DISCRETE ANALOGUE OF A THEOREM OF LITTLEWOOD-PALEY[1)]

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1. One of the most important theorems given by Littlewood-Paley [2] reads as follows.

For $f(\theta) \in L^p\ (0, 1)\ (1 < p < \infty)$, let

$$a_\nu = \int_0^1 f(\theta)\, e^{-2\pi i\nu\theta}\, d\theta,$$

$$f(\theta) \sim \sum_{\nu=-\infty}^{\infty} a_\nu e^{2\pi i\nu\theta}.$$

If

$$\Delta_n(\theta) = \begin{cases} \displaystyle\sum_{\nu=2^{n-1}}^{2^n-1} a_\nu e^{2\pi i\nu\theta} & n = 1, 2, \ldots\ldots, \\ a_0 & n = 0, \\ \displaystyle\sum_{\nu=-2^{-n}+1}^{-2^{-n-1}} a_\nu e^{2\pi i\nu\theta} & n = -1 -2, \ldots\ldots, \end{cases}$$

then[2)]

$$0 < A(p) \leqq \int_0^1 \left\{ \sum_{n=-\infty}^{\infty} |\Delta_n(\theta)|^2 \right\}^{p/2} d\theta \Big/ \int_0^1 |f(\theta)|^p d\theta \leqq A'(p) < \infty.$$

The corresponding result for Walsh-Fourier series was proved by Paley [4]. The discrete analogue of this theorem is as follows.

THEOREM 1. *For $c(k) \in l^2 \cap l^p$ $(1 < p < \infty)$, let*

$$f(\theta) \sim \sum_{k=-\infty}^{\infty} c(k) e^{2\pi ik\theta},$$

$$c(k) = \int_0^1 f(\theta) e^{-2\pi ik\theta}\, d\theta.$$

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2) $A(p)$, $A'(p)$, $B(p)$, ……denote constants depending upon only p, and these are not the same in different occurrences.

If

$$\delta_n(k) = \int_{2^{-n}}^{2^{-(n-1)}} f(\theta) e^{-2\pi i k\theta}\, d\theta \qquad n = 1, 2, \ldots\ldots,$$

then

$$0 < B(p) \leqq \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} |\delta_n(k)|^2 \right\}^{p/2} \Big/ \sum_{k=-\infty}^{\infty} |c(k)|^p \leqq B'(p) < \infty.$$

Hirschman [1] proved the Walsh-Fourier series analogue of this theorem. The object of this paper is to supply the proof of Theorem 1. All known proofs of the Littlewood-Paley theorem depend upon the complex variable methods; For example, see Littlewood-Paley [2], Zygmund [9] [10] and Sunouchi [7]. In the proof of Theorem 1, we can not use the complex methods, because the variable is discrete. But the idea of the complex variable methods survives and the author follows the method given Zygmund [9] [10].

In **2** and **3**, we shall prove preliminary lemmas and in **4** we shall prove that

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} |\delta_n(k)|^2 \right\}^{p/2} \leqq B'(p) \sum_{k=-\infty}^{\infty} |c(k)|^p.$$

This is the most interesting part of the theorem.

In **5** and **6** we prove the converse inequality. From Theorem 1, we get the following l^p-multiplier theorem in **7**.

THEOREM 2. *If* $\lambda(\theta)$ *is a function such that*

$$|\lambda(\theta)| \leqq M, \qquad \int_{2^{-n}}^{2^{-(n-1)}} |d\lambda(\theta)| \leqq M \qquad (n = 1, 2, \ldots\ldots),$$

then, the hypothesis

$$f(\theta) \sim \sum_{k=-\infty}^{\infty} c(k) e^{2\pi i k\theta}, \quad \sum_{k=-\infty}^{\infty} |c(k)|^p < \infty \qquad (1 < p < \infty),$$

implies

$$\lambda(\theta) f(\theta) \sim \sum_{k=-\infty}^{\infty} d(k) e^{2\pi i k\theta}, \qquad \sum_{k=-\infty}^{\infty} |d(k)|^p < \infty,$$

and moreover

$$\sum_{k=-\infty}^{\infty} |d(k)|^p \leqq C(p) \sum_{k=-\infty}^{\infty} |c(k)|^p.$$

This corresponds to the L^p-multiplier theorem of Marcinkiewicz [3].

2. In this section, we prove a discrete analogue of M. Riesz's theorem and related theorems.

LEMMA 2. 1. *Let* $c(k) \in l^2 \cap l^p$ $(1 < p < \infty)$, *and*

$$c(k) = \int^1 f(\theta) e^{-2\pi ik\theta}\, d\theta,$$

$$c_t(k) = \int_0^t f(\theta) e^{-2\pi ik\theta}\, d\theta$$

then

$$\sum_{k=-\infty}^{\infty} |c_t(k)|^p \leqq A(p) \sum_{k=-\infty}^{\infty} |c(k)|^p \qquad (0 \leqq t \leqq 1).$$

PROOF. If we set

$$f_t(\theta) = \begin{cases} f(\theta) & 0 \leqq \theta \leqq t, \\ 0 & \text{elsewhere,} \end{cases}$$

then

$$f_t(\theta) = f(\theta) \chi_t(\theta)$$

where $\chi_t(\theta)$ is the characteristic function of the interval $(0, t)$. These functions have the following Fourier coefficients.

$$c_t(k) = \int_0^1 f_t(\theta) e^{-2\pi ik\theta}\, d\theta$$

$$c(k) = \int_0^1 f(t) e^{-2\pi ik\theta}\, d\theta$$

$$\frac{1 - e^{-2\pi ikt}}{2\pi ik} = \int_0^1 \chi_t(\theta) e^{-2\pi ik\theta}\, d\theta \qquad k \neq 0,$$

$$t = \int_0^1 \chi_t(\theta) d\theta \qquad k = 0.$$

If we denote by Σ' the summation that the term $m = 0$ is omitted, then

$$c_t(k) = \sum_{m=-\infty}^{\infty}{}' c(k-m) \frac{1 - e^{-2\pi imt}}{2\pi im} + tc(k)$$

$$= \sum_{m=-\infty}^{\infty}{}' \frac{c(k-m)}{2\pi im} - e^{-2\pi ikt} \sum_{m=-\infty}^{\infty}{}' \frac{c(k-m) e^{2\pi i(k-m)t}}{2\pi im} + tc(k)$$

$$= c_t^{(1)}(k) + c_t^{(2)}(k) + c_t^{(3)}(k),$$

say. If we denote the discrete Hilbert transform by

$$\tilde{c}(k) = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{c(k+m) - c(k-m)}{m},$$

then the well known M. Riesz's theorem [5] yields for $1 < p < \infty$

$$\sum_{k=-\infty}^{\infty} |\tilde{c}(k)|^p \leqq A(p) \sum_{k=-\infty}^{\infty} |c(k)|^p.$$

Hence we get for $j = 1, 2,$

$$\sum_{k=-\infty}^{\infty} |c_t^{(j)}(k)|^p \leqq B(p) \sum_{k=-\infty}^{\infty} |c(k)|^p$$

and the lemma is proved.

LEMMA 2. 2. *If for* $1 < p < \infty$, *then*

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{\infty} |\tilde{c}(k, l)|^2 \right\}^{p/2} \leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{\infty} |c(k, l)|^2 \right\}^{p/2},$$

where $\tilde{c}(k, l)$ *means the discrete Hilbert transform of* $c(k, l)$ *with respect to* k.

PROOF. The present author [7] showed the following result.

If

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2 dt \right\}^{p/2} dx < \infty, \qquad (1 < p < \infty)$$

then

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |\tilde{f}(x, t)|^2 dt \right\}^{p/2} dx \leqq A(p) \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2 dt \right\}^{p/2} dx$$

where $\tilde{f}(x, t)$ is the conjugate function of $f(x, t)$ with respect to the variable x. We can prove the Lemma by using this fact.

Let $\mathfrak{R}_{k,l}$ denote the square with center $p = p(k, l)$ where k, l are integers and edges of length 1. Given a sequence $c(k, l)$, let $f(x, t)$ denote the function taking the value $c(k, l)$ at the part of $\mathfrak{R}'_{k,l}$ (concentric and similarly situated square with edge 1/2), and equals to zero elsewhere.

Then

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{\infty} |c(k, l)|^2 \right\}^{p/2} \Big/ \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x, t)|^2 dt \right\}^{p/2} dx$$

depends upon p only. we set $[x] = k$, $[t] = l$, then

$$\tilde{f}(x, t) = \sum_{k' \neq k} c(k', l) \int_{k'-1/2}^{k'+1/2} \frac{dy}{x-y} + c(k, l) \int_{k-1/2}^{k+1/2} \frac{dy}{x-y}$$

$$= 2^{-2} \sum_{k' \neq k} c(k', l) \frac{1}{k-k'} + c(k, l) \int_{k-1/2}^{k+1/2} \frac{dy}{x-y}$$

$$+O\left[\sum_{k'\neq k} c(k',l)\left\{\int_{k'-1/2}^{k'+1/2} \frac{dy}{x-y} - 2^{-2}\frac{dy}{k-k'}\right\}\right]$$

$$= 2^{-2}\tilde{c}(k,l) + c(k,l)\chi_k(x) + O\left\{\sum_{k'\neq k} c(k',l)\frac{1}{(k-k')^2}\right\}$$

where $\chi_k(x)$ is the characteristic function of the interval $[k-1/2,\ k+1/2]$. Hence

$$|\tilde{c}(k,l)|^2 \leqq A|\tilde{f}(x,t)|^2 + B|c(k,l)|^2|\tilde{\varphi}_k(x)|^2 + O\left\{\sum_{k'\neq k} c(k',l)\frac{1}{(k-k')^2}\right\}^2.$$

Integrating with respect to t,

$$\sum_{l=-\infty}^{\infty} |\tilde{c}(k,l)|^2 \leqq A\int_{-\infty}^{\infty} |\tilde{f}(x,t)|^2 dt + B|\tilde{\chi}_k(x)|^2 \sum_{l=-\infty}^{\infty} |c(k,l)|^2 + \sum_{l=-\infty}^{\infty} O\left\{\sum_{k'\neq k} c(k',l)\frac{1}{(k-k')^2}\right\}^2.$$

Take the $p/2$-power of both side, and integrate with respect to x, then

$$\sum_{k=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} |\tilde{c}(k,l)|^2\right\}^{p/2}$$

$$\leqq A(p)\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} |\tilde{f}(x,t)|^2 dt\right\}^{p/2} dx + B(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} |c(k,l)|^2\right\}^{p/2}\cdot$$

$$\int_{\mathfrak{R}'_k} |\tilde{\chi}_k(x)|^p dx + C(p)\sum_{k=-\infty}^{\infty}\left[\sum_{l=-\infty}^{\infty}\left\{\sum_{k'\neq k} c(k',l)\frac{1}{(k-k')^2}\right\}^2\right]^{p/2}$$

$$= I + J + K,$$

say, where $\mathfrak{R}'_k$ means interval $[k-1/2, k+1/2]$. From the first remark,

$$I \leqq A'(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} |c(k,l)|^2\right\}^{p/2}$$

and

$$J \leqq B(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} |c(k,l)|^2\right\}^{p/2}\int_{\mathfrak{R}'_k} |\tilde{\chi}_k(x)|^p dx$$

$$\leqq B'(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} |c(k,l)|^2\right\}^{p/2},$$

because $\tilde{\chi}_k$ is the conjugate of the characteristic function of the interval $[k-1/2\ k+1/2]$. We have

$$K = C(p)\sum_{k=-\infty}^{\infty}\left[\sum_{l=-\infty}^{\infty}\left\{\sum_{k'\neq k} c(k',l)\frac{1}{(k-k')^2}\right\}^2\right]^{p/2}$$

$$\leqq C(p)\sum_{k=-\infty}^{\infty}\Big[\sum_{l=-\infty}^{\infty}\Big\{\sum_{k'\neq k}\frac{|c(k',l)|^2}{(k-k')^2}\sum_{k'\neq k}\frac{1}{(k-k')^2}\Big\}\Big]^{p/2}$$

$$\leqq C(p)\sum_{k=-\infty}^{\infty}\Big[\sum_{l=-\infty}^{\infty}\Big\{\sum_{k'\neq k}\frac{|c(k',l)|^2}{(k-k')^2}\Big\}\Big]^{p/2}$$

$$\leqq C(p)\sum_{k=-\infty}^{\infty}\Big[\sum_{k'\neq k}\frac{1}{(k-k')^2}\sum_{l=-\infty}^{\infty}|c(k',l)|^2\Big]^{p/2}$$

For the sake of simplicity, we set

$$\sum_{l=-\infty}^{\infty}|c(k',l)|^2=d(k'),$$

then the last term is

$$C(p)\sum_{k=-\infty}^{\infty}\Big\{\sum_{k'\neq k}\frac{d(k')}{(k-k')^2}\Big\}^{p/2}.$$

In the case $1<p\leqq 2$, applying Jensen's inequality, this is less than

$$C(p)\sum_{k'\neq k}\frac{1}{(k-k')^p}\sum_{k'=-\infty}^{\infty}|d(k')|^{p/2}$$

$$\leqq C(p)\sum_{k'=-\infty}^{\infty}|d(k')|^{p/2}\sum_{k'\neq k}\frac{1}{(k-k')^p}$$

$$\leqq C'(p)\sum_{k'=-\infty}^{\infty}|d(k')|^{p/2}\leqq C'(p)\sum_{k=-\infty}^{\infty}\Big\{\sum_{l=-\infty}^{\infty}|c(k,l)|^2\Big\}^{p/2}$$

Thus the lemma is proved for $1<p\leqq 2$. For the case $p\geqq 2$, we can get the lemma by the generalized conjugacy method for Banach-space valued functions given by Boas and Bochner [0].

By the same method, we can prove following two lemmas

LEMMA 2. 3. *Let $\tilde{c}(k,t)$ the discrese Hilbert transform of $c(k,t)$ with respect to k, then*

$$\sum_{k=-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}|\tilde{c}(k,t)|^2dt\Big\}^{p/2}\leqq A(p)\sum_{k=-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}|c(k,t)|^2dt\Big\}^{p/2}$$

for $1<p<\infty$.

LEMMA 2. 4. *Let $\tilde{c}(k,t)$ the discrete Hilbert transform of $c(k,t)$ with respect to k, then*

$$\sum_{k=-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}|\tilde{c}(k,t)|^2d\mu(t)\Big\}^{p/2}\leqq A(p)\sum_{k=-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}|c(k,t)|^2d\mu(t)\Big\}^{p/2}$$

where $\mu(t)=t$ or has a jump 1 where t is an integer.

LEMMA 2. 5. *For given*

$$c_n(k) \in l^2 \cap l^p(1 < p < \infty) \ (k = 0, \pm 1, \pm 2, \ldots\ldots; n = 1, 2, \ldots\ldots N),$$

we set

$$f_n(\theta) = \sum_{k=-\infty}^{\infty} c_n(k) e^{2\pi i k\theta},$$

$$c_n(k) = \int_0^1 f_n(\theta) e^{-2\pi i k\theta} d\theta,$$

$$c_{t_n}(k;\ c_n(k)) = \int_0^{t_n} f_n(\theta) e^{-2\pi i k\theta} d\theta$$

then

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{N} |c_{t_n}(k; c_n(k))|^2 \right\}^{p/2} \leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{N} |c_n(k)|^2 \right\}^{p/2}.$$

PROOF. From the view of Lemma 2. 1 and 2. 2, this Lemma is immediate.

LEMMA 2. 6 *Under the same assumption of the above Lemma, if we set*[1)]

$$C_{t_n}(\sigma_n + ik; c_n(k)) = \int_0^{t_n} f_n(\theta) e^{-2\pi(\sigma_n + ik)\theta} d\theta,$$

then

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{N} |C_{t_n}(\sigma_n + ik; c_n(k))|^2 \right\}^{p/2} \leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=1}^{N} |c_n(k)|^2 \right\}^{p/2} \quad (1 < p < \infty).$$

PROOF. Using the notation of Lemma 2. 5 and integrating by part

$$C_{t_n}(\sigma_n + ik;\ c_n(k)) = \int_0^{t_n} f_n(\theta) e^{-2\pi(\sigma_n + ik)\theta} d\theta$$

$$= \int_0^{t_n} e^{-2\pi\sigma_n\theta} d_\theta c_\theta(k; c_n(k))$$

$$= [e^{-2\pi\sigma_n\theta} c_\theta(k; c_n(k))]_0^{t_n} + 2\pi\sigma_n \int_0^{t_n} e^{-2\pi\sigma_n\theta} c_\theta(k;\ c_n(k)) d\theta$$

$$= e^{-2\pi\sigma_n t_n} c_{t_n}(k;\ c_n(k)) + 2\pi\sigma_n \int_0^{t_n} e^{-2\pi\sigma_n\theta} c_\theta(k;\ c_n(k)) d\theta.$$

1) We write $C(ik) = c(k)$ and $C_t(ik) = c_t(k)$.

Squaring and summing up with respect to n and applying Jensen's inequality, we have

$$\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} |C_{t_n}(\sigma_n+ik;\ c_n(k))|^2\right\}^{p/2}$$

$$\leqq A(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} |c_{t_n}(k;\ c_n(k))|^2\right\}^{p/2}$$

$$+ A'(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} 2\pi\sigma_n\int_0^{t_n} e^{-2\pi\sigma_n\theta}|c_\theta(k;\ c_n(k)|^2 d\theta\right\}^{p/2}$$

$$\leqq B(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} |c_n(k)|^2\right\}^{p/2}$$

by Lemma 2. 5. and Lemma 2. 4.

LEMMA 2. 7. *Let I_n is a subinterval in $[0, \sigma_n]$, then*

$$\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} C_{t_n}(\sigma_n + ik,\ c_n(k))|^2\right\}^{p/2}$$

$$\leqq A(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N}\frac{1}{|L_n|}\int_{L_n} |C(\sigma + ik;\ c_n(k))|^2 d\sigma\right\}^{p/2},$$

under the same hypothesis with Lemma 2. 5.

PROOF. From Lemma 2. 6, we have

$$\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} |C_{t_n}(\sigma_n + ik,\ c_n(k))|^2\right\}^{p/2}$$

$$\leqq A(p)\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{N} |C(\sigma_n' + ik,\ c_n(k)|^2\right\}^{p/2}$$

where $\sigma_n' \in (0,\ \sigma_n)$. Hence by the definition of Riemann integral, the lemma is immediate.

3. We shall define discrete analogue of Littlewood-Paley function $g(\theta)$ and Marcinkiewicz function $\mu(\theta)$.

Let us set

$$D(k) = \sum_{\nu=0}^{k} c(\nu),$$

$$M(k;\ c) = \left\{\sum_{m=1}^{\infty}\frac{|D(k+m) + D(k-m) - 2D(m)|^2}{m^3}\right\}^{1/2},$$

then we get the following lemma. $M(k; c)$ is a discrete analoge of $\mu(\theta)$.

LEMMA 3. 1. If; $c(k) \in l^p(1 < p \leqq 2)$, then

$$\sum_{k=-\infty}^{\infty} |M(k)|^p \leqq A(p) \sum_{k=-\infty}^{\infty} |c(k)|^p.$$

PROOF. We split the proof into two stages. In the first place, we prove

(1°) If $c(k) \in l^1$, then $M(k)$ is finite for all k, and

$$N_s \leqq \frac{A}{s} \sum_{k=-\infty}^{\infty} |c(k)|,$$

where N_s denotes numbers of k such as $M(k;\, c) > s$. This means that $M(k; c)$ is the operation of weak type (1,1) acting on l^1. But the proof of (1°) is only from words to words repetation of Stein's proof for continuous case [6,Lemma 7, p. 438]. So we omit details.

(2°) $M(k;\, c)$ is the operation of the strong type (2. 2), that is,

$$\sum_{k=-\infty}^{\infty} |M(k, c)|^2 \leqq B \sum_{k=-\infty}^{\infty} |c(k)|^2.$$

From the definition

$$\begin{aligned} &D(k+m) + D(k-m) - 2D(k) \\ &\quad = \int_0^1 f(x) \left\{ \sum_{\nu=0}^{k+m} + \sum_{\nu=0}^{k-m} - 2\sum_{\nu=0}^{k} \right\} e^{-2\pi i \nu x}\, dx \\ &\quad = 2i \int_0^1 f(x) e^{-2\pi i k x} \left(\sum_{\nu=1}^{m} \sin \nu x \right) dx \\ &\quad = 2i \int_0^1 f(x) e^{-2\pi i k x} \frac{\cos \frac{1}{2} x - \cos\left(m + \frac{1}{2}\right) x}{2 \sin \frac{1}{2} x}\, dx. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{k=-\infty}^{\infty} |D(k+m) + D(k-m) - 2D(k)|^2 \\ &\leqq A \sum_{m=1}^{\infty} \frac{1}{m^3} \int_0^1 \frac{|f(x)|^2}{x^2} \left| \cos \frac{1}{2} x - \cos\left(m + \frac{1}{2}\right) x \right|^2 dx \\ &\leqq A \int_0^1 |f(x)|^2 \left(\sum_{m=1}^{\infty} \frac{\left| \cos \frac{1}{2} x - \cos\left(m + \frac{1}{2}\right) x \right|^2}{m^3 x^2} \right) dx. \end{aligned}$$

Since the summation is

$$\sum_{m=1}^{\infty} \frac{\left| \cos \frac{1}{2} x - \cos\left(m + \frac{1}{2}\right) x \right|^2}{m^3 x^2}$$

$$\leqq A \sum_{m=1}^{[1/x]} \frac{(m+1)^4 x^4}{m^3 x^2} + \frac{B}{x^2} \sum_{m=[1/x]}^{\infty} \frac{1}{m^3}$$

$$= O(1),$$

it follows that

$$\sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{k=-\infty}^{\infty} |D(k+m) + D(h-m) - 2D(m)|^2$$

$$\leqq C \int_0^1 |f(x)|^2 dx = C \sum_{k=-\infty}^{\infty} |c(k)|^2.$$

Thus we have proved that $M(k, c)$ is the operation of strong type (2. 2). Hence we get Lemma 3. 1. by applying the interpolation theorem of Marcinkiewicz [10, p. 111].

We define another function $G(n)$, which is a discrete analogue of Littlewood-Paley function $g(\theta)$. Let

$$c(k) = \int_0^1 f(\theta) e^{-2\pi i k \theta} d\theta$$

and for $\sigma > 0$, we set[1)]

$$C(\sigma + ik) = \int_0^1 f(\theta) e^{-2\pi(\sigma + ik)\theta}$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{c(m)[1 - \exp\{2\pi(im - \sigma - ik)\}]}{\sigma + ik - im}$$

$$D_k C(\sigma + ik) = -2\pi \int_0^1 f(\theta) \theta e^{-2\pi(\sigma + ik)\theta} d\theta$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{c(m) \psi(im - \sigma - ik)}{(\sigma + ik - im)^2}$$

where

$$\psi(im - \sigma - ik) = \exp[2\pi(im - \sigma - ik) - 1 - 2\pi(im - \sigma - ik)\exp\{2\pi(im - \sigma - ik)\}]$$

and

$$G(k) = \left\{\int_0^{\infty} \sigma |D_k C(\sigma + ik)|^2 d\sigma\right\}^{1/2}.$$

LEMMA 3. 2. *Under the above notations,*

1) In $C(\sigma+ik)$, the variable k is discrete. Now we suppose k is continuons formally and differentiate $C(\sigma+ik)$ with respect to k. This is $D_k C(\sigma+ik)$.

$$G(k) \leqq AM(k).$$

PROOF. For the sake of simplicity, we denote

$$K(\sigma, k) = \frac{1}{2\pi} \frac{\psi(-\sigma - ik)}{(\sigma + ik)^2}$$

and set this into the definition $D_k C(\sigma + ik)$,

$$D_k C(\sigma + ik) = \sum_{m=-\infty}^{\infty} c(m) K(\sigma, k - m)$$

The partial summation yields

$$\sum_{m=-\infty}^{\infty} D(m) \Delta_m K(\sigma, k - m),$$

because

$$D(m) = o(m), \qquad as \ |m| \to \infty$$

$$\Delta_m K(\sigma, k - m) = O(m^{-3}) \ as \ |m| \to \infty.$$

Since

$$\sum_{m=-\infty}^{\infty} \Delta_m K(\sigma, k - m) = 0,$$

we can write

$$D_k C(\sigma + ik) = \sum_{m=0}^{\infty} \{D(k - m) + D(k + m) - 2D(k)\} \Delta_m K(\sigma, m).$$

On the other hand, from the definition of $K(\sigma, m)$, we have

$$\Delta_m K(\sigma, m) = O(\sigma^{-3}) \qquad for\ \sigma > 0$$

$$\Delta_m K(\sigma, m) = O(m^{-3}) \qquad for\ |m| > \sigma > 0.$$

If we denote

$$E_k(m) = D(k - m) + D(k - m) - 2D(k),$$

then

$$\begin{aligned} &\sigma |D_k C(\sigma + ik)|^2 \\ &= \sigma \left\{ \sum_{m=-\infty}^{\infty} E_k(m) \Delta_m K(\sigma, m) \right\}^2 \\ &= \sigma \left\{ \left(\sum_{m=-\infty}^{-\sigma-1} + \sum_{-\sigma}^{\sigma} + \sum_{\sigma+1}^{\infty} \right) E_k(m) \Delta_m K(\sigma, m) \right\}^2 \\ &\leqq A\sigma/\sigma^6 \left\{ \sum_{m=-\sigma}^{\sigma} E_k(m) \right\}^2 + B\sigma \left\{ \sum_{m=-\infty}^{-\sigma-1} + \sum_{m=\sigma+1}^{\infty} (E_k(m)/m^3) \right\}^2. \end{aligned}$$

By the Schwarz inequality, this last term is less than

$$\leqq A\sigma^{-4}\left\{\sum_{m=-\sigma}^{\sigma}|E_k(m)|^2\right\}+C\sum_{m=\sigma+1}^{\infty}|E_k(m)|^2/m^4$$

$$\leqq \alpha(\sigma)+\beta(\sigma),$$

say. Consequently

$$G^2(k)=\int_0^\infty \sigma|D_kC(\sigma+ik)|^2d\sigma$$

$$\leqq \int_0^\infty \alpha(\sigma)d\sigma+\int_0^\infty \beta(\sigma)d\sigma.$$

The first term is less than

$$\leqq A\int_0^\infty \sigma^{-4}\left\{\sum_{m=1}^{\sigma}|E_k(m)|^2\right\}d\sigma$$

$$\leqq 2A\sum_{m=1}^{\infty}|E_k(m)|^2\int_m^\infty \sigma^{-4}\,d\sigma$$

$$\leqq 3A\sum_{m=1}^{\infty}|E_k(m)|^2/m^3,$$

and the last term is less than

$$\leqq C\int_0^\infty d\sigma\sum_{m=\sigma}^{\infty}|E_k(m)|^2/m^4$$

$$\leqq C\sum_{m=1}^{\infty}|E_k(m)|^2/m^4\int_0^m d\sigma$$

$$\leqq C\sum_{m=1}^{\infty}|E_k(m)|^2/m^3.$$

Summing up these estimates, we get the lemma.

LEMMA 3. 3. *For* $1<p<\infty$,

$$\sum_{k=-\infty}^{\infty}|G(k)|^p\leqq A(p)\sum_{k=-\infty}^{\infty}|c(k)|^p.$$

From Lemma 3. 1 and 3. 2, we can show the lemma for $1<p\leqq 2$.

Next, we shall prove this for $\lambda\geqq 4$. In the following let us assume that $c_k=0$ for $k\leqq 0$; this is no restriction. Let

$$\frac{1}{p}+\frac{2}{\lambda}=1,$$

$$G(k)=G(k,c)$$

and $\xi(k)$ be any non-negative function such that

$$\left\{\sum_{k=1}^{\infty} |\xi(k)|^p\right\}^{1/p} \leqq 1, \qquad \textit{for } 1 < p \leqq 2,$$

and

$$G(k, \xi) = \Gamma(k).$$

Then

$$\left\{\sum_{k=1}^{\infty} |G^2(k)|^{\lambda}\right\}^{2/\lambda} = \left\{\sum_{k=1}^{\infty} |G^2(k)|^{\lambda/2}\right\}^{2/\lambda}$$

$$= \sup_{\xi} \sum_{k=1}^{\infty} G^2(k)\, \xi(k).$$

The last summation is

$$\sum_{k=1}^{\infty} G^2(k)\, \xi(k)$$

$$= \sum_{k=1}^{\infty}\left\{\int_0^{\infty} \sigma |D_k C(\sigma + ik)|^2 d\sigma\right\}\xi(k)$$

$$= \int_0^{\infty} \sigma \left\{\sum_{k=1}^{\infty} |D_k C(\sigma + ik)|^2\, \xi(k)\right\} d\sigma$$

$$= 4\int_0^{\infty} \sigma \left\{\sum_{k=1}^{\infty} |D_k C(2\sigma + ik)|^2\, \xi(k)\right\} d\sigma.$$

On the other hand, since Jensen's inequality yields

$$|D_k C(2\sigma + ik)|^2 \leqq \sum_{m=1}^{\infty} \frac{\sigma |D_k C(\sigma + im)|^2 \{1 - \exp 2\pi(im - \sigma - ik)\}}{|\sigma + ik - im|^2},$$

we can write

$$\sum_{k=1}^{\infty} |D_k C(2\sigma + ik)|^2\, \xi(k)$$

$$\leqq \sum_{k=1}^{\infty} \xi(k) \sum_{m=1}^{\infty} \frac{\sigma |D_k C(\sigma + im)|^2 |1 - \exp 2\pi(im - \sigma - ik)|}{|\sigma + ik - im|^2}$$

$$\leqq \sum_{m=1}^{\infty} |D_m C(\sigma + im)|^2 \cdot \Re\{\xi(\sigma + im)\}.$$

Hence

$$\sum_{k=1}^{\infty} G^2(k)\, \xi(k)$$

$$\leqq 4\int_0^\infty \sigma\left[\sum_{k=1}^\infty |D_kC(\sigma+ik)|^2\,\Re\{\xi(\sigma+ik)\}\right]d\sigma.$$

Applying the well known formula

$$4|F'|^2 = \Delta(F)$$

where Δ means Laplacian operator, it follows that

$$4|D_kC(\sigma+ik)|^2 = \Delta[|C(\sigma+ik)|^2\Re\{\xi(\sigma+ik)\}]$$
$$-2\{|C^2(\sigma+ik)|_\sigma|\Re\,\xi|_\sigma + |C^2(\sigma+ik)|_k|\Re\,\xi|_k\}.$$

$$\sum_{k=1}^\infty G^2(k)\,\xi(k)$$
$$\leqq 4\sum_{k=1}^\infty\int_0^\infty \Delta[|C(\sigma+ik)|^2\Re\,\xi(\sigma+ik)]\,\sigma d\sigma$$
$$+8\sum_{k=1}^\infty\int_0^\infty |C(\sigma+ik)|\,\{|C(\sigma+ik)|_\sigma|\Re\xi|_\sigma + |C(\sigma+ik)|_k|\Re\xi|_k\}\sigma d\sigma$$
$$= I_1 + I_2,$$

say. If we write,

$$|C(\sigma+ik)| \leqq \sup_{\sigma>0} C(\sigma+ik) = C^*(k)$$

then

$$I_2 \leqq 8\sum_{k=1}^\infty C^*(k)\left\{\int^\infty \sigma|D_kC(\sigma+ik)|^2d\sigma\right\}^{1/2}\left\{\int_0^\infty \sigma|D_k\xi(\sigma+ik)|^2d\sigma\right\}^{1/2}$$
$$\leqq 8\sum_{k=1}^\infty C^*(k)G(k)\Gamma(k)$$
$$\leqq 8\,\|C^*(k)\|_\lambda\|G(k)\|_\lambda\|\Gamma(k)\|_p \leqq A(p,\lambda)\|c(k)\|_\lambda\|G(k)_\lambda\|^{1)}.$$

Concerning with I_1, since differentiation with respect to k yields the same with respect to σ,

$$I_1 = 4\sum_{k=1}^\infty\int_0^\infty \sigma\Delta\{|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))\}d\sigma$$
$$= 4\sum_{k=1}^\infty\int_0^\infty \sigma\{|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))\}_{\sigma\sigma}d\sigma$$
$$= 4\sum_{k=1}^\infty\left\{[\sigma\{|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))\}_\sigma]_0^\infty - \int^\infty\{|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))\}_\sigma d\sigma\right\}$$

1) $\|C^*(k)\|_\lambda \leqq A(\lambda)\|c(k)\|_\lambda$, $(1<\lambda<\infty)$ is proved quite similarly to that of continuous case.

$$= -4\sum_{k=1}^{\infty}\int_0^{\infty}\{|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))\}_\sigma d\sigma$$

$$\leqq -4\sum_{k=1}^{\infty}[|C(\sigma+ik)|^2\Re(\xi(\sigma+ik))]_0^{\infty}$$

$$= 4\sum_{k=1}^{\infty}|C(ik)|^2|\xi(k)|$$

$$\leqq \|c(k)\|_\lambda^2\|\xi\|_p \leqq \|C(k)\|_\lambda^2$$

Hence

$$\|G(k)\|_\lambda^2 \leqq A(p,\lambda)\|C(k)\|_\lambda\|G(k)\|_\lambda + \|c(k)\|_\lambda{}^2$$

and we get

$$\|G(k)\|_\lambda \leqq B(p,\lambda)\|c(k)\|_\lambda.$$

This proves Lemma 3. 3 for $\lambda \geqq 4$. And the interpolation theorem of sublinear operations [10, p. 111] yields the Lemma completely.

4. Consider now the proof of the right half inequality. Our definition is

$$c(k) = \int_0^1 f(\theta)e^{-2\pi ik\theta}\,d\theta$$

$$f(\theta) \sim \sum_{\eta=-\infty}^{\infty} c(k)e^{-2\pi ik\theta}$$

$$C(\sigma+ik) = \int_0^1 f(\theta)e^{-2\pi(\sigma+ik)\theta}\,d\theta.$$

Moreover we define

$$c_t(k) = \int_0^t f(\theta)e^{-2\pi ik\theta}\,d\theta$$

$$C_t(\sigma+ik) = \int^t f(\theta)e^{-2\pi(\sigma+ik\theta)}\,d\theta.$$

The notation $D_kC_t(\sigma+ik)$ means that

$$D_kC_t(\sigma+ik)$$
$$= -2\pi\int_0^t f(\theta)\cdot\theta e^{-2\pi(\sigma+ik)\theta}\,d\theta.$$

Differentiating these formula with respect to t, then

$$\frac{\partial c_t(k)}{\partial t} = f(t)e^{-2\pi ikt}$$

$$\frac{\partial}{\partial t} D_k C_t(\sigma + ik) = f(t) t e^{-2\pi(\sigma+ik)t}$$

and

$$\frac{\partial}{\partial t} D_k C_t(\sigma + ik) = \left\{\frac{\partial}{\partial t} C_t(k)\right\} te^{-2\pi\sigma t}.$$

Hence we have

$$\begin{aligned}
\delta_n(k) &= \int_{2^{-n}}^{2^{-(n-1)}} f(\theta) e^{-2\pi i k\theta}\, d\theta \\
&= c_{2^{-(n-1)}}(k) - c_{2^{-n}}(k) \\
&= \int_{2^{-n}}^{2^{-(n-1)}} \frac{\partial c_t(k)}{\partial t}\, dt \\
&= \int_{2^{-n}}^{2^{-(n-1)}} \frac{\partial^{2\pi\sigma t}}{t} \left\{\frac{\partial}{\partial t} D_k C_t(\sigma + ik)\right\} dt \\
&= \left[\frac{e^{2\pi\sigma t}}{t} D_k C_t(\sigma + ik)\right]_{2^{-n}}^{2^{-(n-1)}} \\
&\quad - \int_{2^{-n}}^{2^{-(n-1)}} D_k C_t(\sigma + ik) \frac{2\pi\sigma t e^{2\pi\sigma t} - e^{2\pi\sigma t}}{t^2}\, dt \\
&= \frac{e^{2\pi\sigma 2^{-(n-1)}}}{2^{-(n-1)}} D_k C^{2^{-(n-1)}}(\sigma + ik) \\
&\quad - \frac{e^{2\pi\sigma 2^{-n}}}{2^{-n}} D_k C_{2^{-n}}(\sigma + ik) \\
&\quad - \int_{2^{-n}}^{2^{-(n-1)}} D_k C_t(\sigma + ik) \frac{2\pi\sigma t\ e^{2\pi\sigma t} + e^{2\pi\sigma t}}{t^2}\, dt \\
&= I_1(n, k) + I_2(n, k) + I_3(n, k)
\end{aligned}$$

say.

Then

$$|\delta_n(k)|^2 \leqq A(|I_1(n, k)|^2 + |I_2(n, k)|^2 + |I_3(n, k)|^2).$$

If we take $\sigma_n \in (2^n, 2^{n+1})$, then

$$\begin{aligned}
|I_2(n, k)|^2 &\leqq \left|\frac{e^{2\pi 2^n 2^{-(n-1)}}}{2^{-n}}\right|^2 |D_k C_{2^{-(n-1)}}(\sigma_n + ik)|^2 \\
&\leqq B 2^{2n} |D_k C_{2^{-(n-1)}}(\sigma_n + ik)|^2
\end{aligned}$$

and we may write

$$J_2 = \sum_{k=1}^{\infty} \left\{\sum_{n=1}^{\infty} |I_2(n, k)|^2\right\}^{p/2}$$

$$\leqq C(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}2^{2n}|D_kC_{2-(n-1)}(\sigma_n+ik)|^2\right\}\cdot$$

Let L_n be the interval $(2^n, 2^{n+1})$, and applying Lemma 2. 7, the last terms is less than

$$D(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}2^{2n}\frac{1}{|L_n|}\int_{L_n}|D_kC(\sigma+ik)|^2d\sigma\right\}^{p/2}$$
$$\leqq D(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}2^{n}\int_{L_n}|D_kC(\sigma+ik)|^2d\sigma\right\}^{p/2}$$
$$\leqq E(p)\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\sigma|D_kC(\sigma+ik)|^2d\sigma\right\}^{p/2}\cdot$$

In the same way, it follows that

$$J_1=\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}|I_p(n,k)|^2\right\}^{p/2}$$
$$\leqq E_1(p)\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\sigma|D_kC(\sigma+ik)|^2d\sigma\right\}^{p/2}\cdot$$

Concerning with $I_3(n, \mathrm{k})$, since $\sigma_n\in(2^n, 2^{n+1})$, we have

$$|I_3(n,k)|^2\leqq A\left\{\int_{2-n}^{2-(n-1)}|D_kC_t(\sigma+ik)|^2\frac{2\pi\sigma te^{2\pi\sigma t}+e^{2\pi\sigma t}}{t^2}dt\right\}^2$$
$$\leqq B\left\{\int_{2-n}^{2-(n-1)}|D_kC_t(\sigma_n+ik)|^2\frac{e^{4\tau}|2\pi+1|}{2^{-2(n-1)}}dt\right\}^2$$
$$\leqq C2^{4n}\left\{\int_{2-n}^{2-(n-1)}dt\right\}\left\{\int_{2-n}^{2-(n-1)}|D_kC_t(\sigma_n+ik)|^2dt\right\}$$
$$\leqq C2^{3n}\left\{\int_{2-n}^{2-(n-1)}|D_kC_t(\sigma_n+ik)|^2dt\right\}.$$

If we write

$$J_3=\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}|I_3(n,k)|^2\right\}^{p/2}$$

then

$$J_3\leqq D_3(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}2^{3n}\int_{2-n}^{2-(n-1)}|D_kC_t(\sigma+ik)|^2dt\right\}^{p/2}$$
$$\leqq E_3(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}2^{3n}\int_{2-n}^{2-(n-1)}\left\{\frac{1}{|L_n|}\int_{L_n}|D_kC(\sigma+ik)|^2d\sigma\right\}dt\right\}^{p/2}$$

where $L_n=(2^n, 2^{n+1})$. This is less than

$$\leqq F_3(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty} 2^{3n}\,2^{-n}\,2^{-n}\int_{L_n} |D_kC(\sigma+ik)|^2 d\sigma\right\}^{p/2}$$

$$\leqq G_3(p)\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\sigma|D_kC(\sigma+ik)|^2 d\sigma\right\}^{p/2}.$$

Summing up, these estimates,

$$\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}|\delta_n(k)|^2\right\}^{p/2}$$

$$\leqq A(p)\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}(|I_1(n,k)|^2+|I_2(n,k)|^2+|I_3(n,k)|^2)\right\}^{p/2}$$

$$\leqq B(p)\sum_{i=1}^{3}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{|I_i(n,k)|^2\right\}^{p/2}$$

$$\leqq C(p)\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\sigma|D_kC(\sigma+ik)|^2 d\sigma\right\}^{p/2}$$

$$\leqq D(p)\sum_{k=1}^{\infty}|G(k)|^p$$

by the definition. Hence, from Lemma 3.3, we have

$$\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}|\delta_n(k)|^2\right\}^{p/2}$$

$$\leqq B'(p)\sum_{k=-\infty}^{\infty}|c(k)|^p.$$

5. In this section, we prove the following lemma.

LEMMA 5. 1. *If $c(0)=0$, then for $1<p<\infty$,*

$$\sum_{k=-\infty}^{\infty}|c(k)|^p \leqq A(p)\sum_{k=-\infty}^{\infty}|G(k)|^p.$$

PROOF. Let

$$c_1(k)=\int_0^1 f_1(\theta)e^{-2\pi ik\theta}d\theta$$

$$c_2(k)=\int_0^1 f_2(\theta)e^{-2\pi ik\theta}d\theta$$

$$D_kC_1(\sigma+ik)=\int_0^1 f_1(\theta)\theta e^{-2\pi(\sigma+ik)\theta}\,d\theta$$

$$D_kC_2(\sigma+ik)=\int^1 f_2(\theta)\theta e^{-2\pi(\sigma+ik)\theta}\,d\theta,$$

then

$$\sum_{k=-\infty}^{\infty} D_k C_1(\sigma + ik) D_{-k} C_2(\sigma - ik)$$
$$= \int_0^1 f_1(\theta) f_2(\theta) \theta^2 e^{-2\pi\sigma\theta} d\theta$$
$$= I(\sigma),$$

say. Then

$$\int_\rho^\infty I(\sigma) d\sigma = \int_0^1 f_1(\theta) f_2(\theta) \theta^2 \frac{e^{-2\pi\rho\theta}}{2\pi\theta} d\theta.$$

$$\int_0^\infty d\rho \int_\rho^\infty I(\sigma) d\sigma = \int_0^1 f_1(\theta) f_2(\theta) \theta^2 \left[\frac{e^{-2\pi\rho\theta}}{-(2\pi\theta)^2} \right]_0^\infty$$
$$= A \int_0^1 f_1(\theta) f_2(\theta) \theta^2 \, d\theta.$$

On the other hand, Parseval's identity yields

$$\sum_{k=-\infty}^{\infty} c_1(k) c_2(-k) = \int^1 f_1(\theta) f_2(\theta) d\theta.$$

and we have

$$\left| \sum_{k=-\infty}^{\infty} c_1(k) c_2(-k) \right| \leqq A \int_0^\infty d\rho \int_\rho^\infty \left| \sum_{k=-\infty}^{\infty} D_k C_1(\sigma + ik) D_{-k} C_2(\sigma - ik) \right| d\sigma$$

$$\leqq A \sum_{k=-\infty}^{\infty} \int_0^\infty d\rho \int_\rho^\infty |D_k C_1(\sigma + ik)| \, |D_{-k} C_2(\sigma - ik)| \, d\sigma$$

$$= A \sum_{k=-\infty}^{\infty} \int_0^\infty d\sigma \int_0^\sigma |D_k C_1(\sigma + ik)| \, |D_{-k} C^2(\sigma - ik)| \, d\rho$$

$$= A \sum_{k=-\infty}^{\infty} \int_0^\infty \sigma |D_k C_1(\sigma + ik)| \, |D_{-k}(\sigma - ik)| \, d\sigma$$

$$\leqq A \sum_{k=-\infty}^{\infty} \left\{ \int_0^\infty |D_k C_1(\sigma + ik)|^2 d\sigma \right\}^{1/2} \left\{ \int_0^\infty \sigma |D_{-k} C_2(\sigma - ik)|^2 d\sigma \right\}^{1/2}$$

$$\leqq A \sum_{k=-\infty}^{\infty} G_1(k) G_2(-k).$$

We now set

$$h(k) = |c(k)|^{p-1} \text{sign } c(k), \qquad \Gamma(k) = G(k; h)$$

and p' is the conjugate exponent of p. Then

$$\left\{ \sum_{k=-\infty}^{\infty} |\Gamma(k)|^{p'} \right\}^{1/p'} \leqq A(p') \left\{ \sum_{k=-\infty}^{\infty} |h(k)|^{p'} \right\}^{1/p'}$$

$$\leqq A(p')\left\{\sum |c(k)|^p\right\}^{(p-1)/p},$$

using the result of the preceding section. Hence

$$\sum_{k=-\infty}^{\infty} |c(k)|^p = \sum_{k=-\infty}^{\infty} c(k)h(k) \leqq A \sum_{k=-\infty}^{\infty} G(\sigma)\Gamma(-k)$$

$$\leqq A\left\{\sum_{k=-\infty}^{\infty} |G(\sigma)|^p\right\}^{1/p}\left\{\sum_{k=-\infty}^{\infty} |\Gamma(-\sigma)|^{p'}\right\}^{1/p'}$$

$$\leqq B(p')\left\{\sum_{k=-\infty}^{\infty} |G(k)|^p\right\}^{1/p}\left\{\sum_{k=-\infty}^{\infty} |c(k)|^p\right\}^{(p-1)/p}.$$

This gives

$$\left\{\sum_{k=-\infty}^{\infty} |c(k)|^p\right\}^{1/p} \leqq C(p)\left\{\sum_{k=-\infty}^{\infty} |G(k)|^p\right\}^{1/p}.$$

6. We prove the another part of inequality, that is,

$$\sum_{k=-\infty}^{\infty} |c(k)|^p \leqq B(p) \sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{\infty} |\delta_n(k)|^2\right\}^{p/2}.$$

In the view of the preceding section, it is sufficient to prove

$$\sum_{k=-\infty}^{\infty} |G(k)|^p \leqq A(p) \sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{\infty} |\delta_n(k)|^2\right\}^{p/2}.$$

If we set

$$\int_0^\theta f(t)te^{-2\pi ikt}\,dt = h_k(\theta),$$

then

$$D_kC(\sigma+ik) = \int_0^1 f(\theta)\theta e^{-2\pi(\sigma+ik)\theta}d\theta$$

$$= \int_0^1 e^{-2\pi\sigma\theta}\,dh_k(\theta)$$

$$= [e^{-2\pi\sigma\theta}h_k(\theta)]_0^1 + 2\pi\sigma\int_0^1 e^{-2\pi\sigma\theta}\,h_k(\theta)d\theta$$

$$= e^{-2\pi\sigma}h_k(1) + 2\pi\sigma\int_0^1 e^{-2\pi\sigma\theta}\,h_k(\theta)d\theta$$

$$= I_1(\sigma, k) + I_2(\sigma, \sigma)$$

say. Let us now consider the term containing $I_1(\sigma, k)$. Since

$$I_1(\sigma, k) = e^{-2\pi\sigma}h_k(1),$$

$$\int_0^\infty \sigma |I_1(\sigma, k)|^2 d\sigma = \int_0^\infty \sigma e^{-2\pi\sigma} h_k(1) d\sigma$$
$$\leqq Ah_k(1).$$

We get

$$\sum_{k=-\infty}^{\infty} \left\{ \int_0^\infty \sigma |I_1(\sigma, k)|^2 d\sigma \right\}^{p/2} \leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ |h_k(1)|^2 \right\}^{p/2}.$$

Now

$$|h_k(1)|^2 = \left| \int_0^1 f(t) t e^{-2\pi ikt}\, dt \right|^2$$
$$= \left[\int_0^1 du \left\{ \int_0^1 f(t) e^{-2\pi ikt}\, dt - \int_0^u f(t) e^{-2\pi ikt}\, dt \right\} \right]^2$$
$$\leqq \int_0^1 du \left\{ \int_0^1 f(t) e^{-2\pi ikt}\, dt - \int_0^u f(t) e^{-2\pi ikt} dt \right\}^2$$
$$= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} du \left\{ \int_0^1 f(t) e^{-2\pi ikt}\, dt - \int_0^u f(t) e^{-2\pi tik}\, dt \right\}^2.$$

Hence

$$\sum_{k=-\infty}^{\infty} \left\{ \int_0^\infty \sigma |I_1(\sigma, k)|^2 d\sigma \right\}^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} du \left\{ \int_0^1 f(t) e^{-2\pi ikt}\, dt - \int_0^u f(t) e^{-2\pi ikt}\, dt \right\}^2 \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} 2^{-n} \left\{ \int_0^1 f(t) e^{-2\pi ikt}\, dt - \int_0^{2^{-(n+1)}} f(t) e^{-2\pi ikt} dt \right\}^2 \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} 2^{-n}\ |\delta_0(k) + \cdots\cdots + \delta_n(k)|^2 \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} 2^{-n} \left\{ \sum_{j=0}^{n} |\delta_j(k)| \right\}^2 \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} 2^{-n} \left\{ \sum_{j=0}^{n} |\delta_j(k)|^2 \right\} \cdot n \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{j=0}^{\infty} |\delta_j(k)|^2 \sum_{n=j}^{\infty} \frac{n}{2^n} \right]^{p/2}$$
$$\leqq \sum_{k=-\infty}^{\infty} \left[\sum_{j=0}^{\infty} |\delta_j(k)|^2 \right]^{p/2}.$$

Concerning with the term $I_2(\sigma, k)$, we have

$$
\begin{aligned}
&\int_0^\infty \sigma |I_2(\sigma, k)|^2 d\sigma \\
&= 4\pi^2 \int_0^\infty \sigma^3 \left| \int^1 e^{-2\pi\sigma\theta} h_k(\theta) d\theta \right|^2 d\sigma \\
&= 4\pi^2 \int_0^\infty \sigma^3 d\sigma \left| \int_0^{1/\sigma} e^{-2\pi\sigma\theta} h_k(\theta) \right|^2 d\theta + 4\pi^2 \int_0^\infty \sigma^3 d\sigma \left| \int_{1/\sigma}^\infty e^{-2\pi\sigma\theta} h_k(\theta) d\theta \right|^2 \\
&= 4\pi^2 P(k) + 4\pi^2 Q(k)
\end{aligned}
$$

say. Then

$$
\begin{aligned}
P(k) &= \int_0^\infty \sigma^3 d\sigma \left| \int_0^{1/\sigma} e^{-2\pi\sigma\theta} h_k(\theta) \right|^2 d\theta \\
&\leqq \int_0^\infty \sigma^3 d\sigma \int_0^{1/\sigma} |h_k(\theta)|^2 d\theta \int_0^{1/\sigma} e^{-4\pi\sigma\theta} d\theta \\
&\leqq \int_0^\infty \sigma^2 d\sigma \int_0^{1/\sigma} |h_k(\theta)|^2 d\theta \\
&= \int_0^1 \frac{dx}{x^4} \int_0^x |h_k(\theta)|^2 d\theta \\
&= \int_0^1 |h_k(\theta)|^2 d\theta \int_\theta^1 \frac{dx}{x^4} \leqq \int_0^1 |h_k(\theta)|^2 d\theta \int_\theta^\infty \frac{dx}{x^4} \\
&= \frac{1}{3} \int_0^1 |h_k(\theta)|^3 \theta^{-3} d\theta .
\end{aligned}
$$

$$
\begin{aligned}
Q(k) &= \int_0^\infty \sigma^3 d\sigma \left| \int_{1/\sigma}^\infty e^{-2\pi\sigma\theta} h_k(\theta) \right|^2 d\theta \\
&\leqq \int_0^\infty \sigma^3 d\sigma \left\{ \int_{1/\sigma}^\infty \frac{|h_k(\theta)|^2}{\theta^4} d\theta \right\} \left\{ \int_{1/\sigma}^\infty \theta^4 e^{-4\pi\sigma\theta} d\theta \right\} \\
&\leqq \int_0^\infty \sigma^3 d\sigma \left\{ \int_{1/\sigma}^\infty \frac{|h_k(\theta)|^2}{\theta^4} d\theta \right\} \frac{1}{\sigma^5} \\
&\leqq A \int_0^\infty \sigma^{-2} \int_{1/\sigma}^\infty \frac{|h_k(\theta)|^2}{\theta^4} d\theta \\
&= A \int_0^1 dx \int_x^1 \frac{|h_k(\theta)|^2}{\theta^4} d\theta \\
&= A \int_0^1 \frac{|h_k(\theta)|^2}{\theta^4} \int_0^\theta dx = A \int_0^1 \frac{|h_k(\theta)|^2}{\theta^3} d\theta .
\end{aligned}
$$

Collecting these estimates,

$$
\sum_{k=-\infty}^{\infty} \left\{ \int_0^\infty \sigma |I_2(\sigma, k)|^2 d\sigma \right\}^{p/2}
$$

$$\leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ \int_0^1 \frac{|h_k(\theta)|^2}{\theta^3} d\theta \right\}^{p/2}$$

$$\leqq A(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \frac{|h_k(2^{-(n+1)})|^2}{(2^{-n})^3} d\theta \right\}^{p/2}$$

$$\leqq B(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} |h_k(2^{-(n+1)})|^2 (2^{-n})^{-2} \right\}^{p/2}$$

$$= B(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{\left| \int_0^{2^{-(n+1)}} tf(t) e^{-2\pi ikt} dt \right|^2}{(2^{-n})^2} \right\}^{p/2}$$

$$= B(p) \sum_{k=-\infty}^{\infty} \{U(k)\}^p,$$

say.

$$U^2(k) = \sum_{n=0}^{\infty} \left| \int_0^{2^{-(n+1)}} tf(t) e^{-2\pi ikt} dt \right|^2 / |2^{-n}|^2$$

$$= \sum_{n=0}^{\infty} \left[\int_0^{2^{-(n+1)}} du \left\{ \int_0^{2^{-(n+1)}} f(t) e^{-2\pi ikt} dt - \int_0^u f(t) e^{-2\pi ikt} dt \right\} \right]^2 / |2^{-n}|^2$$

$$\leqq \sum_{n=0}^{\infty} \left[\int_0^{2^{-(n+1)}} \left\{ \int_0^{2^{-(n+1)}} f(t) e^{-2\pi ikt} dt - \int_0^u f(t) e^{-2\pi ikt} dt \right\}^2 \right] / |2^{-n}|$$

$$\leqq \sum_{n=0}^{\infty} \frac{1}{|2^{-n}|} \sum_{m=n}^{\infty} \int_{2^{-(m+1)}}^{2^{-(m)}} du \left\{ \int_0^{2^{-n}} f(t) e^{-2\pi ikt} dt - \int_0^u f(t) e^{-2\pi ikt} dt \right\}^2.$$

Hence, by Lemma 2. 5.

$$\sum_{k=-\infty}^{\infty} |U(k)|^p$$

$$\leqq A(p) \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{|2^{-n}|} \sum_{m=n}^{\infty} \int_{2^{-(m+1)}}^{2^{-m}} du \left\{ \int_0^{2^{-n}} f(t) e^{-2\pi ikt} dt - \int_0^u f(t) e^{-2\pi ikt} dt \right\}^2 \right]^{p/2}$$

$$\leqq A(p) \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{|2^{-n}|} \sum_{m=n}^{\infty} 2^{-m} \left\{ \int_0^{2^{-n}} f(t) e^{-2\pi ikt} dt - \int_0^{2^{-(m+1)}} f(t) e^{-2\pi ikt} dt \right\}^2 \right]^{p/2}.$$

Now, since the interior term is

$$\left| \int_0^{2^{-n}} f(t) e^{-2\pi ikt} dt - \int_0^{2^{-(m+1)}} f(t) e^{-2\pi ikt} dt \right|$$

$$\leqq |\delta_{n+1}(u)| + |\delta_{n+2}(k)| + \cdots\cdots + |\delta_{m+1}(k)|,$$

we get

$$\sum_{k=-\infty}^{\infty} |U(k)|^p$$

$$\leqq B(p) \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{|2^{-n}|} \sum_{m=n}^{\infty} 2^{-m} \{ |\delta_{n+1}(k)| + \cdots\cdots + |\delta_{m+1}(k)| \}^2 \right]^{p/2}$$

The interior sum is

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{|2^{-n}|} \sum_{m=n}^{\infty} 2^{-m} \{ |\delta_{n+1}(k)| + \cdots\cdots + |\delta_{m+1}(k)| \}^2 \\
&\leqq \sum_{n=0}^{\infty} 2^n \sum_{m=n}^{\infty} 2^{-m} \left\{ \sum_{j=n}^{m} |\delta_j(k)| \right\}^2 \\
&\leqq \sum_{n=0}^{\infty} 2^n \sum_{m=n}^{\infty} 2^{-m} \left\{ \sum_{j=n}^{m} |\delta_j(k)| \right\}^2 (m-n) \\
&\leqq \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |\delta_j(k)|^2 \sum_{m=n}^{\infty} 2^{-m} m \\
&\leqq \sum_{j=0}^{\infty} |\delta_j(k)|^2 \sum_{n=j}^{\infty} \sum_{m=n}^{\infty} \frac{m}{2^m} \\
&\leqq \sum_{j=0}^{\infty} |\delta_j(k)|^2 \sum_{n=j}^{\infty} \frac{n}{2^n} \\
&\leqq \sum_{j=0}^{\infty} |\delta_j(k)|^2.
\end{aligned}$$

Thus we have

$$\sum_{k=-\infty}^{\infty} |U(k)|^p \leqq A(p) \left\{ \sum_{j=0}^{\infty} |\delta_j(k)|^2 \right\}^{p|2}.$$

Summing up these estimates, we get the requring result.

7. In the last section, we prove Theorem 2. Let us set

$$f(\theta) \sim \sum_{k=-\infty}^{\infty} c(k) e^{2\pi i k\theta}$$

$$\lambda(\theta) f(\theta) \sim \sum_{k=-\infty}^{\infty} d(k) e^{2\pi i k\theta}$$

$$\int_{2^{-n}}^{t} f(\theta) e^{-2\pi i k\theta} = \delta_{n,t}(k)$$

$$\int_{2^{-n}}^{2^{-(n-1)}} f(\theta) \lambda(\theta) e^{-2\pi i k\theta}\, d\theta = \delta'_n(k),$$

then

$$\delta'_n(k) = \int_{2^{-n}}^{2^{-(n-1)}} f(\theta) \lambda(\theta) e^{-2\pi i k\theta}\, d\theta$$

$$= \lambda(2^{-(n-1)})\,\delta_n(k) - \int_{2^{-n}}^{2^{-(n-1)}} \delta_{n,\theta}(k)\,d\lambda(\theta).$$

Hence

$$|\delta_n'(k)|^2 \leqq |(\lambda 2^{-(n-1)})|^2 |\delta_n(k)|^2 + \int_{2^{-n}}^{2^{-(n-1)}} |\delta_{n,\theta}(k)|^2 |d\lambda(\theta)| \int_{2^{-n}}^{2^{-(n-1)}} |d\lambda(\theta)|,$$

$$\leqq M^2 |\delta_n(k)|^2 + M \int_{2^{-n}}^{2^{-(n-1)}} |\delta_{n,\theta}(k)|^2 |d\lambda(\theta)|,$$

by the hypothesis.

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} |\delta_n'(k)|^2 \right\}^{p/2} \leqq \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} M^2 |\delta_n(k)|^2 + M \int_{2^{-n}}^{2^{-(n-1)}} |\delta_{n,\theta}(k)|^2 |d\lambda(\theta)| \right\}^{p/2}$$

$$\leqq \sum_{k=-\infty}^{\infty} A(p) \left\{ \sum_{n=0}^{\infty} M^2 |\delta_n(k)|^2 + M \int_{2^{-n}}^{2^{-(n-1)}} |d\lambda(\theta)|\, |\delta_n(k)|^2 \right\}^{p/2}$$

$$\leqq \sum_{k=-\infty}^{\infty} A(p) \left\{ \sum_{n=0}^{\infty} M^2 |\delta_n(k)|^2 + M^2 |\delta_n(k)|^2 \right\}^{p/2}$$

$$\leqq B(p) \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=0}^{\infty} |\delta_n(k)|^2 \right\}^{p/2}$$

by the lemma 2.2. In the view of Theorem 1, we have Theorem 2.

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