thus

$$\{[E^n a_{k+1}, E^n a_{k+1}]\} = 0.$$

When u is even or ∞, if there is an element $\alpha \in \pi_{2(k+n)+3}({}^{n+1}A)$ such that

$$H(\alpha) = e, \qquad (e \text{ is the unity}),$$

then

$$\begin{aligned}[E^n a_{k+1}, E^n a_{k+1}] &= \Delta\{E^{n+1} a_{k+1}, E^{n+1} a_{k+1}\} \\ &= \Delta \circ i(\alpha) = 0.\end{aligned}$$

Thus

$$\{[E^{n+1} a_{k+1}, E^{n+1} a_{k+1}]\} = \{0\}.$$

If there is no element $\alpha \in \pi_{2(k+n)+2}({}^{n+1}A)$ such that

$$H(\alpha) = e,$$

then

$$\begin{aligned}2[E^n a_{k+1}, E^n a_{k+1}] &= 2\Delta \{E^{n+1} a_{k+1}, E^{n+1} a_{k+1}\} \\ &= \Delta \circ i[E^{n+2} a_{k+1}, E^{n+2} a_{k+1}] \\ &= 0,\end{aligned}$$

but, $\qquad [E^n a_{k+1}, E^n a_{k+1}] \neq 0,$

therefore, $\{[E^n a_{k+1}, E^n a_{k+1}]\}$ is the cyclic group of order 2.

4. Proof of Theorem III. Let $\widetilde{X} = \{(x, w) | w : (I; 0, 1) \to (Y; Y, Y), w(1) = f(x)\}$ and the map $p: \widetilde{X} \to Y$ defined by $p(x, w) = w(1)$, then, as is well known, triple (X, p, Y) is a fiber space. Since the fiber $p^{-1}(y^0)$ over the base point $y^0 \in Y$ is

$$F = \{(x, w) | x \in f^{-1}(y^0),\ w \in E_{Y, y^0}\},$$

F is a contractible, hence, by the exact sequence of the fiber space, we know that induced homomorphisms by p

$$(4.1) \qquad \begin{aligned} p_* &: \pi_r(X) \to \pi_r(Y) \\ p_\# &: H_r(X) \to H_r(Y)\end{aligned}$$

are onto isomorphisms, for all integers $r \geqq 0$.
Now, we define an onto map

$$p' : \widetilde{X} \to X$$

by

$$p'(x, w) = x,$$

then

$$f \circ p'(x, w) = f(x) = w(1) = p(x, w),$$

thus

$$f \circ p' = P. \tag{4.2}$$

Moreover, X is a deformation retract of $\widetilde{X}$, p' is the retraction. Therefore induced homomorphisms p'_{*}, $p'_{\#}$ of p' are onto isomorphisms, for all integers $r > 0$, thus by (4.1), and (4.2), induced homomorphisms of f

$$f_{*} : \pi_r(X) \to \pi_r(Y)$$

$$f_{\#} : H_r(X) \to H_r(Y)$$

are onto isomorphisms, for all integers $r > 0$.

5. Appendix. James notes that if A is m-connected, then

$$h : \pi_{r+1}(\hat{A}\ ;\ C_+,\ C_-) \to \pi_{r+1}(A*A)$$

are the onto isomorphisms for all integers $r \leqq 3m + 1$. We can prove Lemma 1, without using this Note.

LEMMA 1. *Let A be the k-connected special complex $(k \geqq 1)$ and let Ω be the space of loops in the suspension $\hat{A}$ of A. Let*

$$u\ ;\ A \times A \to \Omega$$

be the map such that

$$u(a, a^0) = u(a^0, a) = a \qquad a \in A$$

and let

$$\theta : \pi_r(\Omega, A) \to \pi_{r+1}(\hat{A}\ ;\ C_+,\ C_-)$$

be the natural isomorphism. Then homomorphisms

$$\psi = \theta \circ u_{*} : \pi_r(A \times A,\ A \vee A) \to \pi_{r+1}(\hat{A}\ ;\ C_+,\ C_-)$$

are isomorphisms onto, for all integers $r \leqq 3k + 1$.

Outline of Proof.

Consider the commutative diagram

$$\begin{array}{ccccccc} & \overset{\alpha_*}{\nearrow} & \pi_r(\Omega, A) & \xrightarrow{\theta} & \pi_{r+1}(\hat{A}\,;\, C_+, C_-) & \xrightarrow{h} & \pi_{r+1}(A*A) \\ \pi_r(A_\infty, A) & & & & \uparrow \psi & & \uparrow E \\ & \underset{i}{\nwarrow} & \pi_r(A_2, A) & \xleftarrow{f_*} & \pi_r(A \times A, A \vee A) & \xrightarrow{\chi_*} & \pi_r(A \times\!\!\!\!\times A) \end{array}$$

where, α_* is the canonical isomorphim, i_* is the injection, f_* is the homomorphism induced by the identification map

$$f: (A \times A, A \vee A) \rightarrow (A_2, A).$$

We will find that i^* is the onto isomorphism if $r \leqq 3k + 2$, and by Theorem III, f^* is the onto isomorphism if $r \leqq 3k$, and onto homomorphism if $r = 3k + 1$.

Therefore, ψ is the onto isomorphism if $r \leqq 3k$, and the onto homomorphism if $r = 3k + 1$.

E and χ_* is the onto isomorphism when at least $r \leqq 3k + 1$, then ψ is the onto isomorphism when $r = 3k + 1$.

BIBLIOGRAPHY

[1] A. L. BLAKERS AND W. S. MASSEY, The homotopy groups of a triad II, Ann. of Math., 55(1952), 142-201.

[2] ________________, Products in homotopy theory. Ann. of Math., 58 (1953), 295-324.

[3] ________________, The homotopy groups of a triad III, Ann. of Math., 58(1953), 409-417.

[4] I. M. JAMES, On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford (2), 5(1954), 260-270.

[5] ________, On the suspension triad, Ann. of Math., 63(1956), 191-247.

[6] ________, On the suspension sequence, Ann. of Math. 65(1957), 74-107.

[7] H. TODA, Generalized Whitehead products and homotopy groups of spheres, J. Inst. Polyt. Osaka, 3(1952), 43-82.

[8] G. W. WHITEHEAD, A generalization of the Hopf invariant, Ann. of Math., 15(1950), 192-237.

TŌKAI UNIVERSITY.