SUPPLEMENTS TO MY FORMER PAPER : "ON AHLFORS' DISCS THEOREM AND ITS APPLICATION"

(This Journal, vol. 8(1956), 101-107.)

KAZUO IKOMA

(Received February 20, 1961)

In my paper mentioned in the title, Ahlfors' disce theorem, and as its application, Bloch's theorem were extended to the certain family of K-pseudoanalytic functions in the sense of Grötzsch.

Now, the above theorem hold good also for the wider certain family of K-pseudoanalytic functions in the sense of Pfluger-Ahlfors[1], because the same lemmas as Lemmas 1,2 and 3 used for Ahlfors' discs theorem in the former paper[2] can be deduced almost similarly also for this family.

Mentioning especially the theorem of Bloch type, it is as follows.

THEOREM 1. (*An extension of Bloch's theorem*). *Let* $w = f(z)$ *be a non-constant K-pseudoanalytic function of* $|z| < 1$ *in the sense of Pfluger-Ahlfors and suppose that* $f(0) = 0$ *and* $\lim_{z\to 0} |f(z)|/|z|^{1/K} = 1$, *then the Riemann surface generated by* $w = f(z)$ *on the w-sphere contains a schlicht spherical disc whose radius* $\geqq \beta > 0$, β *being a constant independent of* $f(z)$.

Next, impose on $f(z)$ the more general condition that $f(0) = 0$ and the finite positive $\lim_{z\to 0} |f(z)|/|z|^{\alpha}$ (α is real) exists, then there holds $1/K \leqq \alpha \leqq K$ as was proved in Ikoma-Shibata [1].[3] In other words, the family of K-pseudoanalytic functions satisfying such condition is empty for both $\alpha < 1/K$ and $\alpha > K$.

Now, there arises, in the case $\alpha \neq 1/K$, a question whether or not. the analogous conclusion to one in the above Theorem 1 will hold good under the normal condition that $f(0) = 0$ and $\lim_{z\to 0} |f(z)|/|z|^{\alpha} = 1$.

This question is answered negatively as follows even for the family $\mathfrak{S}^{\alpha}$

1) Such a K-pseudoanalytic function means a constant or an interior transformation $w = f(z)$ from a domain of the z-plane into the Riemann covering surface spread over a domain of the w-plane which is a quasiconformal mapping with the maximal dilatation $\leqq K$ in the sense of Pfluger-Ahlfors.

2) This means the paper mentioned in the title.

3) Ikoma-Shibata [1]: On distortions in certain quasiconformal mappings, to appear in Tôhoku Math. Journ., 13 (1961).

($\alpha \neq 1/K$) of K-quasiconformal mappings in the sense of Grötzsch satisfying the above normal condition.

In the case $1/K < \alpha \leqq 1$, take the following mapping of $|z| < 1$ onto $|w| < r_n$ given in [1][4]:

$$w = |z|^{\alpha} \{1 - (1 - r_n)\ |z|^{(\alpha K - 1) r_n / K(1 - r_n)}\} e^{i \arg z}, \tag{1}$$

where $0 < r_n < 1$ and $r_n \to 0$ as $n \to \infty$, then we found that it belongs to the family $\mathfrak{S}_{\alpha}$.

In the case $1 < \alpha \leqq K$, consider the mapping $w = f_n(z)$ of $|z| < 1$ onto $|w| < r_0^{\alpha-1}$ composed of

$$\frac{r_0 s}{(1-s)^2} = \frac{z}{(1-z)^2}, \tag{2}$$

$$t = |s|^{\alpha} e^{i \arg s}, \tag{3}$$

$$\frac{r_0 t}{(1-t)^2} = \frac{r_0^{\alpha-1} w}{(r_0^{\alpha-1} - w)^2}, \tag{4}$$

where $r_0 = 4rn/(n+r)^2$ ($n = 1, 2, \ldots\ldots$) and r is fixed arbitrarily in the open interval $(0, 1)$, then obviously it belongs to $\mathfrak{S}_{\alpha}$.

By making $n \to \infty$ in each case, it is immediately seen that there exists no so-called Bloch's constant for $\mathfrak{S}_{\alpha}$ ($\alpha \neq 1/K$).

From the above results, we can state the following

THEOREM 2. (*A precision of Theorem* 1). *For the family of non-constant K-pseudoanalytic functions $w = f(z)$ of $|z| < 1$ in the sense of Pfluger-Ahlfors satisfying that $f(0) = 0$ and $\lim_{z \to 0} |f(z)|/|z|^{\alpha} = 1$, where α is real, there exists the so-called Bloch's constant if and only if $\alpha = 1/K$. Further, if $\alpha \neq 1/K$, then there exists no Bloch's constant even for the family $\mathfrak{S}_{\alpha}$.*

4) Ikoma-Shibata, loc. cit. 3).

DEPARTMFNT OF MATHEMATICS, YAMAGATA UNIVERSITY.